Strongly Coupled $\mathcal{N} = 4$ SYM via Integrability

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2406.02698 with Nikolay Gromov and Paul Ryan

KING'S College LONDON

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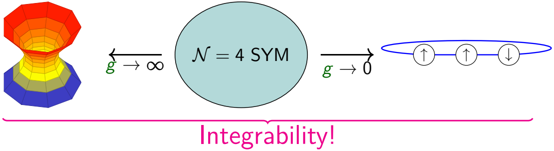
The Big Picture

- Planar $\mathcal{N} = 4$ Super-Yang Mills exhibits integrability [Minahan,Zarembo '03,Beisert '03,Beisert Staudacher '03...]



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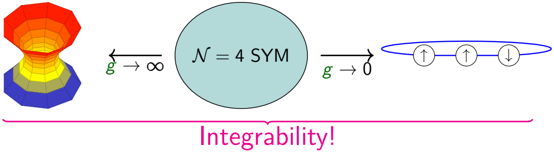

- We have a strong handle on the spectrum!

$$\langle \mathcal{O}(x)\bar{\mathcal{O}}(y)\rangle \sim \frac{1}{|x-y|^{2\Delta}} \quad \leftarrow \text{Computable!}$$

$$\overbrace{\mathcal{O} \simeq \operatorname{tr} \nabla^S Z^L}^{\text{Focus of today}}$$

Z ↳ Scalar field

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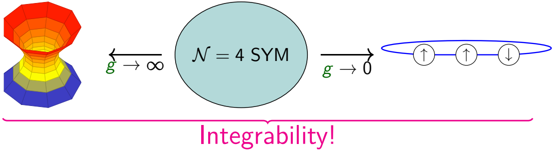

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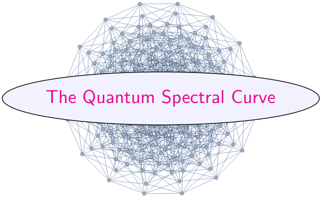
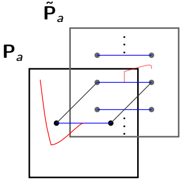
- Most powerful tool on the market? QSC! [Gromov, Kazakov,Leurent,Volin '13'14]

$$\tilde{\mathbf{P}}_a = \mu_{ab}\mathbf{P}^b$$



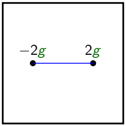


Content of the QSC

- The QSC is a set of 256 Q-functions, they depend on 1 complex parameter: u. The simplest Q-functions are called $\mathbf{P}_a$, $a = 1, \dots, 4$

$\mathbf{P}_a(u)$ ← Spectral parameter

$\mathbf{P}_a$



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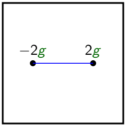

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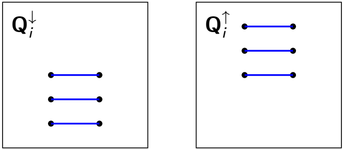
- From $\mathbf{P}_a$ we can build new functions $\mathbf{Q}_i^{\uparrow/\downarrow}$ from a 4th order Baxter equation

D_i are functions of $\mathbf{P}_a$

$$D_2\mathbf{Q}_i(u+2i) + D_1\mathbf{Q}_i(u+i) + D_0\mathbf{Q}_i(u) + D_{-1}\mathbf{Q}_i(u-i) + D_{-2}\mathbf{Q}_i(u-2i) = 0$$



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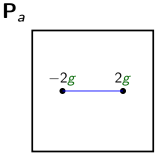

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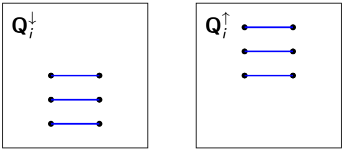
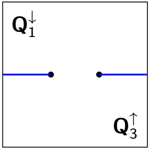


- Glue together $\{\mathbf{Q}_1^{\uparrow/\downarrow}, \mathbf{Q}_3^{\uparrow/\downarrow}\}$ and $\{\mathbf{Q}_2^{\uparrow/\downarrow}, \mathbf{Q}_4^{\uparrow/\downarrow}\}$ to form a long-cut function



Finding Δ

- Where is Δ? Asymptotics $\left(\mathcal{O} \sim \operatorname{tr} \nabla^S Z^L\right)$

$$\mathbf{P}_a \sim \left\{u^{-\frac{L}{2}-1}, u^{-\frac{L}{2}}, u^{\frac{L}{2}-1}, u^{\frac{L}{2}}\right\}, \quad \mathbf{Q}_i \sim \left\{u^{\frac{\Delta-S}{2}+1}, u^{\frac{\Delta+S}{2}}, u^{\frac{-\Delta-S}{2}-1}, u^{\frac{-\Delta+S}{2}-2}\right\}.$$

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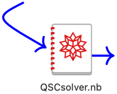
- Name of the game: Find all Q-functions and read of Δ!

Elevator pitch for the $\mathcal{N} = 4$ QSC

- Analytic weak coupling computations available ("Black box")

  [Marboe,Volin 18']

$\mathcal{O}_{\mathcal{K}} \propto \mathrm{tr}\, \Phi_I \Phi^I$



$$\gamma = 12g^2 - 48g^4 + 336g^6 + (-2496 + 576\zeta_3 - 1440\zeta_5)g^8 + \ldots$$

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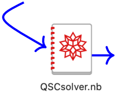

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- Analytic continuation in spin

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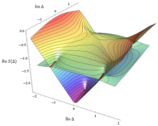

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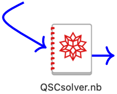

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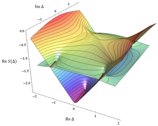
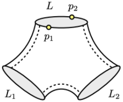


- Structure constants

  [Basso, Georgoudis, Klemenchuk Sueiro '22]



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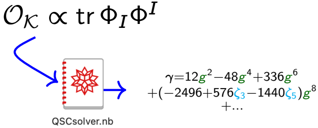

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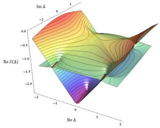

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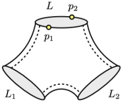
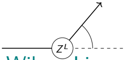


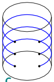
- There also exists many exciting variations and deformations:

[Gromov,Levkovich-Maslyuk '15]



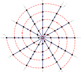
Wilson Lines

[Klabbers,van Tongeren '17]



η- deformations

[Gromov et al '17]



