Exact overlaps for boundary states

Tamas Gombor

Based on: arXiv:2110.07960
arXiv:2311.04870
and recent unpublished work

ELTE
EÖTVÖS LORÁND
UNIVERSITY

HUN
REN

WIGNER

Contents

- Motivation
- Overlaps of two site states for gl(2) spin chains
- Overlaps of two site states for gl(N) spin chains
- Overlaps of matrix product states for gl(N) spin chains

Motivation

Why boundary state overlaps?

boundary state

$$\langle \Psi | \bar{u} \rangle$$

Bethe state

Motivation

Why boundary state overlaps?

boundary state

$$\langle \Psi | \bar{u} \rangle$$

Bethe state

In statistical physics ⟶

Time evolution from initial state $|\Psi\rangle$

The overlaps are input for Quench Action

Motivation

Why boundary state overlaps?

boundary state

$$\langle \Psi | \bar{u} \rangle$$

Bethe state

In statistical physics ⟶ Time evolution from initial state $|\Psi\rangle$

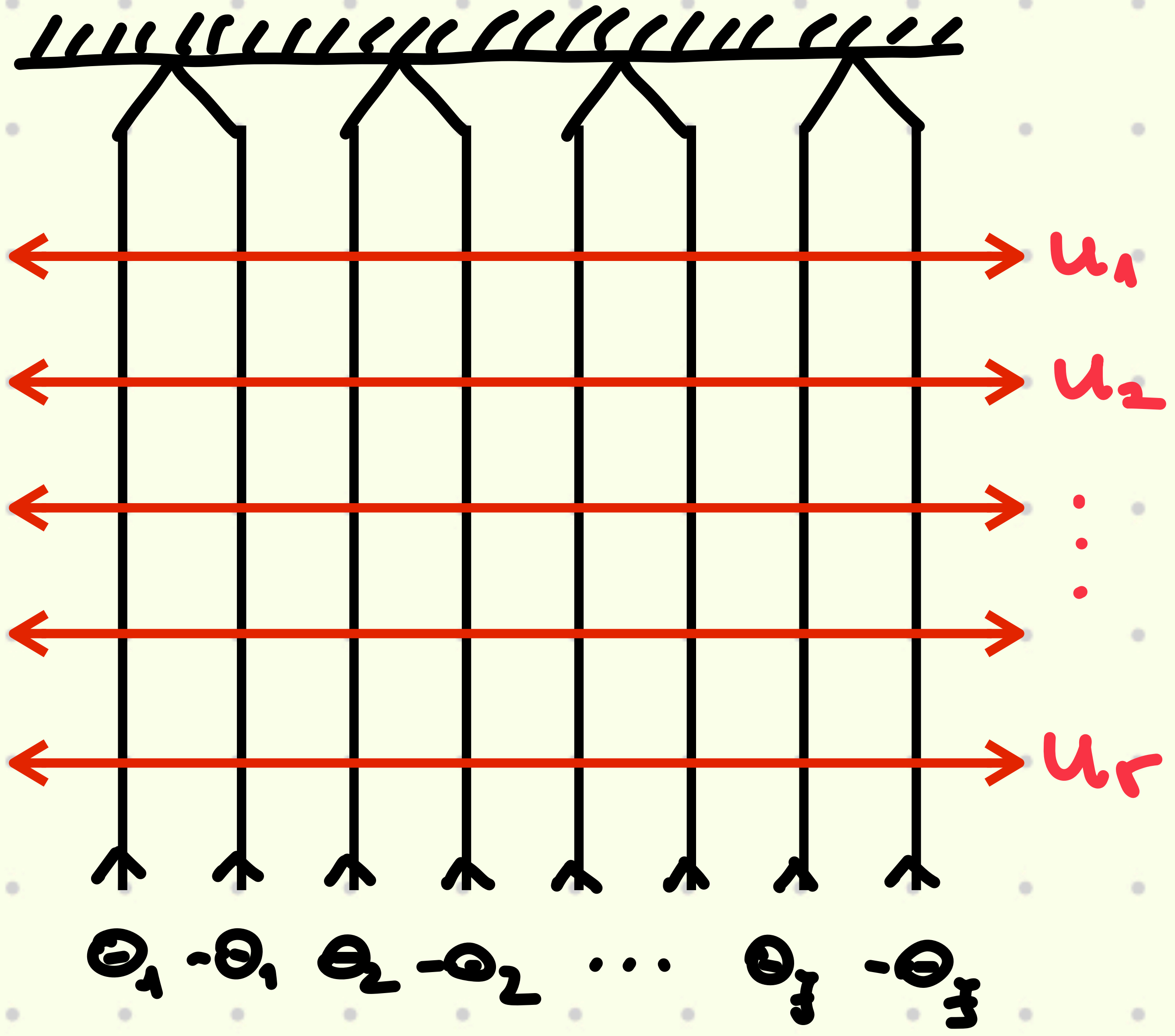
The overlaps are input for Quench Action

In AdS/CFT duality ⟶ The overlaps correspond to 1-pt functions of defect theories

