Scaling limit of the ground state Bethe roots for the inhomogeneous XXZ spin - $\frac{1}{2}$ chain

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Integrability, Q-systems and Cluster Algebras

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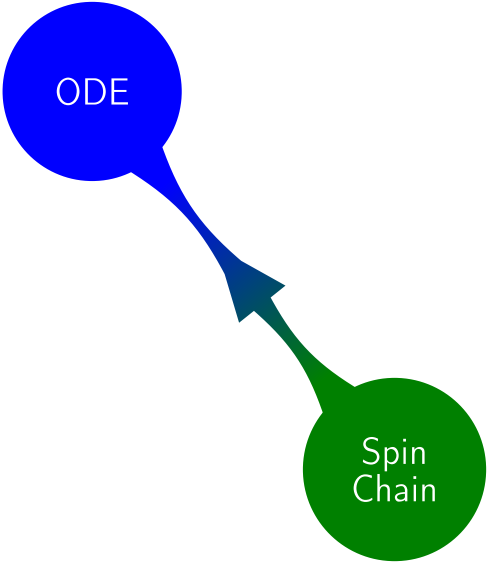
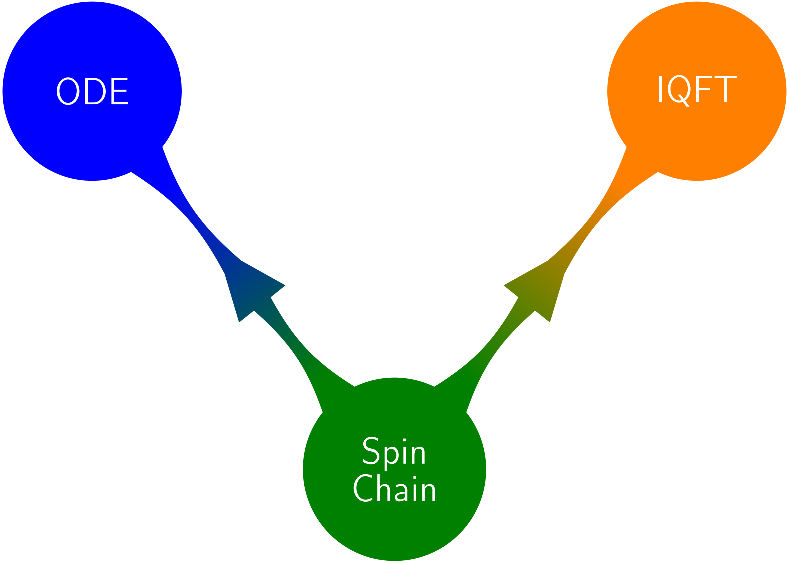
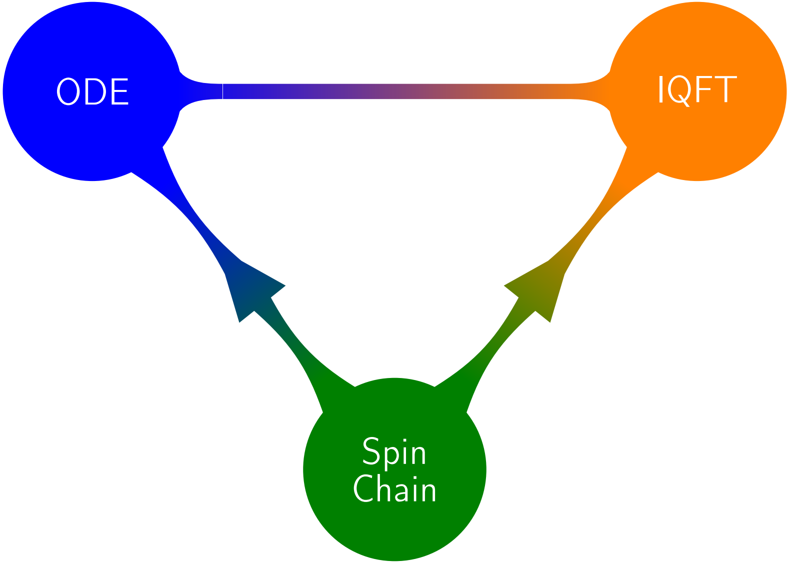
In Kotousov talk introduced were a wide class of ODEs
but the integrable structure remains mostly unknown

$$\Big[-\partial_y^2+p^2+\mathrm{e}^{(n+r)y}-(-1)^A\,E^r\;\mathrm{e}^{ry}-\sum_{(\mu,j)\in\,\boldsymbol{\Xi}_{r,A}}c_{\mu,j}\,E^{\mu}\;\mathrm{e}^{\left((A\mu-rj)\,\frac{n+r}{r}+\mu\right)y}\Big]\,\psi=0$$


ODE
Spin Chain



ODE
IQFT
Spin
Chain


ODE
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Chain

Outline

- Scaling limit XXZ and connection to Schrödinger equation of 3D anharmonic oscillator/qKdV
- Scaling limit of inhomogenous XXZ
  - Definition of the scaling limit
  - Free fermion point
  - Main conjecture
- Concluding remarks and possible future directions