Fishnets

Some More Recent Topics

- The Hagedorn Temperature using QSC [Harmark,Wilhelm '17-'21,SE,Minahan,Thull'23,Harmark '24]

$$T_H^{\mathrm{AdS}_{d+1}} = \frac{1}{2\pi\sqrt{2\alpha'}} + \frac{d}{8\pi}$$

← [Urbach '22,Maldacena (unpublished)]

$$+\sqrt{\alpha'}\,\frac{d(d+1)-8d\log(2)}{16\sqrt{2}\pi} + \alpha'\,\frac{(d+2)(4d-1)d}{256\pi} + \mathcal{O}((\alpha')^{\frac{3}{2}})$$

[SE,Minahan,Thull '23] ⟶
[Bigazzi,Canneti,Cotrone,Mück+Castellani '23,'24]

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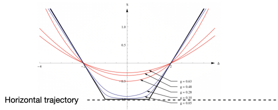
- The higher twist BFKL spectrum of $\mathcal{N} = 4$ [Klabbers, Preti,Szécsényi '23, SE,Gromov,Preti '24]

Twist 3 ex: (matches perfectly with [Kotikov, Lipatov, Rej, Staudacher, Velizhanin '07])



$$S(\Delta) + 2 = 2g - 4g^2\chi(\Delta) - \frac{2\pi^2}{3}g^3 + 24\left(\frac{\pi^2}{18}\chi(\Delta) + \chi''(\Delta) + \frac{7\zeta_3}{6}\right)g^4 + \mathcal{O}(g^5)$$

$$\chi(\Delta) = \Psi\left(\frac{1-\Delta}{2}\right) + \Psi\left(\frac{1+\Delta}{2}\right) + 2\gamma_E$$

Strong coupling

- Today: Strong Coupling. Why interesting? Quantization of free strings on a curved background! ($AdS_5 \times S^5$).

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- Various methods exists for "long operators" $L, S \to \infty$. Ex: Bethe Anstaz [Gleb Arutyunov,Frolov, Staudacher '04] and semi-classical quantisation [Gromov, Serban, Shenderovich, Volin '11]

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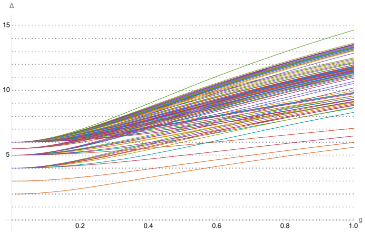
- We will focus on the most difficult regime:

Focus: Strong Coupling and Short Operators.
(ex. Konishi $\mathcal{O}_K = \text{tr}\, \nabla^2 Z^2$).

Strong coupling using QSC

- The QSC can produce numerical results [Gromov,Levkovich-Maslyusk,Sizov '15,Hegedus,Konczer '16,Gromov,Hegedus,Julius,Sokolova '23]



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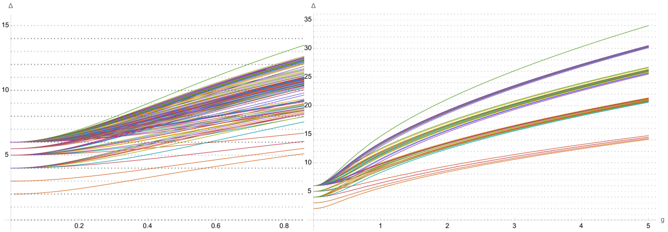

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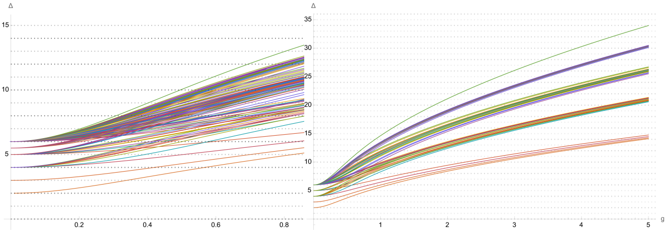

- Some results for the simplest $\mathfrak{sl}(2)$ family.

$$\Delta_{\text{Konishi}} = 2\lambda^{\frac{1}{4}} + 2\frac{1}{\lambda^{\frac{1}{4}}} + \left(\frac{1}{2} - 3\zeta_3\right)\frac{1}{\lambda^{\frac{3}{4}}} + \left(\frac{15}{2}\zeta_5 + 6\zeta_3 + \frac{1}{2}\right)\frac{1}{\lambda^{\frac{5}{4}}}$$

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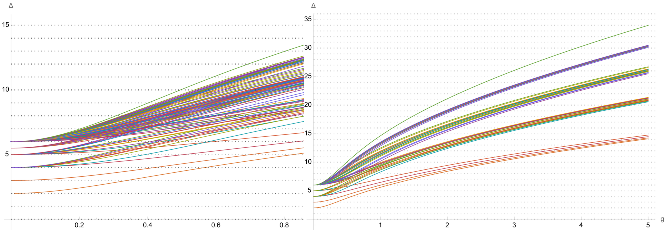

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- But very little is known about most other states

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- Goal of this talk:

A new strong coupling friendly approach to the QSC

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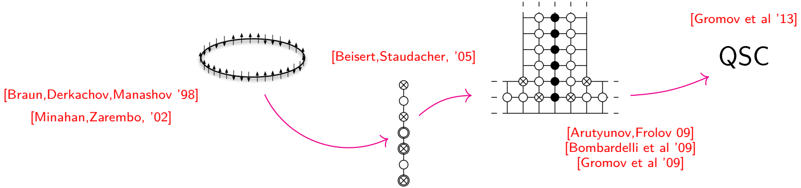
Crash Course On Q-Systems

Cheating and taking short-cuts

- To properly derive the QSC, we must go through a long chain of derivations. Historically:



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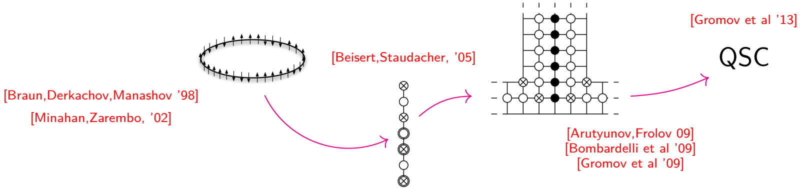

- To speed up I will follow the quicker path

$$\text{Spin Chains} \implies \text{Q-functions} \implies \text{QSC} \implies \text{Results!}$$