two site states of gl(2) spin chains

Definitions

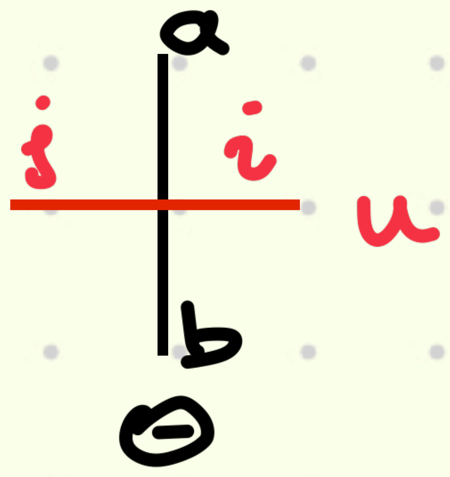
6-vertex model

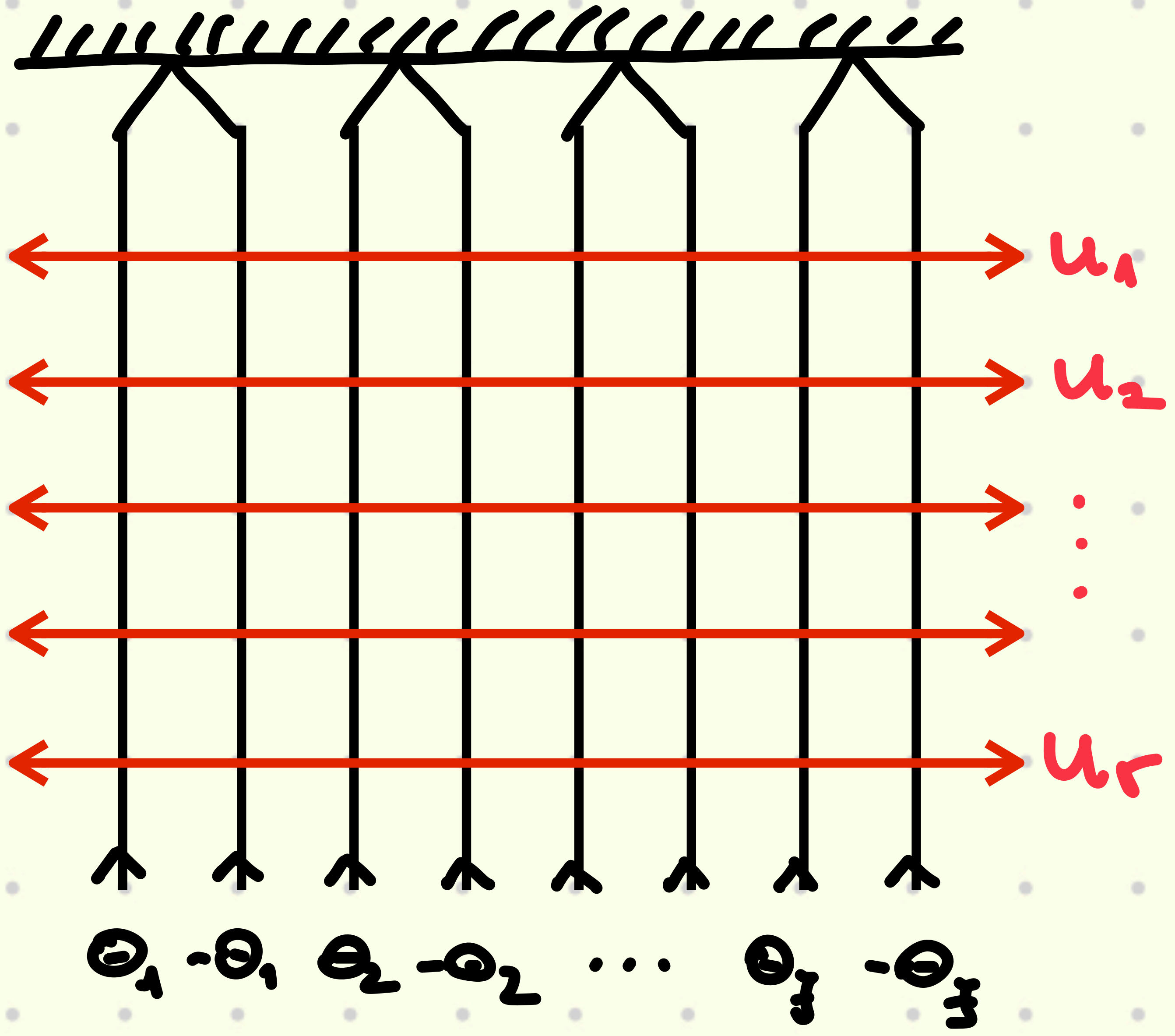


Definitions

6-vertex model

$L^{j,b}_{i,a}(u-\theta) =$

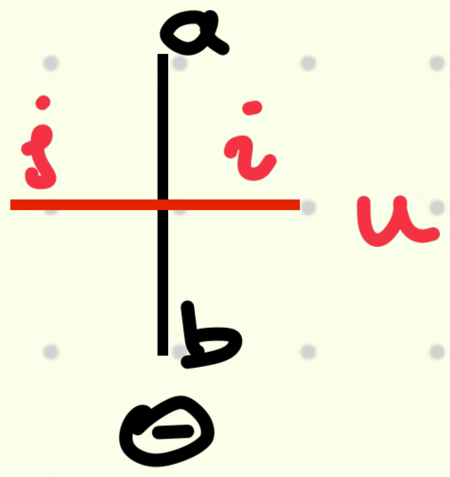


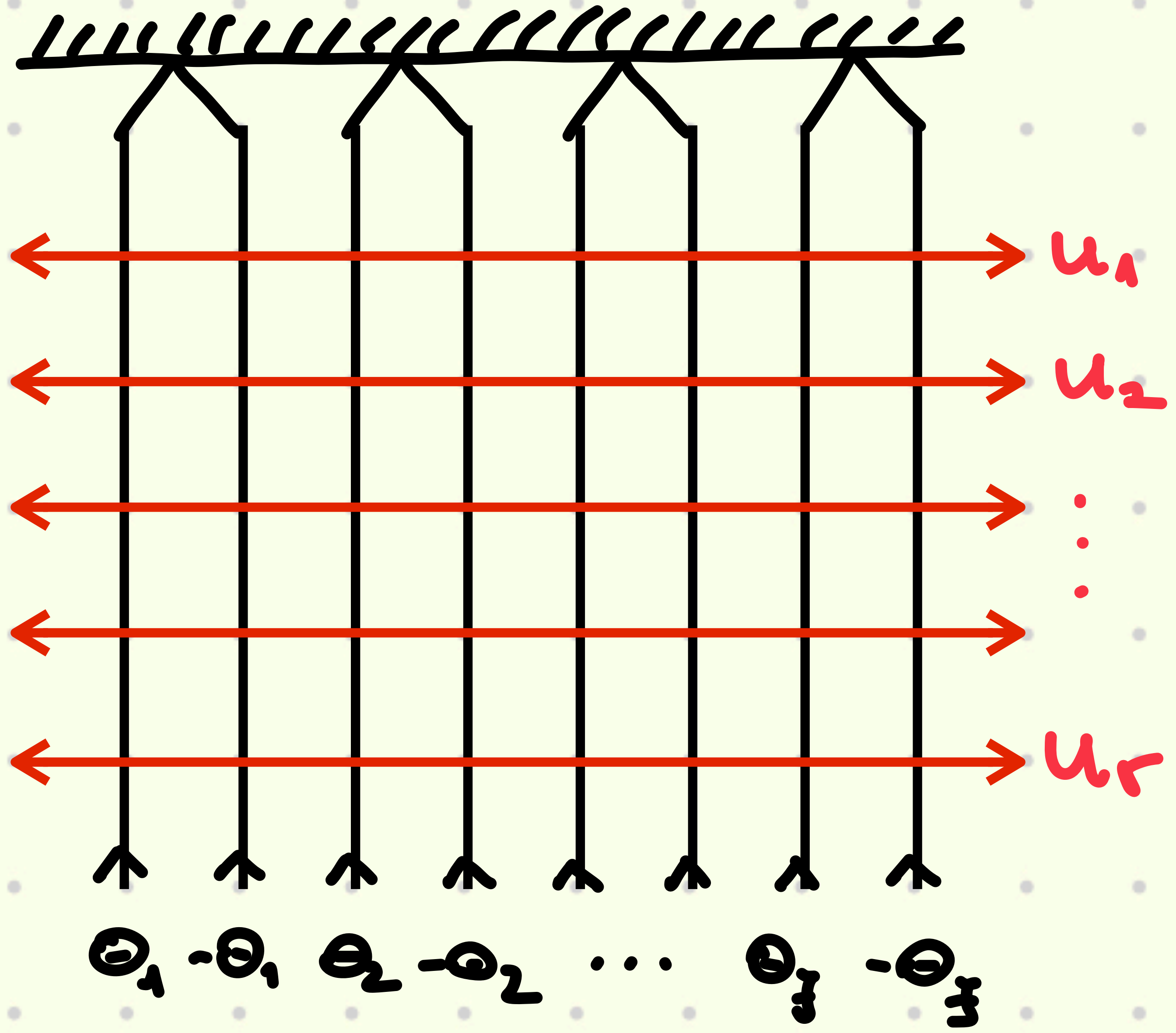


Definitions

6-vertex model