Paradigm example: Homogeneous XXZ $\frac{1}{2}$-spin chain

Hamiltonian of length N

$$\mathbb{H}_{\mathrm{XXZ}} = -\sum_{m=1}^{N} \left(\sigma_m^x \sigma_{m+1}^x + \sigma_m^y \sigma_{m+1}^y + \frac{q+q^{-1}}{2}\, \sigma_m^z \sigma_{m+1}^z \right) .$$

quasi-periodic BCs

$$\sigma_{N+m}^x \pm \mathrm{i}\, \sigma_{N+m}^y = \mathrm{e}^{\pm 2\mathrm{i}\pi \mathtt{k}} \left(\sigma_m^x \pm \mathrm{i}\, \sigma_m^y \right), \qquad \sigma_{N+m}^z = \sigma_m^z \,,$$

energy spectrum described by solutions of the Bethe Ansatz equations:

$$\left(\frac{1 + q^{+1}\, \zeta_j}{1 + q^{-1}\, \zeta_j} \right)^N = -\mathrm{e}^{2\mathrm{i}\pi \mathtt{k}}\, q^{2S^z} \prod_{i=1}^{N/2-S^z} \frac{\zeta_i - q^{+2}\, \zeta_j}{\zeta_i - q^{-2}\, \zeta_j} \,, \qquad E = -\sum_{j=1}^{N/2-S^z} \frac{2(q-q^{-1})}{\zeta_j + \zeta_j^{-1} + q + q^{-1}}$$

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Critical behavior

For q being unimodular, i.e. $q = \mathrm{e}^{\mathrm{i}\gamma}$, the Hamiltonian possesses gapless excitations.

1D critical spin chains: low energy spectrum organizes into conformal towers [Cardy '86]

$$E \asymp Ne_\infty + \frac{2\pi v_F}{N}\left(-\frac{c}{12} + \Delta + \bar{\Delta} + \mathtt{L} + \bar{\mathtt{L}}\right) + \dots$$

Δ, $\bar{\Delta}$ conformal dimensions, $\mathtt{L}, \bar{\mathtt{L}} \geq 0$ levels, c central charge.

XXZ: Scaling limit is governed by compact boson $R = \sqrt{2\gamma}^{-1}$, i.e. [Luther-Peschel '75; Kadanoff-Brown '79; Alcaraz-Barber-Batchelor '88]

$$c = 1, \quad \Delta = P^2 = \frac{1}{8\pi}\left(\frac{S^z}{R} + 2\pi R(\mathtt{w} + \mathtt{k})\right)^2, \quad \bar{\Delta} = \bar{P}^2 = \frac{1}{8\pi}\left(\frac{S^z}{R} - 2\pi R(\mathtt{w} + \mathtt{k})\right)^2,$$

$$\mathcal{H} = \bigoplus_{\mathtt{w}, S^z \in \mathbb{Z}} \mathcal{F}_P \otimes \bar{\mathcal{F}}_{\bar{P}} \qquad (\mathcal{F}_P \ - \ \text{Fock space})$$

Integrable structure

$$\mathbf{\Psi}_N \xrightarrow{\quad N \to \infty \quad} \psi \otimes \bar{\psi} \in \mathcal{F}_P \otimes \bar{\mathcal{F}}_{\bar{P}}$$

Low energy Bethe
state in spin chain

Integrability: ψ ($\bar{\psi}$) eigenvector of commuting operators acting in $\mathcal{F}_P$ ($\bar{\mathcal{F}}_{\bar{P}}$)
$\Longrightarrow$ (qKdV see Kotousov's talk)

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Exploring integrable structures by scaling limit of spin chains!

How does this work in practice?

Consider the example of the ground state!

The ground state ($S^z = 0$)

$$\left(\frac{1+q^{+1}\,\zeta_j}{1+q^{-1}\,\zeta_j}\right)^N = -\mathrm{e}^{2\mathrm{i}\pi\mathtt{k}}\,q^{2S^z}\prod_{i=1}^{N/2-S^z}\frac{\zeta_i - q^{+2}\,\zeta_j}{\zeta_i - q^{-2}\,\zeta_j}\,,$$

For small enough twist parameter $\mathtt{k}$ the root configuration of the ground state can be proven to be real [Yang-Yang '66]:

$$0 < \zeta_1 < \zeta_2 < \ldots < \zeta_{N/2}\,.$$

In the β-plane ($\zeta = \mathrm{e}^{-2\beta}$) we can depict the ground state configuration as

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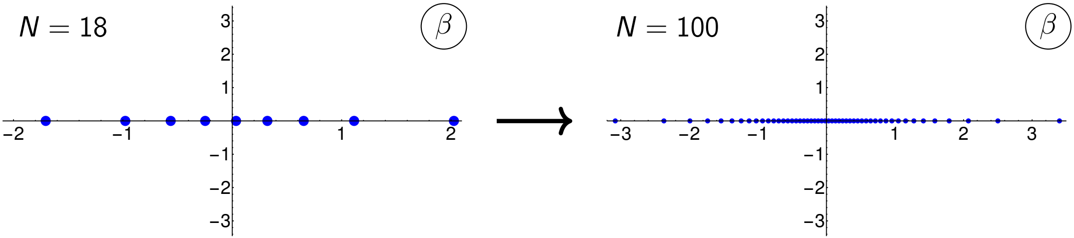
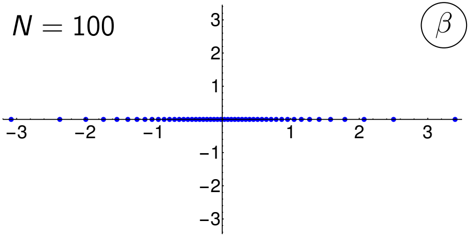
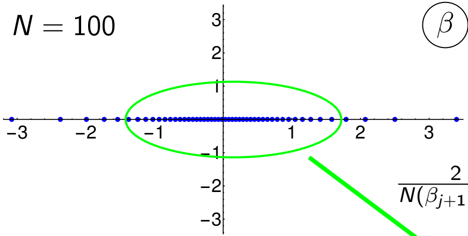