$\mathcal{N} = 4$ and spin chains

- Local gauge-invariant operators in $\mathcal{N} = 4$:

$$\mathcal{O} = \operatorname{tr} \mathcal{V}_1 \mathcal{V}_2 \ldots \mathcal{V}_L \qquad \mathcal{V} = \{\underset{\hookleftarrow \text{ 6 scalars}}{\Phi_I}, \psi_{\alpha,i}, \bar{\psi}^i_{\dot{\alpha}}, \mathcal{F}^+_{\alpha\beta}, \mathcal{F}^-_{\dot{\alpha}\dot{\beta}}\} + \text{Derivatives}$$

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- Recall, we want to compute the conformal dimension. In perturbation theory $\Delta = \Delta^0 + g^2\,\gamma_{(2)} + \mathcal{O}(g^4)$

$$\mathcal{H}\,\mathcal{O} = g^2\gamma_{(2)}\,\mathcal{O}$$

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- In the simplest possible case $\mathcal{V} = \{Z, X\} = \{\Phi_1 + \mathtt{i}\Phi_2, \Phi_5 + \mathtt{i}\Phi_6\}$

$$\mathcal{O} = \mathrm{tr}\,ZXZZ\ldots X \leftrightarrow |\uparrow\downarrow\uparrow\uparrow\ldots\downarrow\rangle \qquad \mathcal{H} = 2\,g^2\sum_{l=1}(\mathbb{1} - \mathbb{P})$$

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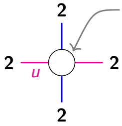
- $\mathcal{H}$ is an integrable Hamiltonian.

$\mathfrak{su}_2$ spin chain I

- Consider a homogeneous $\mathfrak{su}_2$ spin chain. This model has an R-matrix from which we can find a Lax matrix and then a transfer-matrix



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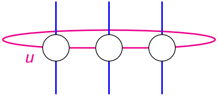
$t(u) =$ 

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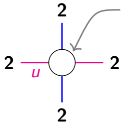

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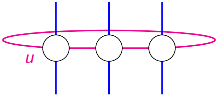
$t(u) =$ 

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where $f^{[n]} = f(u + \frac{\mathrm{i}}{2} n)$.

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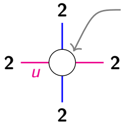

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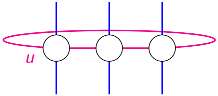
$t(u) =$ 

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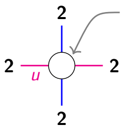

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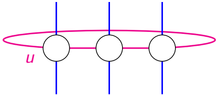

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- In particular:

$$\gamma_{(2)} = 2g^2 \sum_{i=1}^{M} \frac{1}{u_i^2 + \frac{1}{4}}$$

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- Can introduce polynomial Q_2 and write $t(u)$ in polynomial form:

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Q_1, Q_2 must satisfy the QQ/Wronskian-relation

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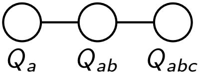
- Yet another way to attack the same problem is to use a Baxter equation. It is given as

$$(u - \frac{\mathtt{i}}{2})^L Q_a^{[2]} - t(u) Q_a + (u + \frac{\mathtt{i}}{2})^L Q_a^{[-2]} = 0\,, \quad \begin{vmatrix} Q_a^{[2]} & Q_a & Q_a^{[-2]} \\ Q_1^{[2]} & Q_1 & Q_1^{[-2]} \\ Q_2^{[2]} & Q_2 & Q_2^{[-2]} \end{vmatrix} = 0$$

$\mathfrak{su}_N$ Q-systems

- To go to $\mathfrak{su}_n$ attach a Q-"vector" to nodes on the Dynkin diagram

A_3:



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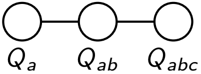

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- The various Q-functions are related by functional equations QQ-relations:
  $A = 1, 2, 3, 4, 12, 13, 14, \ldots$

$$Q^+_{Aa} Q^-_{Ab} - Q^-_{Aa} Q^+_{Ab} = Q_{Aab} Q_A \,, \qquad Q_{\bar{\emptyset}} = u^L \,.$$

Source term

$\bar{\emptyset} = 1234$

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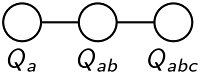

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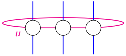
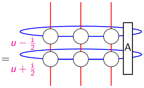
- We can once again formulate a Baxter equation. For simplicity $\mathfrak{su}_3$:

$$(u - \mathtt{i})^L Q_a^{[3]} - t_1(u)Q_a^+ + t_2(u)Q_a^- - (u + \mathtt{i})^L Q_a^{[-3]} = 0$$

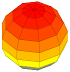
$t_1(u) =$ 

$t_2(u) =$ 

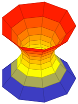
$$\begin{vmatrix} Q_a^{[3]} & Q_a^{[1]} & Q_a^{[-1]} & Q_a^{[-3]} \\ Q_1^{[3]} & Q_1^{[1]} & Q_1^{[-1]} & Q_1^{[-3]} \\ Q_2^{[3]} & Q_2^{[1]} & Q_2^{[-1]} & Q_2^{[-3]} \\ Q_3^{[3]} & Q_3^{[1]} & Q_3^{[-1]} & Q_3^{[-3]} \end{vmatrix} = 0$$

Supersymmetric Q-systems

- For $\mathcal{N} = 4$ SYM we need a supersymmetric $\mathfrak{psu}(2,2|4)$ Q-system. Start from two separate $\mathfrak{su}(4)$ Q-systems.



$$\underbrace{Q_{a|\emptyset} = \mathbf{P}_a}_{\text{compact}} \qquad \underbrace{Q_{\emptyset|i} = \mathbf{Q}_i}_{\text{non-compact}} .$$



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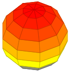

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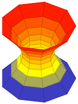

- We connect them through additional QQ-relations

$$\mathbf{P}_a=-Q^{\pm}_{a|i}\mathbf{Q}^i\,,\qquad \mathbf{Q}_i=-Q^{\pm}_{a|i}\mathbf{P}^a\,,\qquad Q^+_{a|i}-Q^-_{a|i}=\mathbf{P}_a\mathbf{Q}_i\,,$$