$$L^{j,b}_{i,a}(u-\theta) =$$



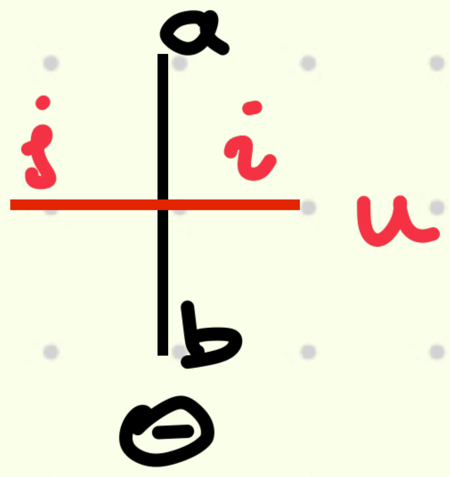


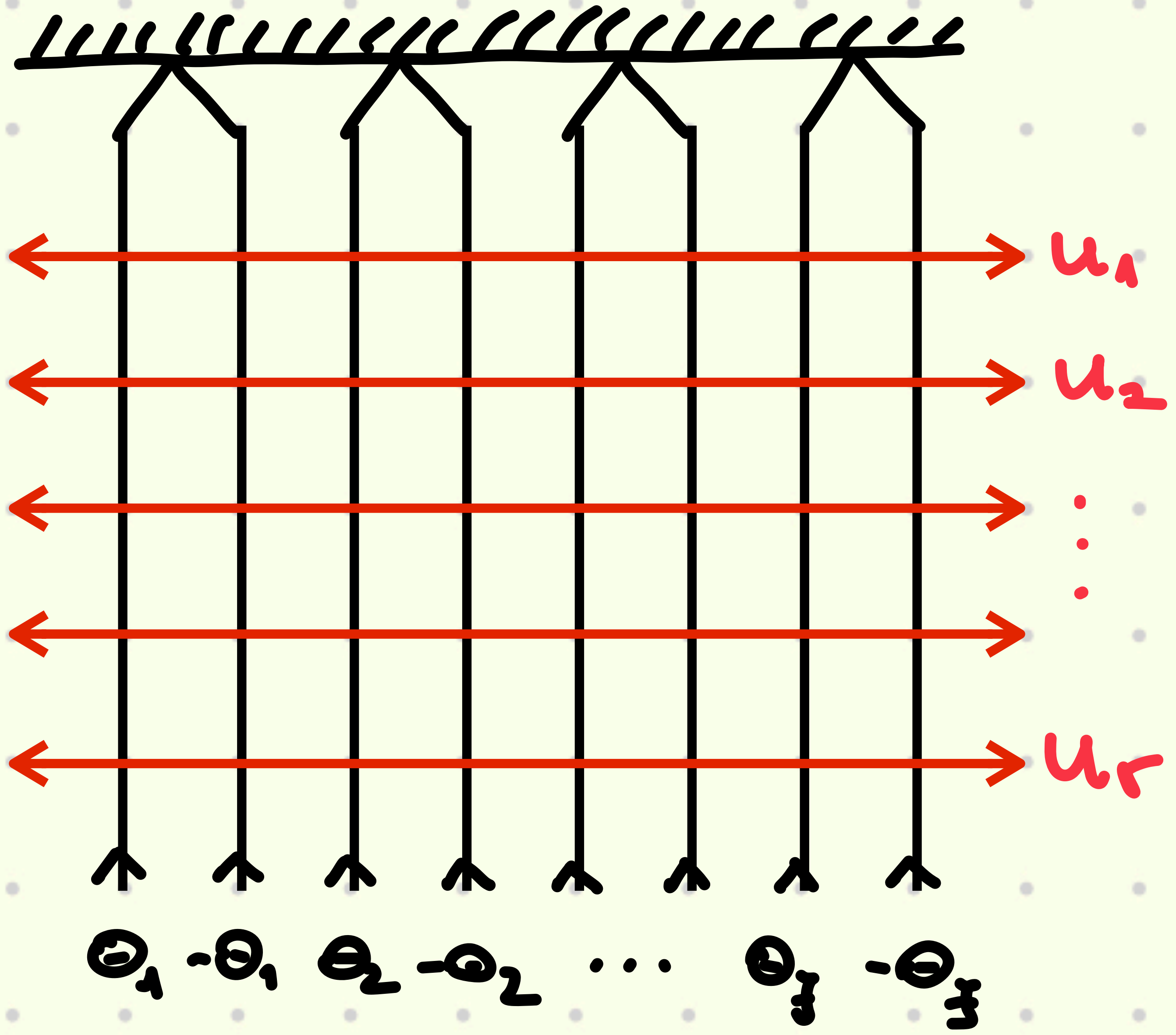
$$= Z_{J,r}(\bar{\theta},\bar{u})$$

Definitions

6-vertex model

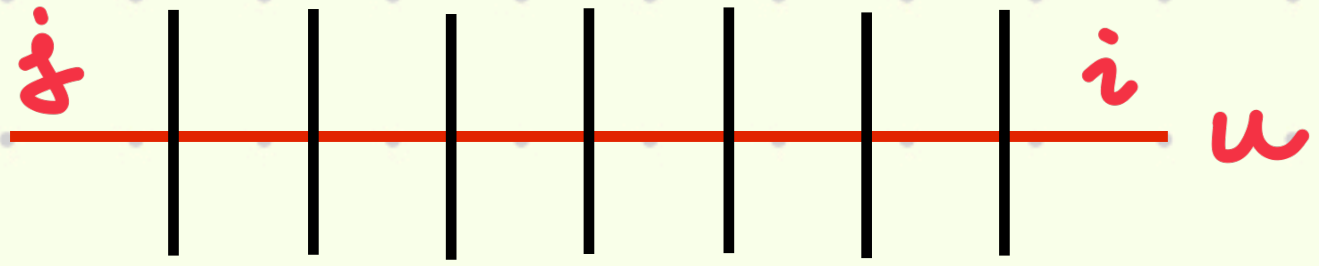
$L_{i,a}^{j,b}(u-\theta) =$





$= Z_{J,r}(\bar{\theta}, \bar{u})$

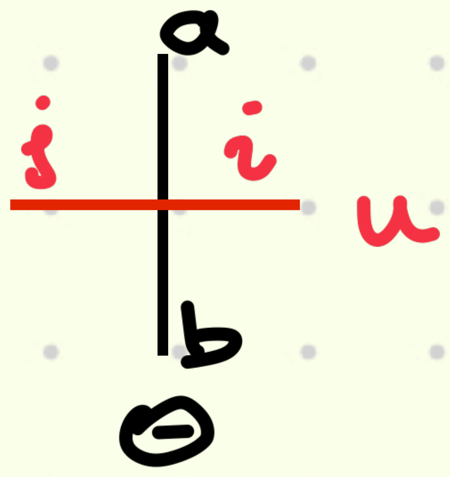