$N = 18$
β
3
2
1
−2
−1
1
2
−1
−2
−3
$N = 100$
β
3
2
1
−3
−2
−1
1
2
3
−1
−2
−3



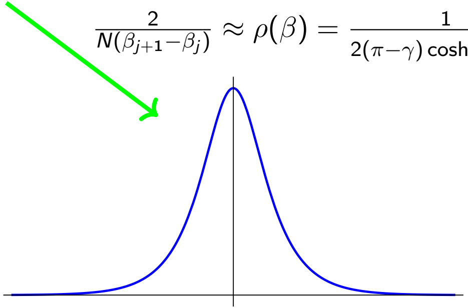
N = 100
β
3
2
1
−1
−2
−3
−3
−2
−1
1
2
3

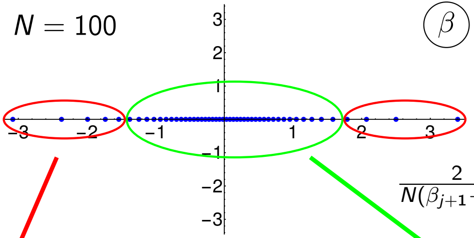
$N = 100$



β

$$\frac{2}{N(\beta_{j+1}-\beta_j)} \approx \rho(\beta) = \frac{1}{2(\pi-\gamma)\cosh\left(\frac{\beta}{\pi-\gamma}\right)}$$





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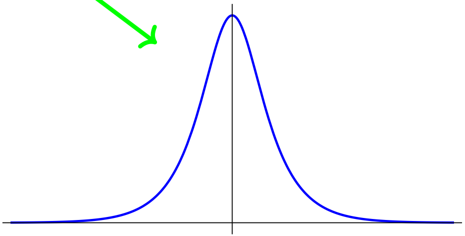

Roots at edges develop scaling behaviour:

$$s_j \sim \lim_{N\to\infty} N^{2(1-\frac{\gamma}{\pi})}\,\zeta_j$$

$$\bar{s}_j \sim \lim_{N\to\infty} N^{2(1-\frac{\gamma}{\pi})}\,\zeta^{-1}_{N/2-S_z-j}$$

$$\mathbf{\Psi}_N(\{\zeta_j\}) \xrightarrow{N\to\infty} \psi(\{s_j\}) \otimes \bar{\psi}(\{\bar{s}_j\})$$

ODE for vacuum [BLZ '98]

Remarkably, the scaled edge roots

$$s_j = \lim_{N\to\infty} (N/N_0)^{2(1-\frac{\gamma}{\pi})} \zeta_j \ .$$

coincides with the eigenvalues E_j of the Schrödinger equation of the anharmonic oscillator

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{\ell(\ell+1)}{x^2} + x^{2\alpha} - E \right) \Psi = 0 \ .$$

if one sets

$$\alpha = \frac{\pi}{\gamma} - 1 \ , \qquad \ell + \frac{1}{2} = \frac{\pi \mathtt{k}}{\gamma}, \qquad N_0 = \frac{\sqrt{\pi}\Gamma\left(\frac{1}{2} + \frac{\pi}{2\pi - 2\gamma}\right)}{2\Gamma\left(1 + \frac{\pi}{2\pi-2\gamma}\right)} \ .$$

The spectral determinant

The observation that

$$s_j = E_j$$

can be reformulated as

$$\operatorname*{slim}_{N\to\infty} A_+\left(\left(N/N_0\right)^{-2(1-\frac{\gamma}{\pi})} E\right) = D_+(E) \qquad \left(\tfrac{\pi}{\gamma} > 2\right)$$

where

$$A_+(\zeta) = \prod_{m=1}^{\frac{N}{2}-S_z} \left(1 - \frac{\zeta}{\zeta_m}\right), \qquad D_+(E) = \prod_{m=1}^{\infty} \left(1 - \frac{E}{E_m}\right)$$

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!!! The LHS is the eigenvalue of the generating function of the qKdV integrable structure !!!

Scaling limit of the Q operator

introduce the lattice Q operator

$$\mathbb{A}_+(\zeta)\,|\Psi\rangle = \prod_{m=1}^{\frac{N}{2}-S_z}\left(1-\frac{\zeta}{\zeta_m}\right)|\Psi\rangle$$

then

$$\operatorname*{slim}_{N\to\infty}\mathbb{A}_+\Big(\big(N/N_0\big)^{-2(1-\frac{\gamma}{\pi})}\,E\Big) = \mathbb{A}_+^{(\mathrm{QFT})}(\lambda) \qquad \text{with} \qquad E\propto\lambda^2 \qquad\qquad (\tfrac{\pi}{\gamma}>2)$$

$$\mathbb{A}_+^{(\mathrm{QFT})}(\lambda) = Z^{-1}\,\mathrm{Tr}\left[\mathrm{e}^{2\mathrm{i}\pi\beta p\mathcal{H}}\,\overleftarrow{\mathcal{P}}\exp\left(\lambda\int_0^{2\pi}\mathrm{d}x\,\left(\mathrm{e}^{+2\mathrm{i}\beta\phi}\,q^{\frac{\mathcal{H}}{2}}\,\mathcal{E}_+ + \mathrm{e}^{-2\mathrm{i}\beta\phi}\,q^{-\frac{\mathcal{H}}{2}}\,\mathcal{E}_-\right)\right)\right]$$

(defined in Kotousov talk)

What about the inhomogenous case?