Where

$$\mathbf{P}^a=\chi^{ab}\mathbf{P}_b\,,\quad \mathbf{Q}^i=\chi^{ij}\mathbf{Q}_j\,,\quad \chi=\begin{pmatrix}0&0&0&-1\\0&0&1&0\\0&-1&0&0\\1&0&0&0\end{pmatrix}. \tag{1.1}$$

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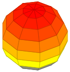

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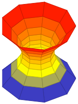

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- Once again there is a Baxter equation:

$$\mathbf{Q}_i^{[4]} D_0 - \mathbf{Q}_i^{[2]} D_1 + \mathbf{Q}_i D_2 + D_3 \mathbf{Q}_i^{[-2]} + D_4 \mathbf{Q}_i^{[-4]} = 0$$

$$D_0 = \begin{vmatrix} \mathbf{P}_1^{[2]} & \mathbf{P}_1 & \mathbf{P}_1^{[-2]} & \mathbf{P}_1^{[-4]} \\ \mathbf{P}_2^{[2]} & \mathbf{P}_2 & \mathbf{P}_2^{[-2]} & \mathbf{P}_2^{[-4]} \\ \mathbf{P}_3^{[2]} & \mathbf{P}_3 & \mathbf{P}_3^{[-2]} & \mathbf{P}_3^{[-4]} \\ \mathbf{P}_4^{[2]} & \mathbf{P}_4 & \mathbf{P}_4^{[-2]} & \mathbf{P}_4^{[-4]} \end{vmatrix}$$

Quick Recap of $\mathfrak{psu}_{2,2|4}$ Q-systems

- Important set of Q-functions:
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$\mathfrak{so}_6$ quantum numbers

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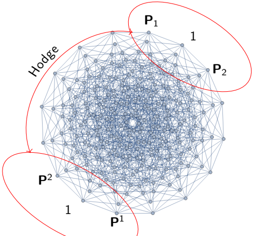
$\mathfrak{su}_{2,2}$ quantum numbers

- **Q** and **P** are related through a Baxter equation.
- One can also define a plethora of additional Q-functions related through finite difference equations. We won't need them.



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The Quantum Spectral Curve

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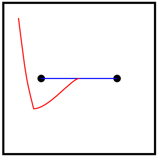
- New feature: Analytic Properties!. Simplest for $\mathbf{P}_a(u)$:

$\mathbf{P}_a$



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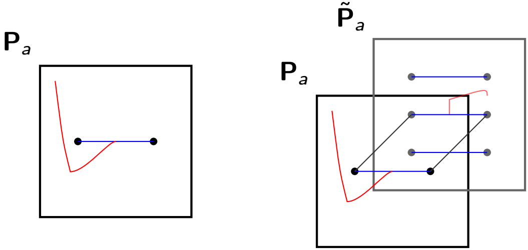

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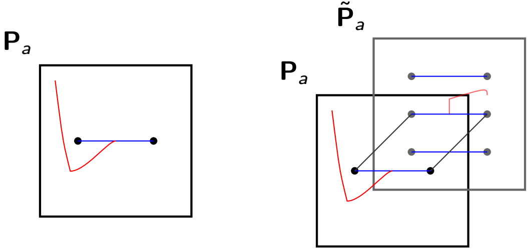
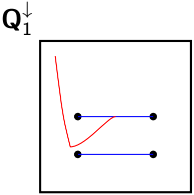


- Slightly more complicated for $\mathbf{Q}_i$



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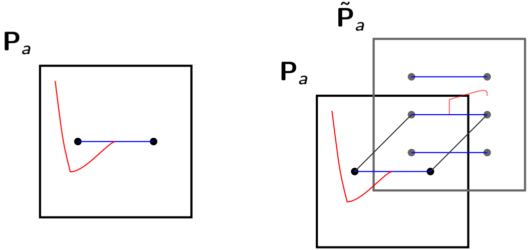

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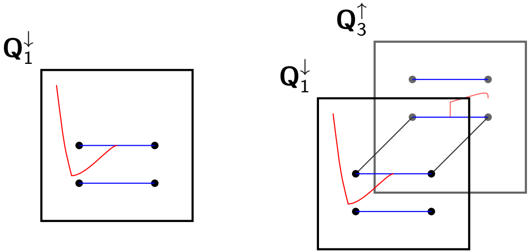

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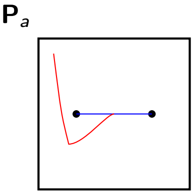

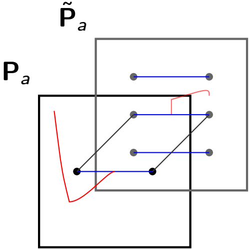

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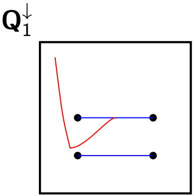

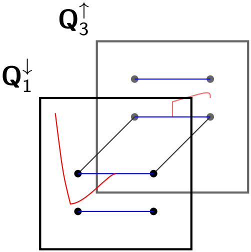

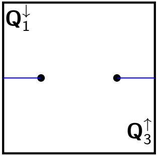

Zhukovsky

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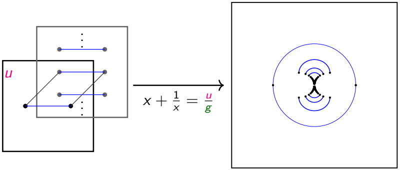

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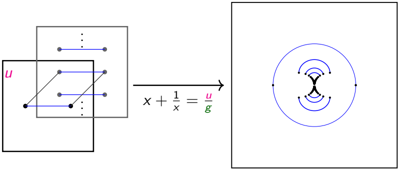
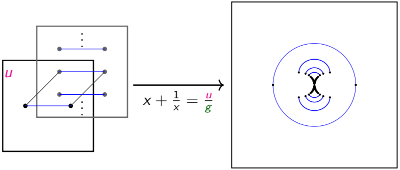


- Parameterise $\mathbf{P}_a$ as

$$\mathbf{P}_a = \sum_n \frac{c_{a,n}}{x^n}\,.$$

Zhukovsky

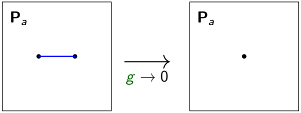
- It will be very convenient to work with the Zhukovsky variable $x : x + \frac{1}{x} = \frac{u}{g}$:



- Parameterise $\mathbf{P}_a$ as

$$\mathbf{P}_a = \sum_n \frac{c_{a,n}}{x^n} \, .$$

- At weak coupling, $g \to 0, x \to \frac{u}{g} + \ldots$, the cuts collapses and we find a a rational spin chain.