$T_{ij}(u) =$



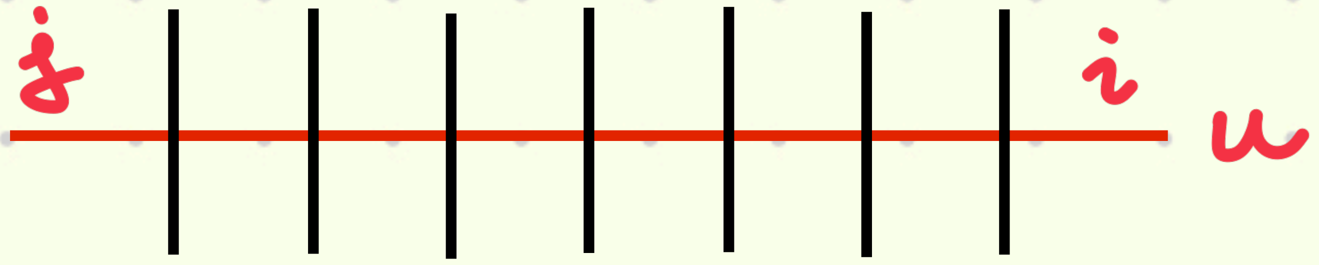
Definitions

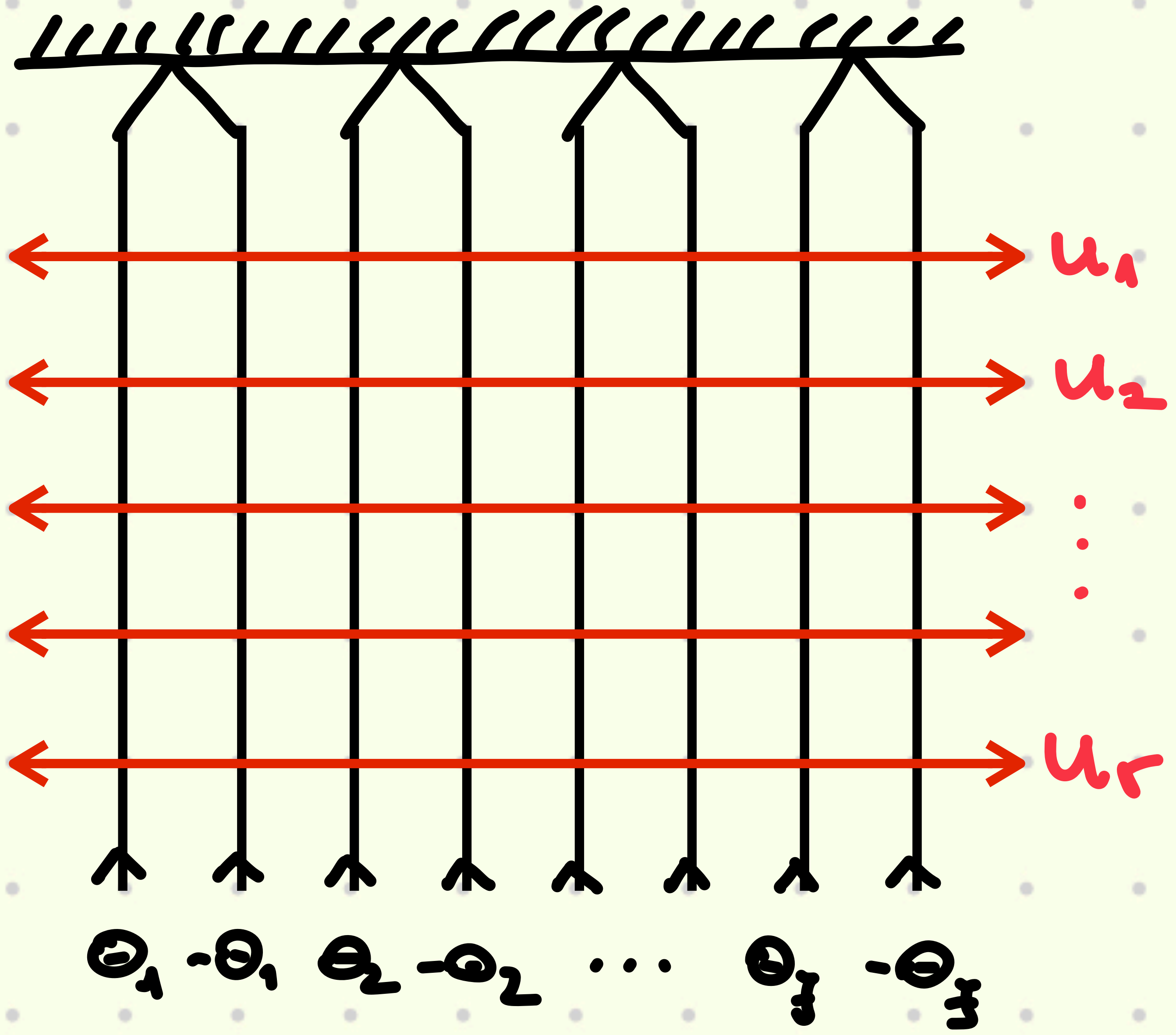
6-vertex model

$L_{i,a}^{j,b}(u-\theta) =$



$T_{ij}(u) =$





$= Z_{J,r}(\bar{\theta}, \bar{u})$

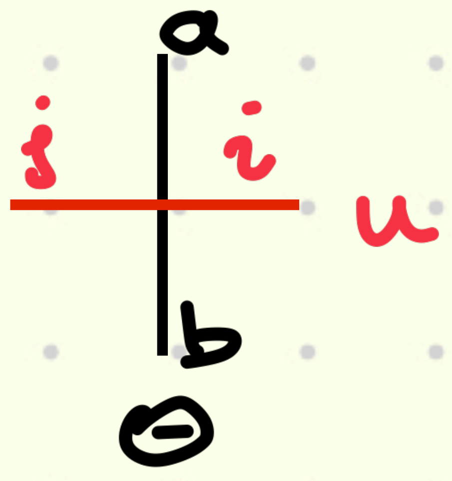
$|\bar{u}\rangle = \prod_{j=1}^{r} T_{1,2}(u_j)|0\rangle$