Multiparameteric generalisation

The XXZ $\frac{1}{2}$-spin chain is related to the homogeneous six-vertex model, a 2D classical statistical system [Baxter '71]. The latter admits an integrable multiparameteric generalisation. The BAE read

$$\prod_{J=1}^{N} \frac{\eta_J + q^{+1}\,\zeta_j}{\eta_J + q^{-1}\,\zeta_j} = -\mathrm{e}^{\pm 2\mathrm{i}\pi \mathrm{k}}\, q^{2S^z} \prod_{i=1}^{N/2-S^z} \frac{\zeta_i - q^{+2}\,\zeta_j}{\zeta_i - q^{-2}\,\zeta_j} \qquad \left(j = 1, 2, \ldots, \tfrac{N}{2} - S^z \right)$$

The complex parameters $\{\eta_J\}_{J=1}^{N}$ are called inhomogeneities. We take N divisible by $r \in \mathbb{N}$ and we have the periodicity condition

$$\eta_{J+r} = \eta_J$$

Then one can introduce local Hamiltonian $\mathbb{H}$

- $r = 1 \implies$ standard XXZ
- $r = 2$ [(Ikhelf)-Jacobsen-Saleur '05('06,'11); Frahm-Seel '14; Bazhanov-Kotousov-Koval-Lukyanov '19'21'21]
- $\mathbb{H}$ includes interaction up to $r + 1$ adjacents spins.

Starting point: $\mathcal{Z}_r$ invariant case

extra $\mathcal{Z}_r$ symmetry by setting

$$\eta_\ell = (-1)^r \, \mathrm{e}^{\frac{\mathrm{i}\pi}{r}(2\ell-1)} \qquad (\ell = 1, \ldots, r)$$

system critical when $q = \mathrm{e}^{\mathrm{i}\gamma}$ but the critical behaviour described differently in each sector

$$\frac{\pi}{r} A < \gamma < \frac{\pi}{r} (A+1) \, .$$

useful parameterization of γ

$$\gamma = \frac{\pi}{r} A + \frac{\pi}{n+r} \qquad (A = 0, 1, \ldots, r-1 \, , \; n > 0) \; .$$

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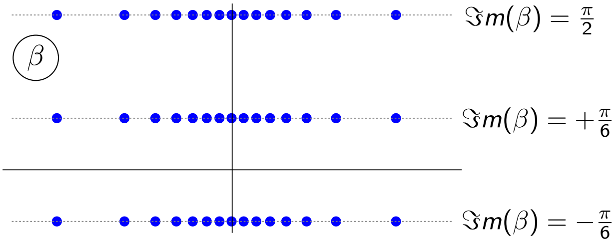
The $\mathcal{Z}_r$ invariant case: Bethe roots

The ground state Bethe roots arrange a simple pattern

$$\arg(\zeta) = \frac{\pi}{r}\left(2\,(a-1) - A\right) \qquad a = 1, \ldots, r$$



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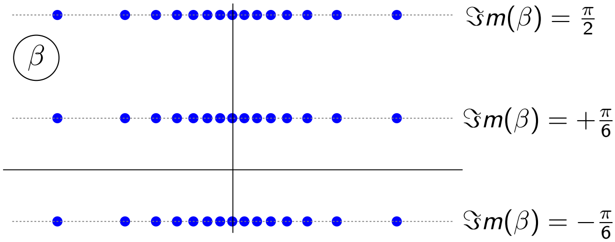

Reduction for $|\zeta_j^{(a)}|^r$ to XXZ

with $q \mapsto e^{\frac{i\pi r}{n+r}}$, $N \mapsto N/r$

Immediated consequences

The limits

$$E_m^{(a)} = \lim_{\substack{N\to\infty \\ m-\text{fixed}}} \left(\frac{N}{rN_0}\right)^{\frac{2n}{r(n+r)}} \zeta_m^{(a)}$$

exist and are non-vanishing.

The numbers $E_m^{(a)}$ are then expressed in terms of the spectrum of an ODE deducted from the reduction to XXZ.

The vacuum ODE for $\mathcal{Z}_r$ invariant case

XXZ:

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{\ell(\ell+1)}{x^2} + x^{2\alpha} - E\right)\Psi = 0 ,$$

$$\alpha = n , \qquad \ell + \frac{1}{2} = (n+1)\mathtt{k}.$$

$$\Big\downarrow \qquad q = \mathrm{e}^{\frac{\mathrm{i}\pi \mathbf{1}}{n+\mathbf{1}}} \mapsto \mathrm{e}^{\frac{\mathrm{i}\pi r}{n+r}}$$

$\mathcal{Z}_r$ invariant case:

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{\ell(\ell+1)}{x^2} + x^{2\alpha} - (-1)^A E^r\right)\Psi = 0 ,$$

$$\alpha = \frac{n}{r} , \qquad \ell + \frac{1}{2} = \frac{n+r}{r}\mathtt{k}.$$

Away from the $\mathcal{Z}_r$ invariant case

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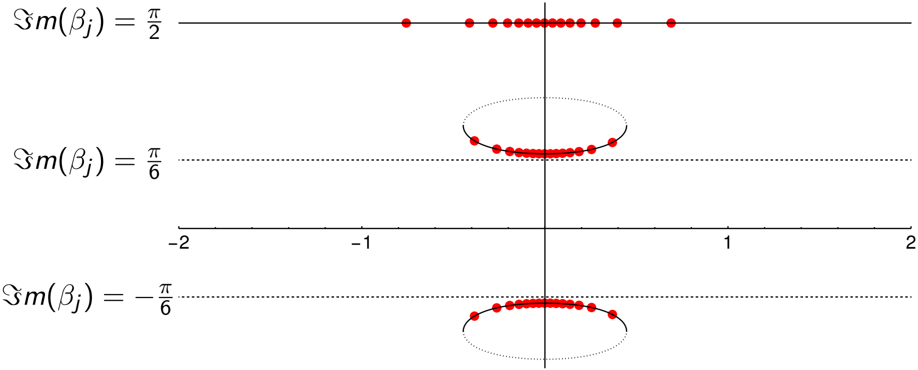
$$\eta_\ell \neq (-1)^r \, \mathrm{e}^{\frac{\mathrm{i}\pi}{r}(2\ell-1)} \qquad\qquad (\ell = 1, \ldots, r)$$