QSC at strong coupling

What Happens at Strong Coupling?

- Search for an equivalent picture at Strong Coupling [Hegedűs,Konczer '16]!

$$\mathbf{P}_a \propto \sum_m^\infty \frac{d_{a,m}}{(x^2-1)^m} \longrightarrow \underbrace{\tilde{p}_{1,2} = -\tilde{p}_{3,4}}_{\text{quasi-momenta}} = 2\pi\mathcal{L}\frac{x}{x^2-1}$$

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- Information at $x = \pm 1$ is crucial. We capture it using densities:

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- Why?

$$\begin{cases} \text{Classically} & \tilde{p}_1(\frac{1}{x}) - \tilde{p}_3(x) = 0 \\ \text{Quantum} & \underbrace{\mathbf{P}_1(\frac{1}{x}) - \mathbf{P}_3(x) \simeq_{x\neq\pm 1} 0}_{\mathbf{P}\mu\text{-system}} \end{cases}$$

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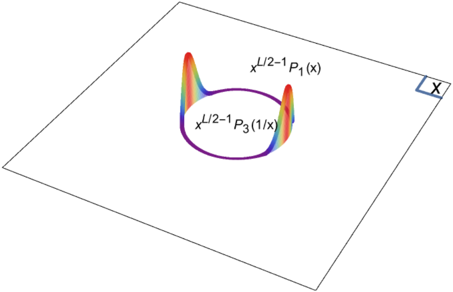

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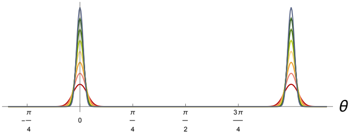

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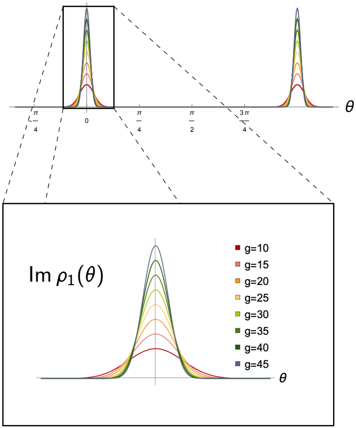

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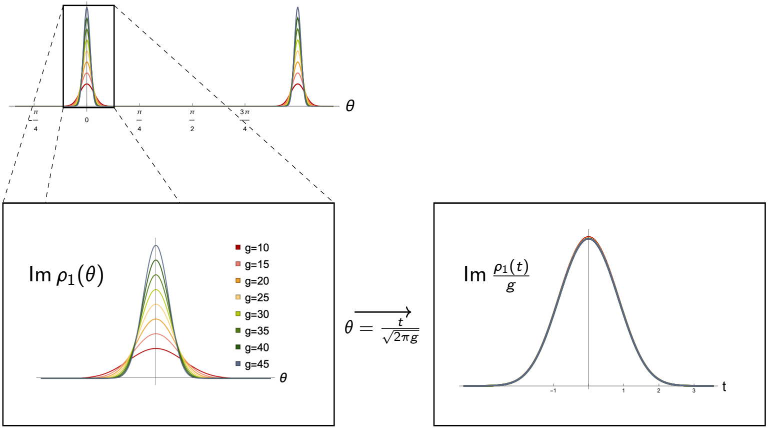

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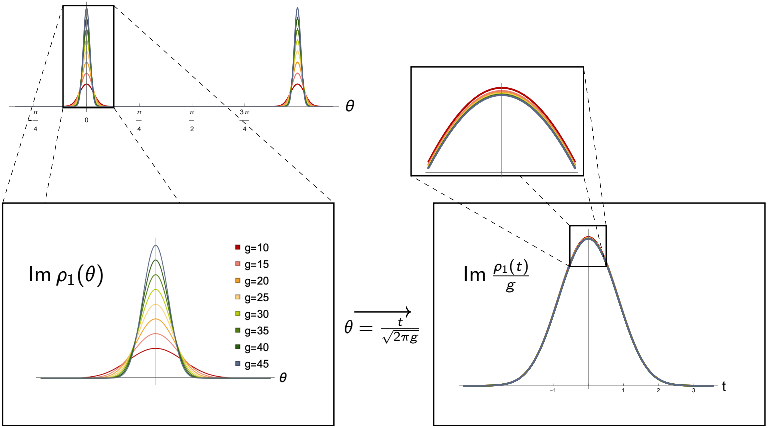

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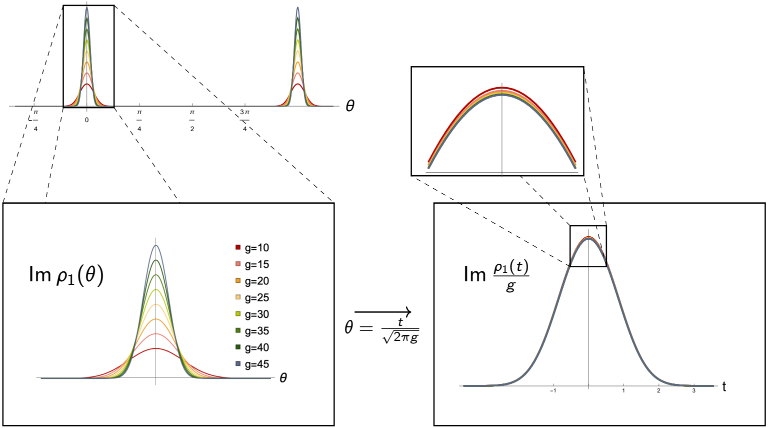
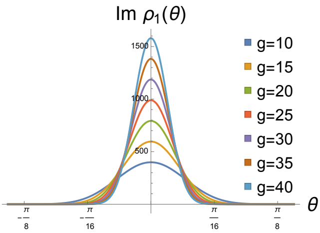
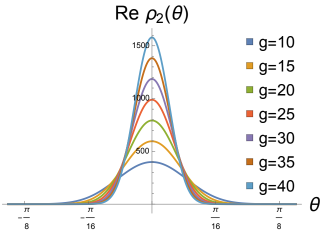


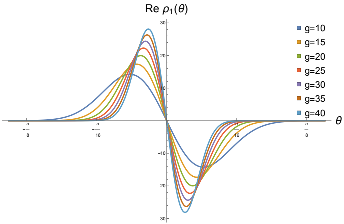
- There exist a finite limiting shape for ρ at $g \to \infty$! We can expand