$|0\rangle =$ ↑ ↑ ↑ ↑ ↑ ↑ ↑

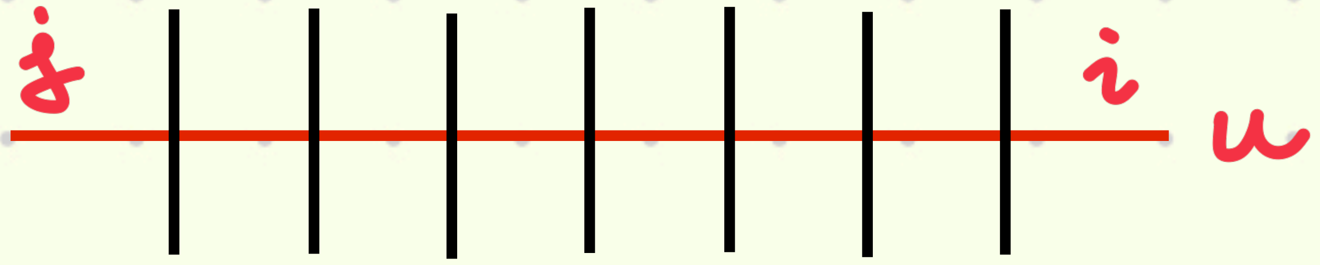
Definitions

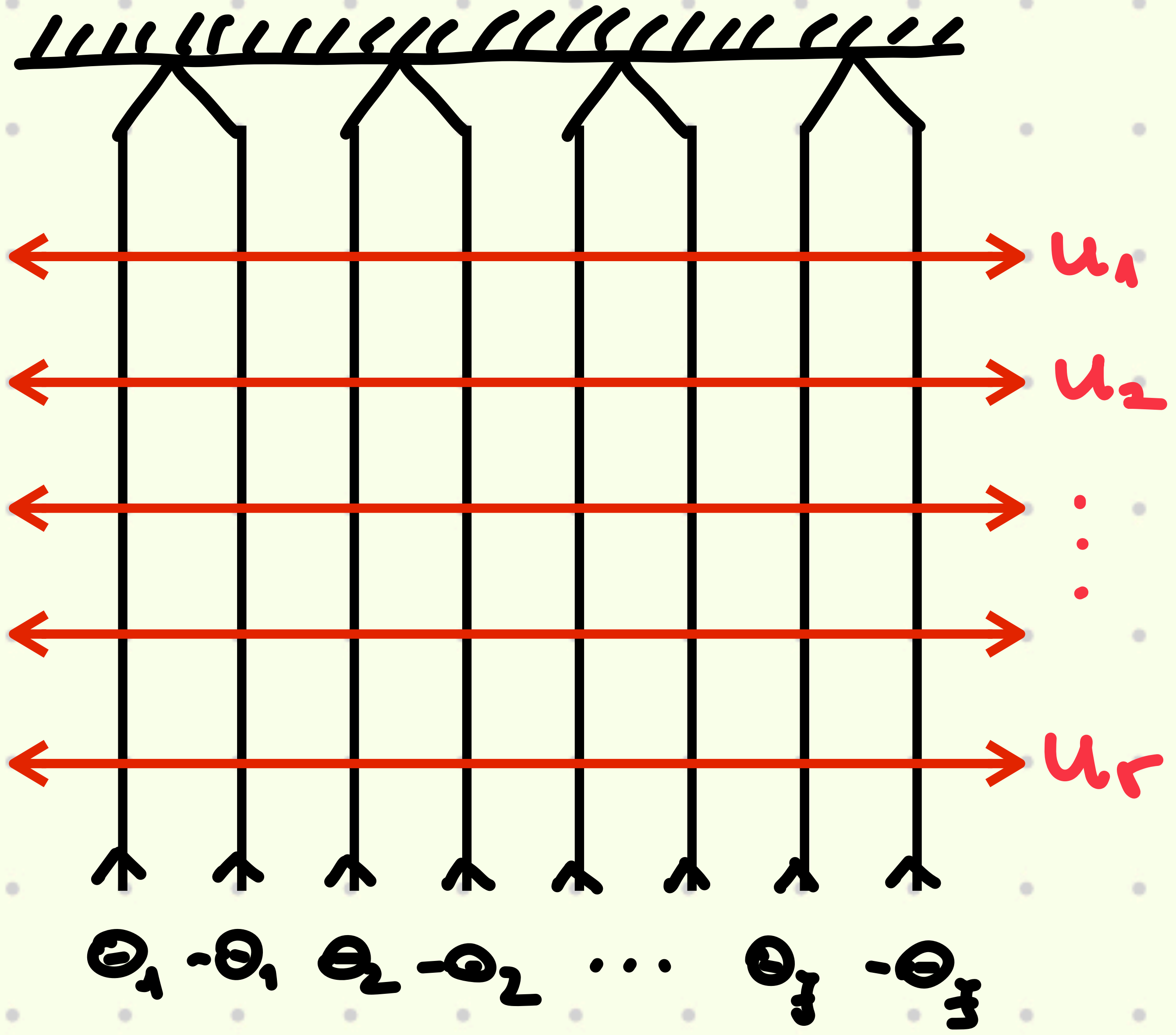
6-vertex model

$L_{i,a}^{j,b}(u-\theta) =$



$T_{ij}(u) =$



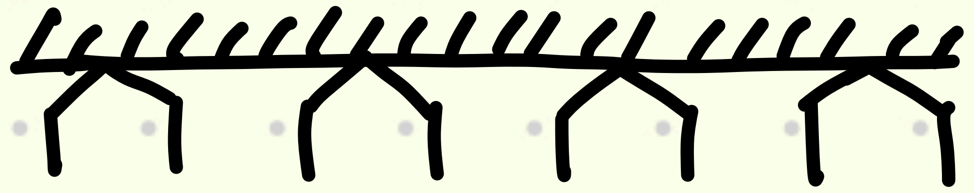


$= Z_{J,r}(\bar\theta,\bar u)$

$$|\bar u\rangle = \prod_{j=1}^{r} T_{1,2}(u_j)\,|0\rangle$$

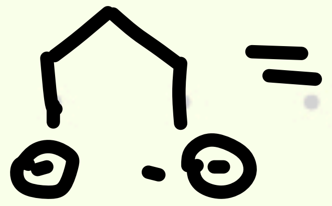
$$|0\rangle = \uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow$$