The Bethe roots align on certain loci on the complex plane, yielding no scaling behaviour of the roots e.g. at the free fermionic point for $r = 3$, $\eta_1 = -\mathrm{e}^{\mathrm{i}\delta}, \eta_2 = -\mathrm{e}^{-\mathrm{i}\delta}, \eta_3 = 1$ and $\delta = \frac{\pi}{3} - \frac{1}{10}$



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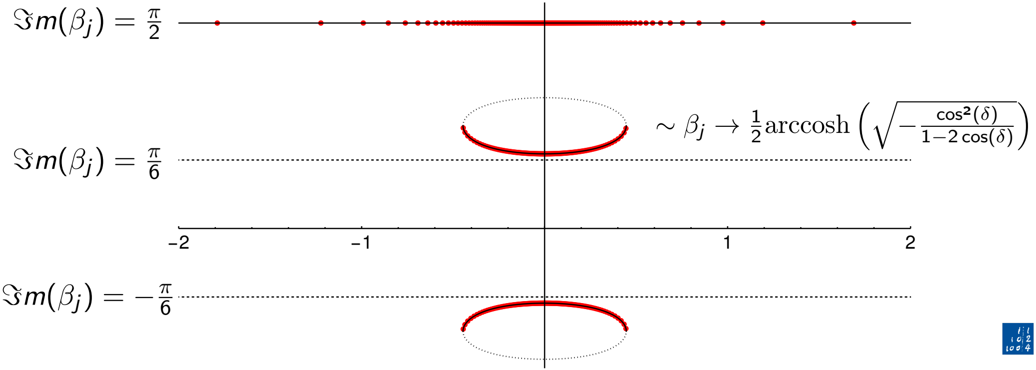

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Idea:

Assign a dependence of the system size N to the inhomogeneities such that

$$\lim_{N\to\infty} \eta_\ell(N) = (-1)^r \, \mathrm{e}^{\frac{\mathrm{i}\pi}{r}(2\ell-1)} \qquad (\ell = 1, \ldots, r)$$

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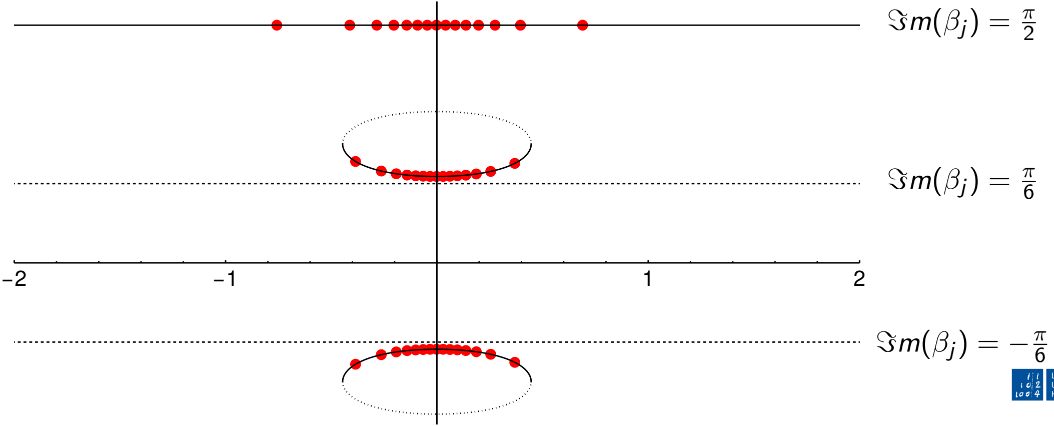

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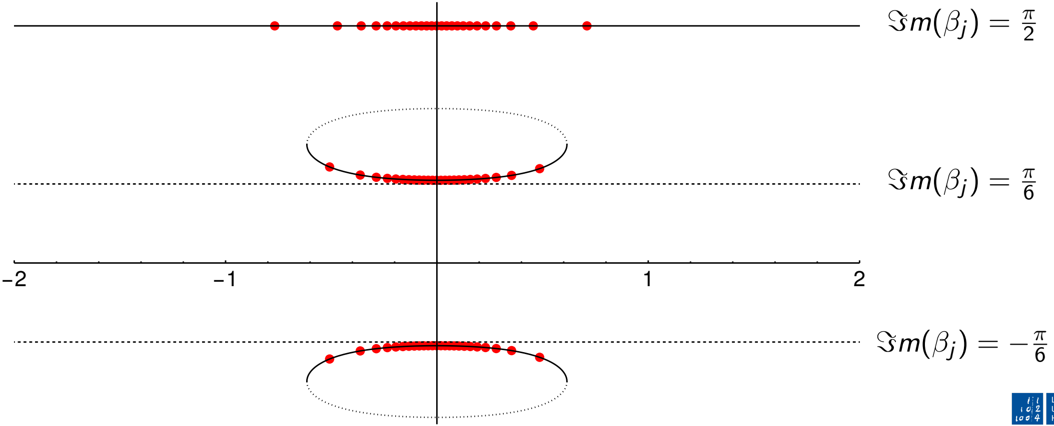