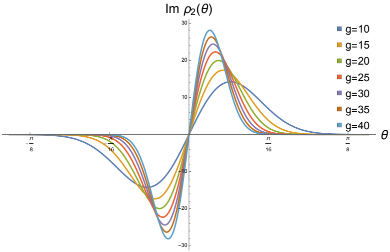
$$\rho_a = g\,\rho_a^{(1)} + \sqrt{g}\,\rho_a^{(\frac{1}{2})} + \rho_a^{(0)} + \dots$$

Pictures of ρ



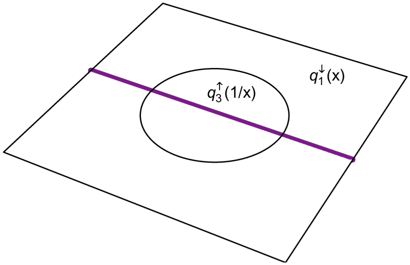






What about Q?

- We can find a similar representation for $\mathbf{Q}$. Define $\mathbf{q}_1(x) = x^{\frac{-\Delta+S-4}{2}}\mathbf{Q}_1(x), \mathbf{q}_3(x) = x^{\frac{\Delta-S+4}{2}}\mathbf{Q}_3(x)$



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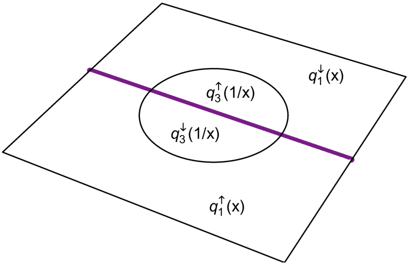

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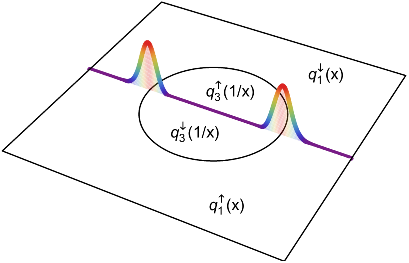

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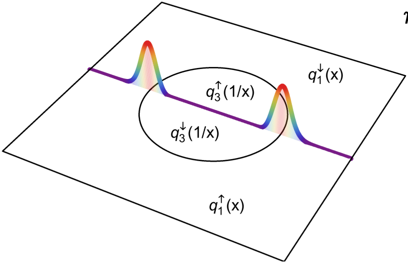

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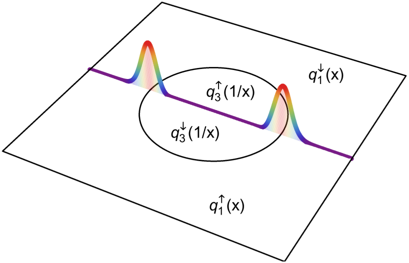

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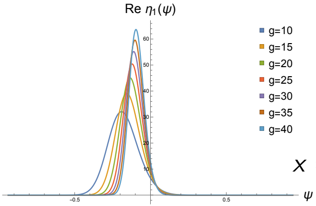

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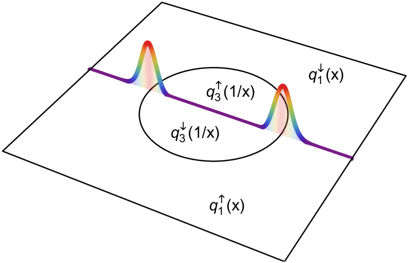

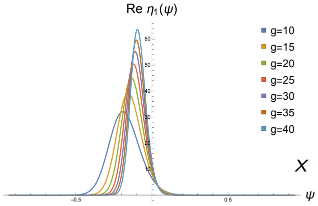

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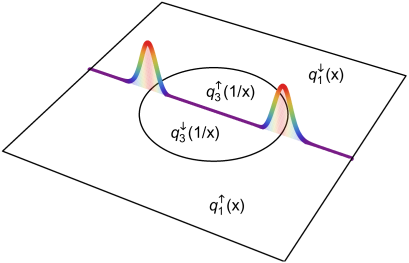

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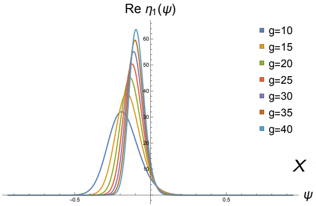

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- The density once again have a well defined expansion:

$$\eta_i(s) = \sqrt{g}\,\eta_i^{(\frac{1}{2})}(s) + \frac{\eta_i^{(-\frac{1}{2})}(s)}{\sqrt{g}} + \mathcal{O}(g^{-\frac{3}{2}}) \qquad x = e^{\frac{s}{\sqrt{2\pi g}}}$$

Closing the Equations

- Recap: We can express all Q-functions using the densities η_i, ρ_a:
  $(\mathrm{tr}\,\nabla^S Z^L, \Delta \sim \sqrt{2S}\,\lambda^{\frac{1}{4}} + \mathcal{O}(\lambda^{-\frac{1}{4}}))$

$$\rho_a(t) \sim g\,\rho_a^{(1)}(t) + \sqrt{g}\,\rho_a^{(\frac{1}{2})}(t) + \dots \quad \eta_i(s) \sim \sqrt{g}\,\eta^{(\frac{1}{2})}(s) + \frac{1}{\sqrt{g}}\,\eta^{(-\frac{1}{2})}(s) + \dots$$

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- Ideas:
  - Asymptotic Bethe Ansatz?
  - Numerics

Numerics

The Numerical Algorithm: Main Idea

- For numerics we parameterize the densities: → Polynomials

$$\rho_a = e^{2\pi(u-2g)} \sum c_{a,n}\, \mathcal{P}_n(\theta)\,, \qquad \eta_i = e^{-2\pi(u-2g)} \sum d_{i,n}\, \mathcal{Q}_n(\psi)\,.$$

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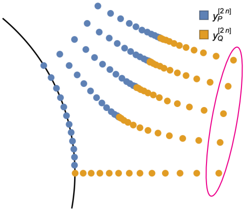
- Compute $\mathbf{P}_a, \mathbf{Q}_i$ on a set of probe points and find $\mathbf{Q}^{(B)}/\mathbf{P}^{(B)}$ on the real line/unit circle using Baxter