$\langle\Psi| =$



$$= \langle\varphi(\theta_1)|\otimes\dots\otimes\langle\varphi(\theta_J)|$$

$\langle\varphi(\theta)| =$

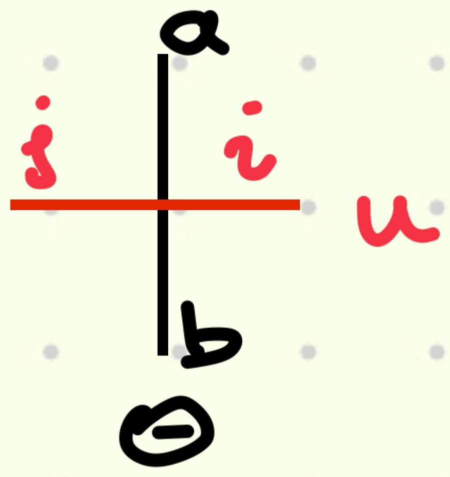


$= \sum_{a,b}\varphi_{a,b}(\theta)\langle a|\otimes\langle b|$

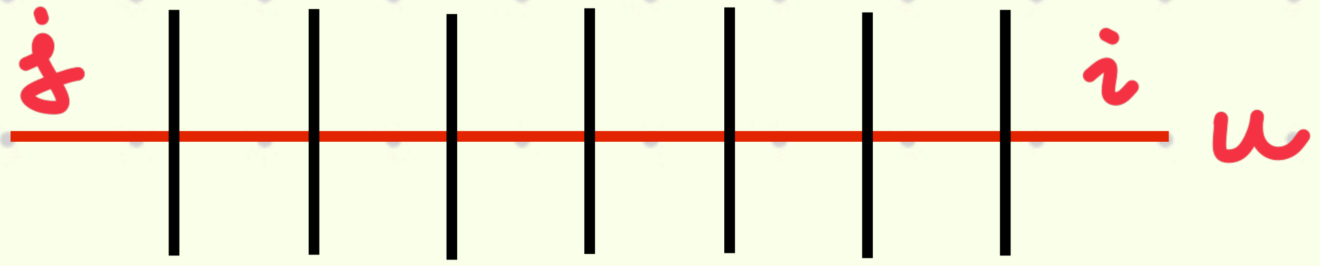
Definitions

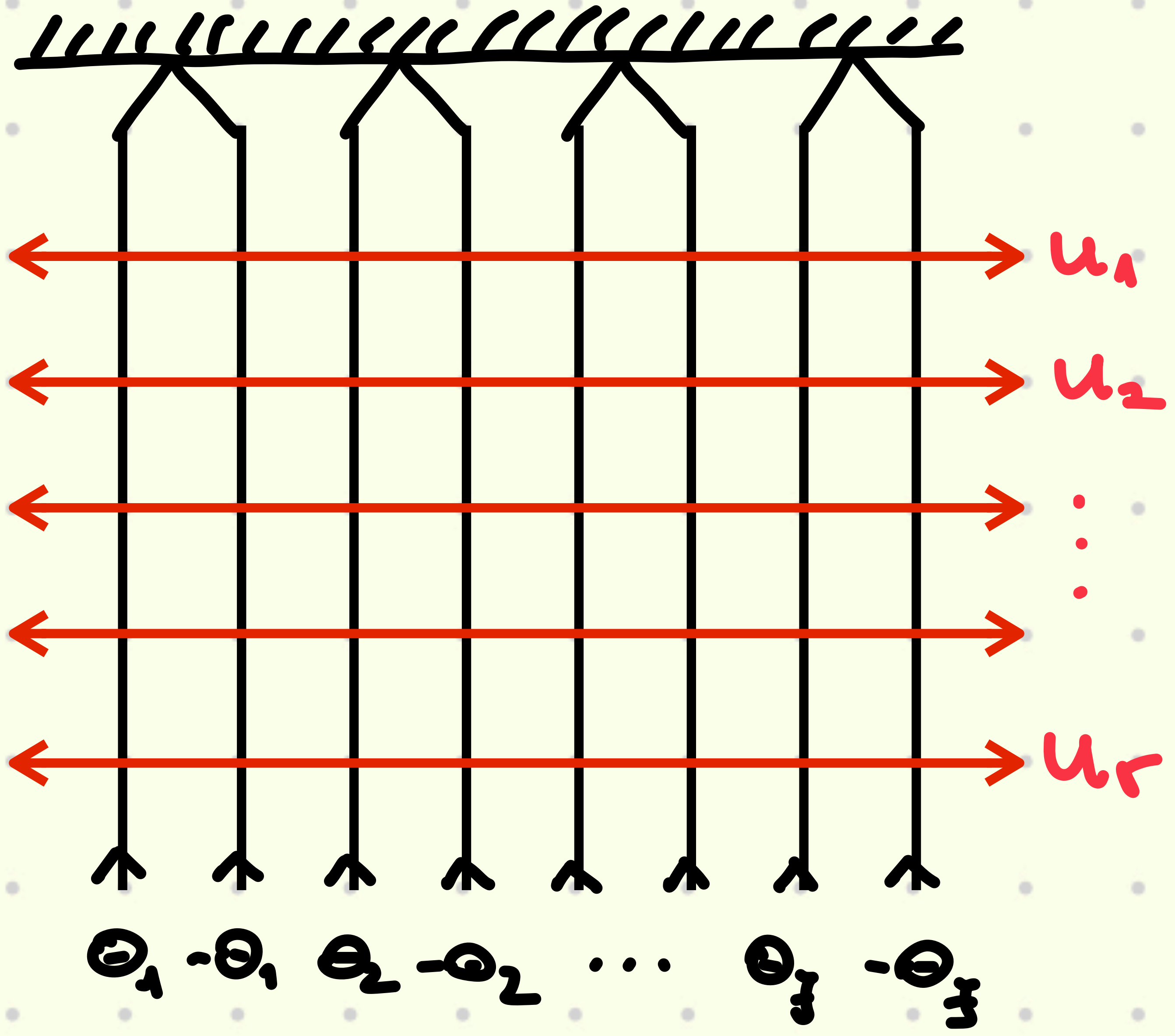
6-vertex model

$L_{i,a}^{j,b}(u-\theta) =$



$T_{ij}(u) =$



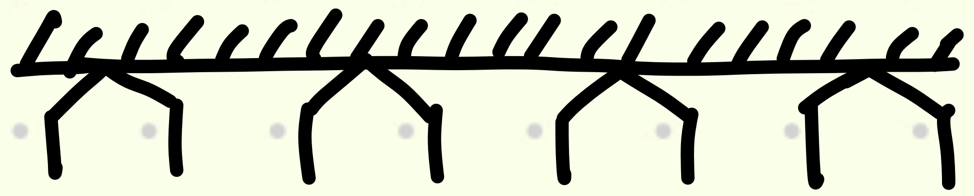


$= Z_{J,r}(\bar{\theta},\bar{u}) = \langle\psi|\bar{u}\rangle$

$|\bar{u}\rangle = \prod_{j=1}^{r} T_{1,2}(u_j)|0\rangle$

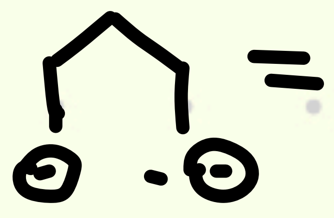
$|0\rangle = \uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow$