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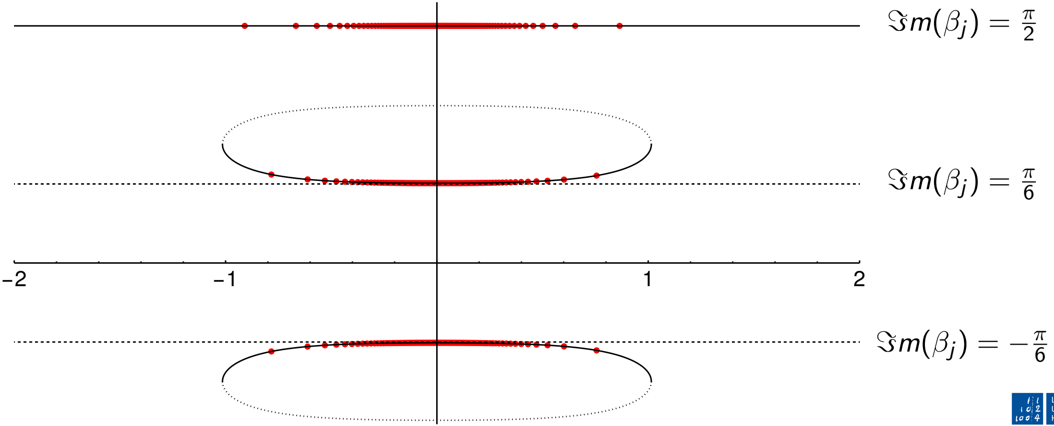

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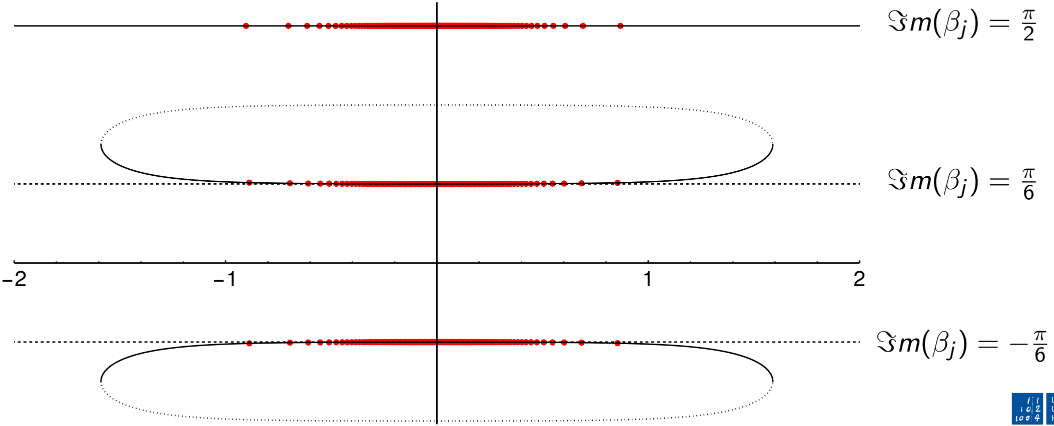

Goal:

Arrange the scaling procedure with

$$\eta_\ell(N) \to (-1)^r \, \mathrm{e}^{\frac{\mathrm{i}\pi}{r}(2\ell-1)}$$

such that the following conditions are met

1. $\lim_{N\to\infty} \left(\frac{N}{rN_0}\right)^{\frac{2n}{r(n+r)}} \zeta_m^{(a)}$ exists
2. The limiting value is different from the $\mathcal{Z}_r$ invariant case and described by a deformed ODE

$$\left[-\partial_y^2 + p^2 + \mathrm{e}^{(n+r)y} - (-1)^A E^r \, \mathrm{e}^{ry} - \delta U(y)\right] \psi = 0$$

$(y = \frac{2}{r}\log(x) + \frac{2}{n+r}\log(\frac{r}{2})$ and $\psi = x^{-\frac{1}{2}}\,\Psi)$ $\qquad p = \frac{n+r}{2}\,\mathtt{k}$

We call this 'softly breaking' the $\mathcal{Z}_r$ symmetry.

The free fermion point: $A = \frac{r-1}{2}$ and $n = r$ (r odd)

For the ground state at the free fermion point we have the Bethe ansatz equations

$$\frac{N}{\mathrm{i}\pi r} \log\left(\prod_{\ell=1}^{r} \frac{1+\mathrm{i}\zeta_m^{(a)}/\eta_\ell}{1-\mathrm{i}\zeta_m^{(a)}/\eta_\ell}\right) = 2m - 1 + 2\mathrm{k} \qquad (m = 1, 2, \ldots, N/(2r))$$

To perform the scaling limit, we write ζ into the form

$$\zeta = \left(\frac{\pi}{2N}\right)^{\frac{1}{r}} E$$

and keep E fixed as $N \to \infty$ to obtain

$$\sum_{k=0}^{A} (-1)^k \mathfrak{a}_{2k+1} (E_k^{(a)})^{2k+1} = 2m - 1 + 2\mathrm{k}$$

where the following are also kept fixed

$$\mathfrak{a}_{2k+1} = \frac{1}{2k+1}\left(\frac{2N}{\pi}\right)^{1-\frac{2k+1}{r}} \frac{1}{r}\sum_{\ell=1}^{r} \eta_\ell^{-2k-1} \qquad (k = 0, \ldots, (r-1)/2)$$