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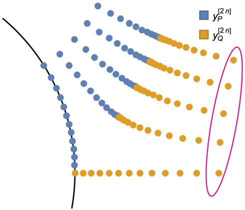

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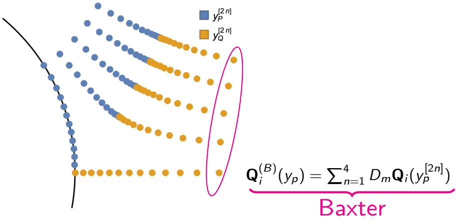

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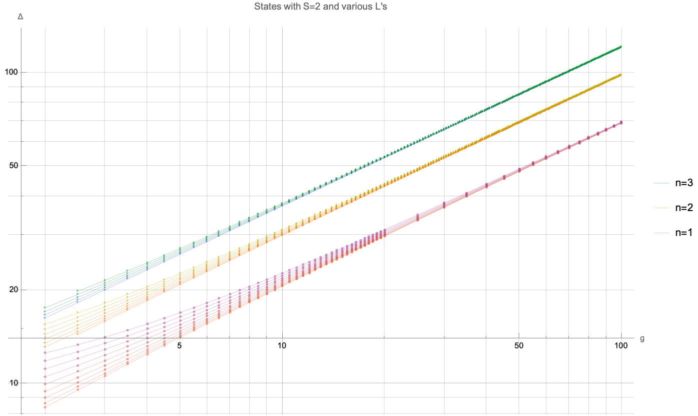
- Finally, find new $c_{a,n}$ and $d_{i,n}$ by minimizing

$$\rho_a - \rho_a^{(B)}\,, \quad \eta_i - \eta_i^{(B)}\,.$$

Numerical Algorithm: Result

Example for $\mathrm{tr}\,\nabla^2 Z^L$



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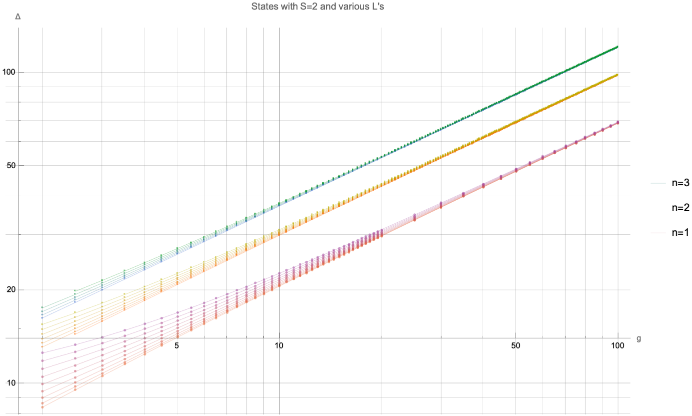

- This is really strong coupling $\lambda = 16\pi^2 g^2 \implies \lambda \sim 10^6$

Analytic Results

Konishi and Friends

- Consider operators $\mathcal{O} = \mathrm{tr}\, \nabla^S Z^L$, $\Delta \sim \sqrt{2S}\, \lambda^{\frac{1}{4}}$

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$$(\Delta+2)^2 - L^2 = S\left(\sqrt{\lambda}\,A_1 + A_2 + \frac{A_3}{\sqrt{\lambda}} + \ldots\right) + S^2\left(B_1 + \frac{B_2}{\sqrt{\lambda}} + \frac{B_3}{\lambda} + \ldots\right)$$

$$+ S^3\left(\frac{C_1}{\sqrt{\lambda}} + \frac{C_2}{\lambda} + \frac{C_3}{\lambda^{\frac{3}{2}}}\right) + S^4\left(\frac{D_1}{\lambda} + \frac{D_2}{\lambda^{\frac{3}{2}}}\right) + S^5\,\frac{E_1}{\lambda^{\frac{3}{2}}} + \mathcal{O}\left(\frac{1}{\lambda^2}\right)$$

Quasi-classics [Gromov,Valatka '11]

To be fixed!

Konishi and Friends

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$$+ S^3\left(\frac{C_1}{\sqrt{\lambda}} + \frac{C_2}{\lambda} + \frac{C_3}{\lambda^{\frac{3}{2}}}\right) + S^4\left(\frac{D_1}{\lambda} + \frac{D_2}{\lambda^{\frac{3}{2}}}\right) + S^5\,\frac{E_1}{\lambda^{\frac{3}{2}}} + \mathcal{O}\left(\frac{1}{\lambda^2}\right)$$

Quasi-classics [Gromov,Valatka '11]

To be fixed!

- From our numerics we find with accuracy 10^{-21}

$$C_3 = \frac{13}{16}\,L^2 - \frac{15}{4}\zeta_5 + 21\zeta_3 - 9\zeta_3^2 + \frac{131}{128}$$

Konishi and Friends

- Consider operators $\mathcal{O} = \mathrm{tr}\, \nabla^S Z^L$, $\Delta \sim \sqrt{2S}\, \lambda^{\frac{1}{4}}$
- General expansion: Slope [Basso '11] Curvature [Gromov,Levkovich-Maslyuk,Sizov,Valatka '14]

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- Konishi ($L = 2, S = 2$)

$$\Delta_{\text{Konishi}} = 2\lambda^{\frac{1}{4}} - 2 + 2\frac{1}{\lambda^{\frac{1}{4}}} + \left(\tfrac{1}{2} - 3\zeta_3\right)\frac{1}{\lambda^{\frac{3}{4}}} + \left(6\,\zeta_3 + \tfrac{15}{2}\zeta_5 + \tfrac{1}{2}\right)\frac{1}{\lambda^{\frac{5}{4}}}$$

$$\left(-\frac{81\,\zeta_3^2}{4} + \frac{\zeta_3}{4} - 40\,\zeta_5 - \frac{315\,\zeta_7}{16} - \frac{27}{16}\right)\frac{1}{\lambda^{\frac{7}{4}}}$$

New!