$\langle\psi| =$



$= \langle\varphi(\theta_1)|\otimes\dots\otimes\langle\varphi(\theta_J)|$

$\langle\varphi(\theta)| =$

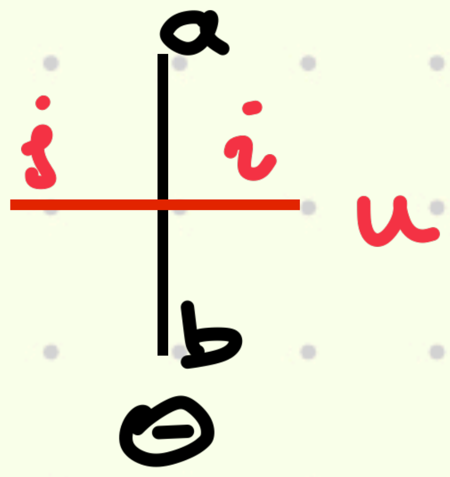


$= \sum_{a,b}\varphi_{a,b}(\theta)\langle a|\otimes\langle b|$

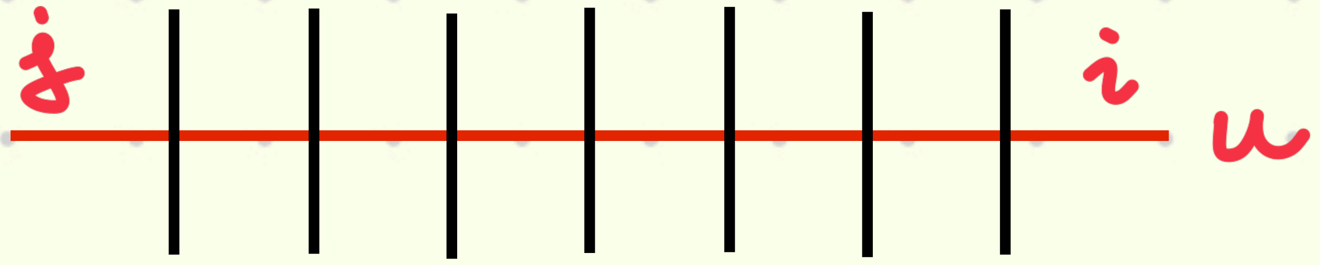
Definitions

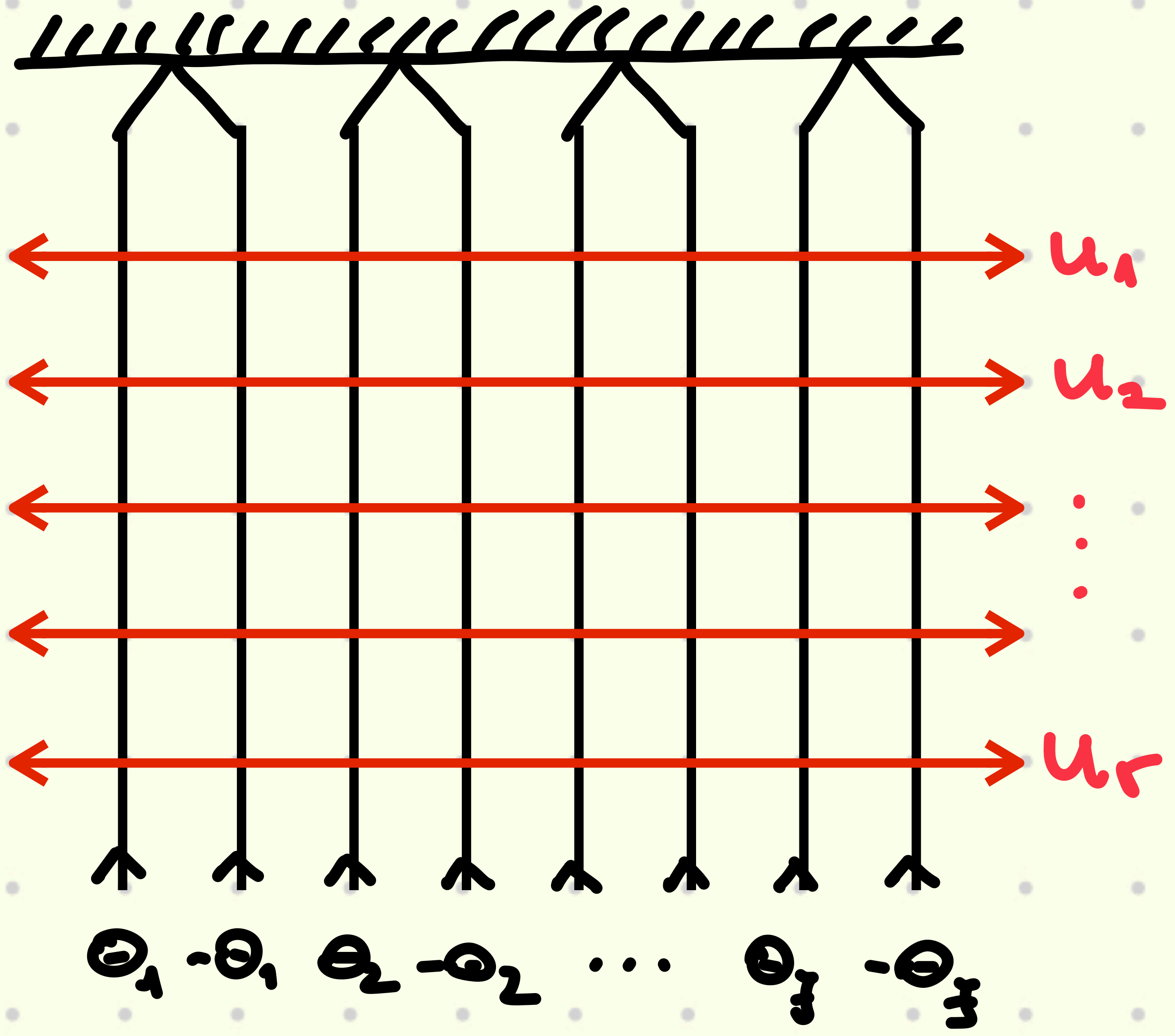
6-vertex model

$L^{j,b}_{i,a}(u-\theta) =$



$T_{ij}(u) =$



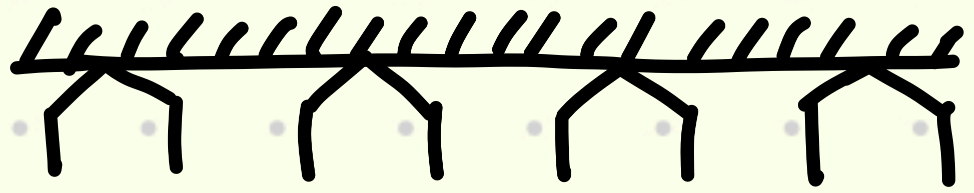


$= Z_{J,r}(\bar\theta, \bar u) = \langle\Psi|\bar u\rangle$

$|\bar u\rangle = \prod_{j=1}^{r} T_{1,2}(u_j)\,|0\rangle$

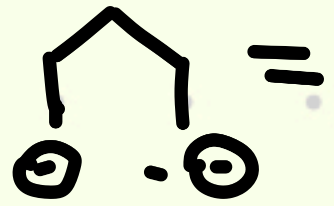
$|0\rangle = \uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow$

$\langle\Psi| =$



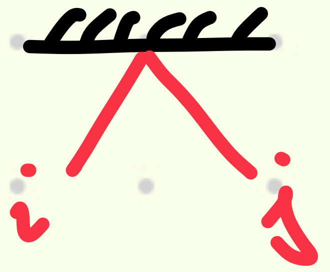
$= \langle\psi(\theta_1)|\otimes\ldots\otimes\langle\psi(\theta_J)|$

$\langle\psi(\theta)| =$



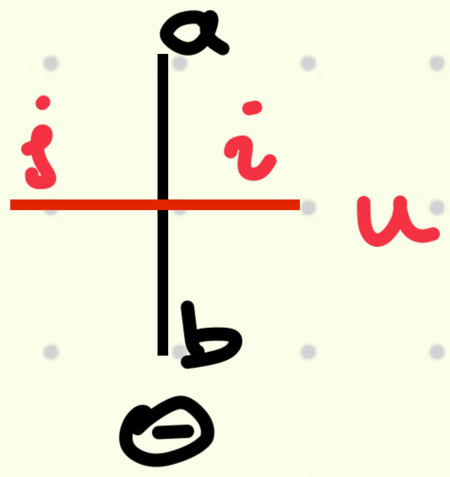
$= \sum_{a,b}\psi_{a,b}(\theta)\langle a|\otimes\langle b|$