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Introducing new ODE

Now: come up with a ODE with 'eigenvalues'* $E_m^{(a)}$ obeying

$$\sum_{k=0}^{A} (-1)^k \mathfrak{a}_{2k+1} (E_k^{(a)})^{2k+1} = 2m - 1 + 2\mathfrak{k}$$

Start from the harmonic oscillator in disguise (confluent hypergeometrical equation)

$$\left[-\partial_y^2 + p^2 + \mathrm{e}^{2ry} - r\lambda(E)\,\mathrm{e}^{ry}\right]\psi = 0$$

$$\psi_p = \mathrm{e}^{py}\,\mathrm{e}^{-\frac{1}{r}\mathrm{e}^{ry}}\,{}_1F_1\left(\frac{1}{2} + \frac{p}{r} - \frac{\lambda}{2}, 1 + \frac{2p}{r}, \frac{2}{r}\mathrm{e}^{ry}\right)$$

*By eigenvalues we mean that the ODE possesses a normalizable solution for $E = E_m^{(a)}$.

The asymptotic behaviour for $y \to \infty$ is given by

$$\psi_p(y) \asymp (2r)^{\frac{p}{r}+\frac{1}{2}} \, \frac{\Gamma(1+\frac{p}{r})}{2\sqrt{\pi}} \left(\frac{r}{2}\right)^{\frac{\lambda}{2}} D_+ \, \exp\left\{\frac{1}{r}\mathrm{e}^{ry} - (\lambda+1)\frac{ry}{2}\right\}$$

where

$$D_+ = \frac{\Gamma(\frac{1}{2}+\frac{p}{r})}{\Gamma(\frac{1}{2}+\frac{p}{r}-\frac{1}{2}\lambda)}$$

The function ψ_p vanishes at large y only if $D_+ = 0$, i.e. we have pole of the Γ function in the denominator $\Longrightarrow$ spectrum is defined by the equation

$$\lambda(E) = 2m - 1 + \frac{2p}{r}$$

Hence, we simply must set in the ODE

$$\lambda(E) = \sum_{k=0}^{A} (-1)^k \mathfrak{a}_{2k+1} (E_k^{(a)})^{2k+1}$$

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The general case for $A = \frac{r-1}{2}$

away for the free fermion point, we propose the ODE

$$\left[-\partial_y^2 + p^2 + \mathrm{e}^{(n+r)y} - (-1)^A E^r \, \mathrm{e}^{ry} - \sum_{j=0}^{A-1} c_{2j+1} \, E^{2j+1} \, \mathrm{e}^{\left(\frac{n+r}{2} - \frac{n-r}{2r} (2j+1) \right) y} \right] \psi = 0$$

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if $c_{2j+1} = 0 \quad \Longrightarrow \quad \mathcal{Z}_r$ invariant case

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if $c_{2j+1} = 0 \quad \Longrightarrow \quad \mathcal{Z}_r$ invariant case

if $n = r$ and we set $c_{2j+1} = (-1)^j\, r\, \mathfrak{a}_{2j+1} \quad \Longrightarrow \quad \mathcal{Z}_r$-softly broken free fermions

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The general case for $A = \frac{r-1}{2}$

away for the free fermion point, we propose the ODE

$$\left[-\partial_y^2 + p^2 + \mathrm{e}^{(n+r)y} - (-1)^A E^r\, \mathrm{e}^{ry} - \sum_{j=0}^{A-1} c_{2j+1}\, E^{2j+1}\, \mathrm{e}^{\left(\frac{n+r}{2} - \frac{n-r}{2r}\,(2j+1)\right) y} \right] \psi = 0$$

if $c_{2j+1} = 0 \quad \Longrightarrow \quad \mathcal{Z}_r$ invariant case

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in general connection of c's and $\mathfrak{a}$'s is rather cumbersome. For example for $r = 5$ we have

$$c_1 = C_0^{(0)}\, \mathfrak{a}_1 + \frac{n-5}{20n} \left(\left(C_1^{(1)} \right)^2 - 5\, C_0^{(0)} \right) \mathfrak{a}_3^2, \qquad c_3 = C_1^{(1)}\, \mathfrak{a}_3,$$

$$C_j^{(j)} = (-1)^j \frac{r \Gamma\left(\frac{r}{2n}\right) \Gamma\left(1 - \frac{(2j+1)(n-r)}{2rn}\right)}{\Gamma\left(\frac{1}{2} + \frac{r}{2n}\right) \Gamma\left(\frac{1}{2} - \frac{(2j+1)(n-r)}{2rn}\right)}\, .$$

The numerical validation: sum rules

$$\underset{N\to\infty}{\text{slim}}\, A_+\left((N/(rN_0))^{-\frac{2n}{r(n+r)}}\, E\right) = D_+(E)\,, \qquad (n > r)$$

expand both sites for small E to get

$$\underset{N\to\infty}{\text{slim}}\left(\frac{N}{rN_0}\right)^{-\frac{2sn}{r(n+r)}} h_s^{(N)} = J_s\,,$$

$$\log(D_+(E)) = -\sum_{s=1}^{\infty} J_s E^s \qquad \text{and} \qquad \log(A_+(\zeta)) = -\sum_{j=1}^{\infty} h_j^{(N)}\, \zeta^j\,.$$