$n = 2$

- Naive generalisation for $\operatorname{tr} \nabla^S Z^L$, $\Delta \sim \sqrt{4S}\,\lambda^{\frac{1}{4}} + \mathcal{O}(\lambda^0)$, $(n = 2)$

$$\Delta^{\text{Expected}}_{n=2,S=2} = 2\sqrt{2}\,\lambda^{\frac{1}{4}} - 2 + \frac{(L^2+4)}{4\sqrt{2}\,\lambda^{\frac{1}{4}}} + \frac{-4\,L^4 + 96\,L^2 - 64(96\,\zeta_3 + 11)}{512\,\sqrt{2}\,\lambda^{\frac{3}{4}}}$$

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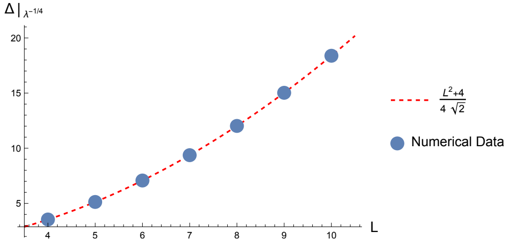
- Third term vs data



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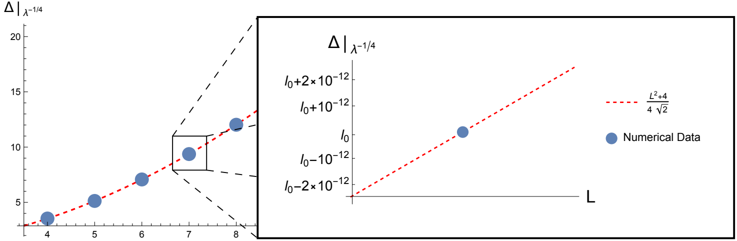

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- The fourth term?

$$512\sqrt{2}\Delta^{\text{Fit}}\big|_{\lambda^{-\frac{3}{4}}} = -4.0386\,L^4 + 102.2702\,L^2 - 8370.5094$$

$$512\sqrt{2}\Delta^{\text{Expected}}\big|_{\lambda^{-\frac{3}{4}}} = -4.0000\,L^4 + 96.0000\,L^2 - 8089.4376$$

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- The fourth term? Add $\frac{1}{L^2}$!

$$512\sqrt{2}\Delta^{\text{Fit}}\big|_{\lambda^{-\frac{3}{4}}} = -4.0000\,L^4 + 96.0003\,L^2 - 8089.4 - \frac{3067.8}{L^2} - \frac{6317.6}{L^4}$$

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A little bit of analytics

- Where is L injected into the QSC?

$$\mathbf{P}_a \simeq \mathbb{A}_a\, u^{-M_a}\,, \quad \underbrace{\Longrightarrow}_{\text{QQ-relations}} \begin{cases} \mathbb{A}_1 \mathbb{A}_4 = \frac{64\,\mathtt{i}\,\pi^2}{L(L+1)}\, g^2 + \dots \\ \mathbb{A}_2\, \mathbb{A}_3 = \frac{64\,\mathtt{i}\,\pi^2}{L(L-1)}\, g^2 + \dots\,. \end{cases}$$

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$$\left(\int_{-\infty}^{\infty} dt\, t\, \rho_1^{(\frac{3}{2})}(t) \right)^2 \propto \frac{L^2 - 18 \pm \sqrt{L^4 - 4L^2 + 36}}{(L^2-9)(L^2-1)}$$

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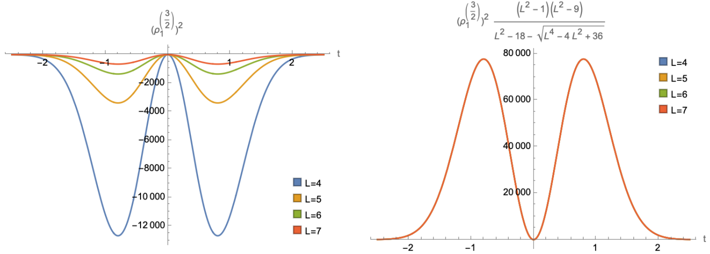

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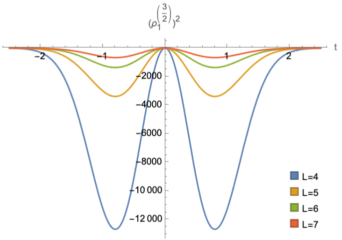

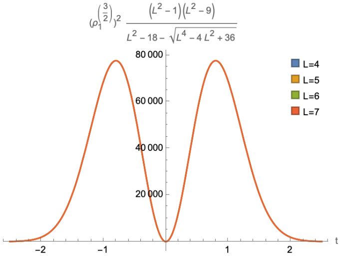

- Natural guess: Fit $\{L^4, L^2, 1, \sqrt{L^4 - 4L^2 + 36}\}$.

Result and Mixing

- Numerics gives

$$(\Delta_{n=2,S=2}+2)^2 = 8\,\lambda^{\frac{1}{2}} + (L^2+4) + \frac{\frac{5L^2}{2} - \frac{3}{2}\sqrt{L^4-4L^2+36} - 48\zeta_3 - 8}{\sqrt{\lambda}}$$

$$+ \frac{6L^2 - \frac{3(2L^4-5L^2+18)}{\sqrt{L^4-4L^2+36}} + 240\zeta_5 + 24\zeta_3 - 1}{\lambda} + \mathcal{O}(\frac{1}{\lambda^{\frac{3}{2}}})$$

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- For higher n we expect much more intricate mixing, how do we understand this? Is $\mathcal{M}$ some integrable Hamiltonian?

Conclusion and Outlook

Conclusion

- The QSC can be formulate in terms of a novel set of densities.
- The densities have a well behaved limit as $g \to \infty$
- We developed a new numerical algorithm which remains efficient for stong coupling.
- Using high precision data we deduced new corrections to Konishi and found a novel analaytic structure for other states.

Outlook

- Main Task: Develop efficient analytic algorithm
- Missing Ingredience: How to fix ρ, η to leading order. Ideas: ABA, (quasi-classical) string theory or conformal bootstrap?
- Understand the new "mixing" structure. Look at higher trajectories.
- Can we understand densities as "transverse coordinates" [Passerini, Plefka, Semenoff, Young '10]
- Input strong coupling data into bootstrap? [Caron-Huot,Coronado,Trinh,Zahraee '22] Generalise to Wilson lines? [Cavaglia,Gromov,Julius,Preti '21]
- Improv upon AdS_4 [Bombardelli, Cavaglià,Conti,Tateo '18]? AdS_3? Make contact with AdS_3 TBA? [Frolov,Sfondrini'21]
- Hexagons at strong coupling? [Basso,Georgoudis,Klemenchuk Sueiro '22][Bercini,Homrich,Vieira'22]

And before its all over