where

$$h_j^{(N)} = \frac{1}{j}\sum_{m=1}^{N/2} \zeta_m^{-j}\,.$$

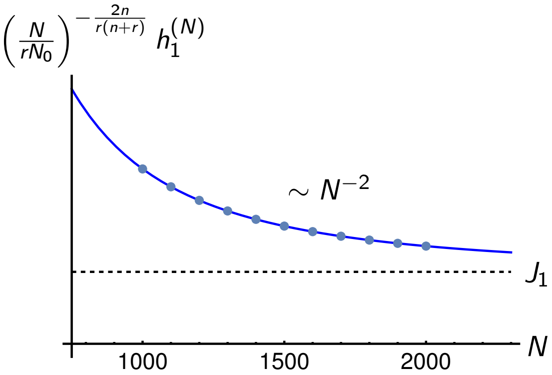
The numerical validation: sum rules

$$J_s = \operatorname{slim}_{N\to\infty} \left(\frac{N}{rN_0}\right)^{-\frac{2sn}{r(n+r)}} \frac{1}{s} \sum_{m=1}^{N/2} \zeta_m^{-s}$$

J's from perturbation theory of ODE

$\frac{1}{s} \sum_{m=1}^{N/2} \zeta_m^{-s}$ from Bethe ansatz

Example: $r = 3,\ A = 1,\ h_1^{(N)}$



$$J_1 = c_1 \rho_1\ f_1(h, g_0)$$

$$f_1(h,g) = \frac{\pi\Gamma(1-2g)}{\sin(\pi g)}\ \frac{\Gamma(g+2h)}{\Gamma(1-g+2h)}$$

$$\rho_1 = \frac{(n+r)^{2g_0-2}}{\Gamma^2(1-g_0)}$$

$$g_0 = \frac{1}{2} - \frac{n-r}{2r(n+r)}$$

General conjecture

For given r and $A = 0, 1, \ldots, r-1$

$$\gamma = \frac{\pi}{r} A + \frac{\pi}{n+r} \qquad\qquad (n > 0)$$

and fixed

$$\mathfrak{a}_s = \frac{1}{s} \left(\frac{N}{rN_0} \right)^{d_s} \frac{1}{r} \sum_{\ell=1}^{r} (\eta_\ell)^{-s} ,$$

we can adjust the d_s such that the scaling limit is governed by the ODE

$$\left[-\partial_y^2 + p^2 + \mathrm{e}^{(n+r)y} - (-1)^A E^r \, \mathrm{e}^{ry} - \sum_{(\mu,j) \in \Xi_{r,A}} c_{\mu,j} \, E^{\mu} \, \mathrm{e}^{\left((A\mu - rj) \frac{n+r}{r} + \mu \right) y} \right] \psi = 0$$

with a polynomial relation between $\{\mathfrak{a}_s\}$ and $\{c_{\mu,j}\}$.

General conjecture

The set $\Xi_{r,A}$ is given by

$$\Xi_{r,0} = \Big\{(\mu,j) \;:\; \mu = 1,2,\dots,r-1 \;\;\&\;\; j=0\Big\}, \qquad \text{for} \qquad A=0$$
$$\Xi_{r,A} = \Big\{(\mu,j) \;:\; \tfrac{rj}{A} < \mu < \tfrac{r}{A+1}\,(j+1) \;\;\&\;\; j\geq 0\Big\} \qquad \text{for} \qquad A=1,2,\dots,r-2,$$
$$\Xi_{r,r-1} = \Big\{(\mu,j) \;:\; \mu = j+1 \;\;\&\;\; j=0,1,\dots,r-2\Big\} \qquad \text{for} \qquad A=r-1\;(GAGM).$$

the exponents d_s must be found case by case!

$A=1, r-1$ and r odd $A=\frac{r-1}{2}$ and r even $A=\frac{r}{2}-1, \frac{r}{2}$ we have classified the d_s and present explicit relation between c's and $\mathfrak{a}$'s

also how to organise the scaling limit to reach ODE:

$$\Big[-\partial_y^2 + p^2 + \mathrm{e}^{(n+r)y} - (-1)^A E^r\,\mathrm{e}^{ry} - c_{\mu,j}\,E^\mu\,\mathrm{e}^{\left((A\mu-rj)\,\frac{n+r}{r}+\mu\right)y}\Big]\psi = 0$$

Summary and conclusion

- We described how to organise scaling limit of inhomogeneous XXZ spin chain to obtain the ODE

$$\Big[-\partial_y^2 + p^2 + \mathrm{e}^{(n+r)y} - (-1)^A\, E^r\, \mathrm{e}^{ry} - \sum_{(\mu,j)\in\,\boldsymbol{\Xi}_{r,A}} c_{\mu,j}\, E^{\mu}\, \mathrm{e}^{\big((A\mu-rj)\,\frac{n+r}{r}+\mu\big)y} \Big]\,\psi = 0\ .$$

- This means the ODE is involved in an ODE/IQFT correspondence for a new multiparametric integrable structure in CFT!!!
- Despite that we do not have a field theory construction of the IQFT, the scaling limit of the lattice system provides a definition of the field theory Q operator which generates the integrals of motion.

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Future directions

- Study the CFT/integrable structure of the critical inhomogeneous XXZ spin chain (including excited states)

  $\Longrightarrow$ uncovering the field theories governing the critical behaviours of inhomogenous six-vertex model? [Lukyanov]

- Higher spin generalizations? [Kotousov, Frahm]

- Different underlying symmetry algebras such as $A_2^{(2)}$ [Jacobsen, Retore]

  $\Longrightarrow$ Higher rank such as $D_n^{(2)}$ [Nepomechie, Retore]

Thank you!