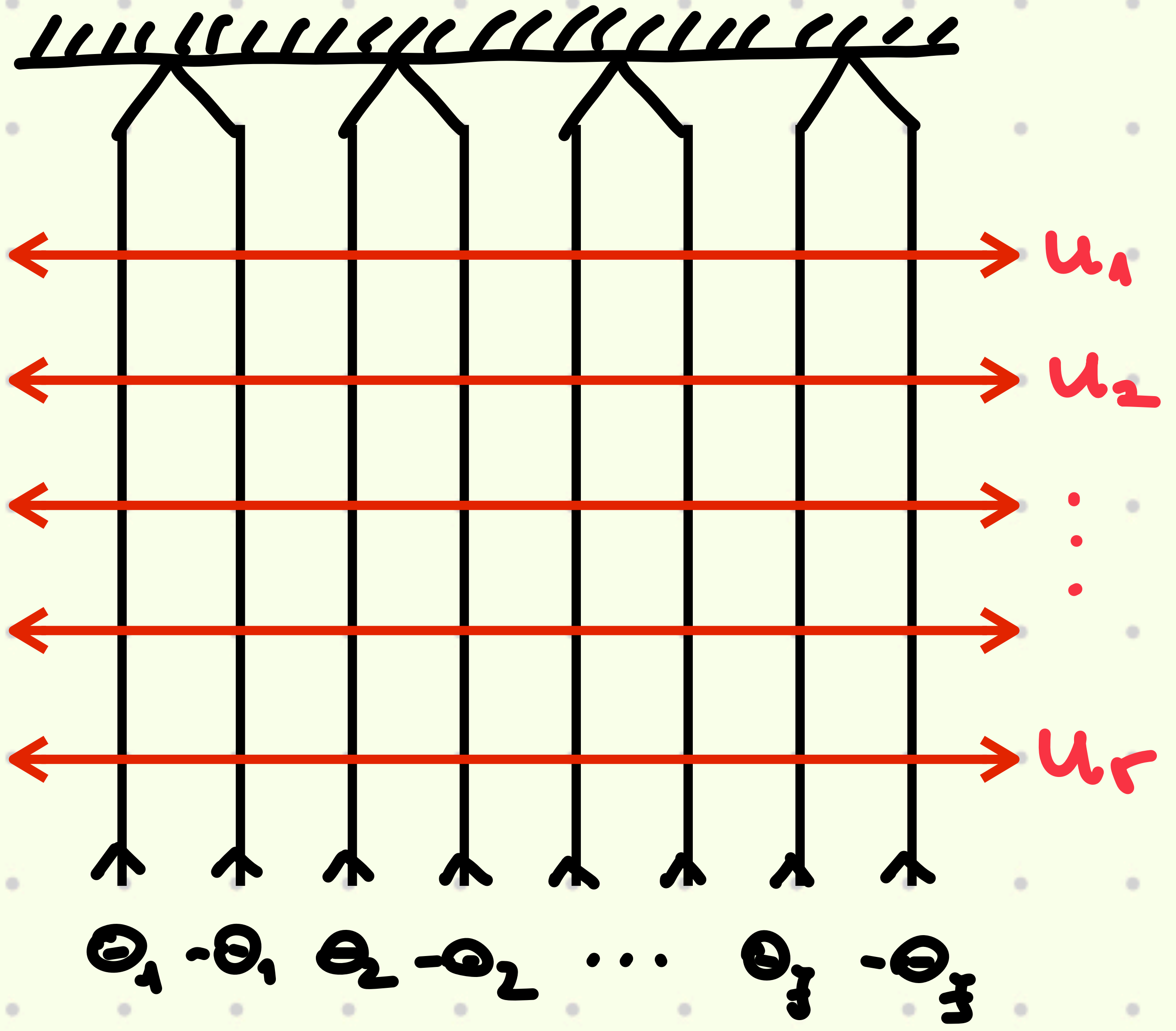
$K_{i,j}(u) =$



Definitions

6-vertex model

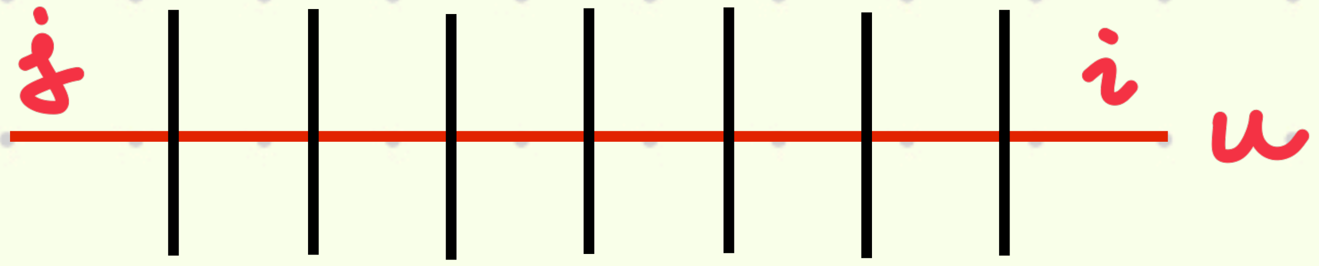
$L^{j,b}_{i,a}(u-\theta) =$ 



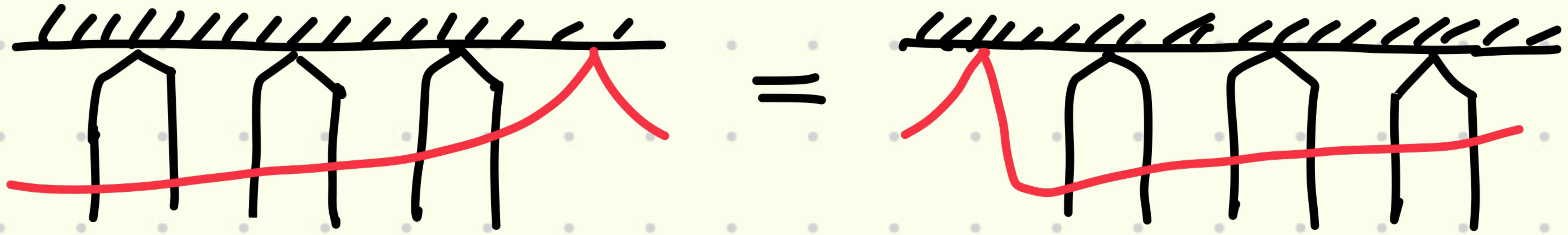
$$= Z_{J,r}(\bar{\theta}, \bar{u}) = \langle \Psi | \bar{u} \rangle$$

$$|\bar{u}\rangle = \prod_{j=1}^{r} T_{1,2}(u_j)\,|0\rangle$$

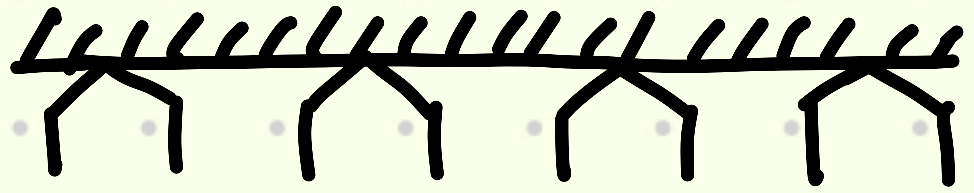
$|0\rangle =$ ↑ ↑ ↑ ↑ ↑ ↑ ↑

$T_{i,j}(u) =$ 

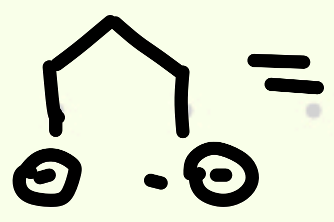
Integrable boundary states



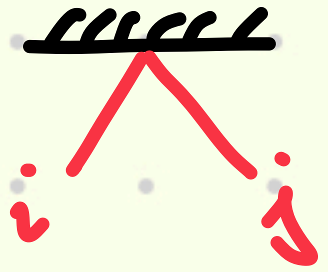
$$\boxed{\sum_k K_{i,k}(u)\,\langle \Psi | T_{k,j}(u) = \sum_k \langle \Psi | T_{i,k}(-u)\, K_{k,j}(u)}$$

$\langle \Psi | =$ 

$= \langle \varphi(\theta_1)| \otimes \ldots \otimes \langle \varphi(\theta_J)|$

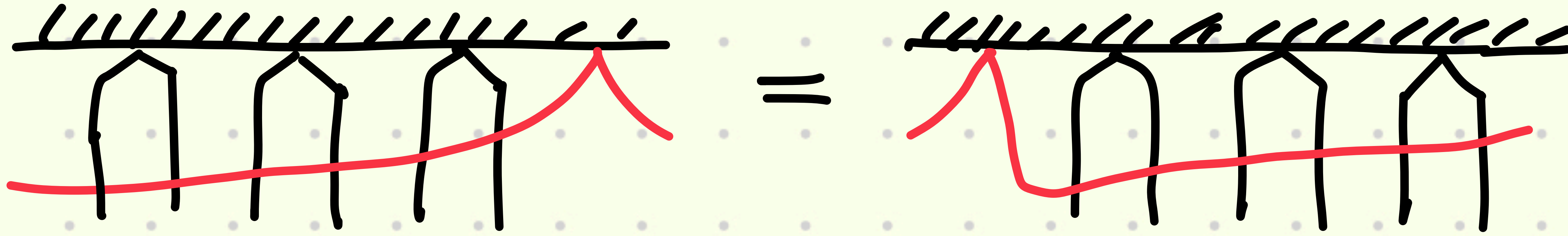
$\langle \varphi(\theta)| =$ 

$= \sum_{a,b} \varphi_{a,b}(\theta) \langle a| \otimes \langle b|$

$K_{i,j}(u) =$ 

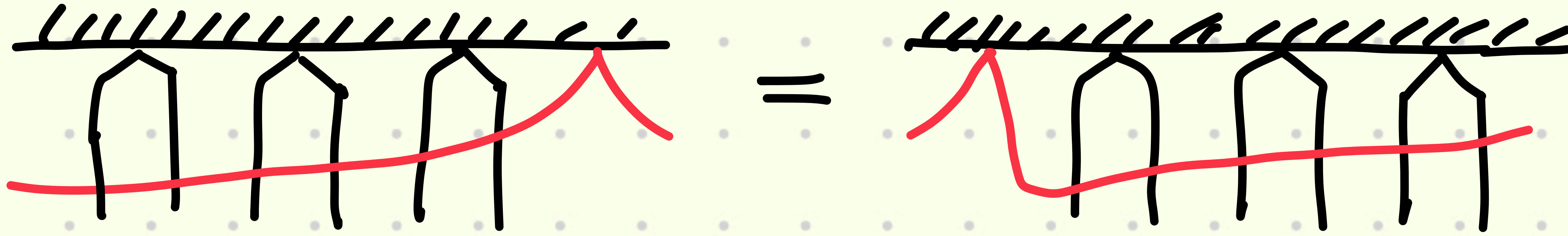
Properties of the KT-relation

$K_0(z)\langle\Psi|T_0(z) = \langle\Psi|T_0(-z)K_0(z)$



Properties of the KT-relation

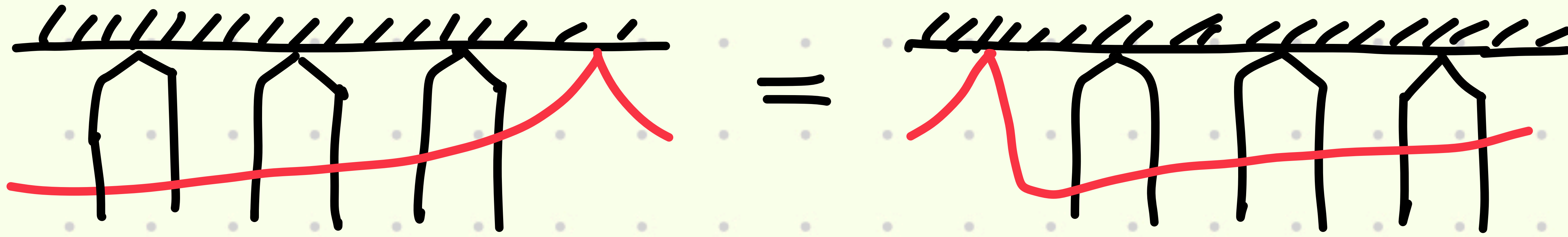
$$K_0(z)\langle\Psi|T_0(z) = \langle\Psi|T_0(-z)K_0(z)$$



Compatibility with the RTT-relation $R_{12}(u-v)\,T_1(u)T_2(v) = T_2(v)T_1(u)R_{12}(u-v)$

Properties of the KT-relation

$$K_0(z)\langle\Psi| T_0(z) = \langle\Psi| T_0(-z) K_0(z)$$

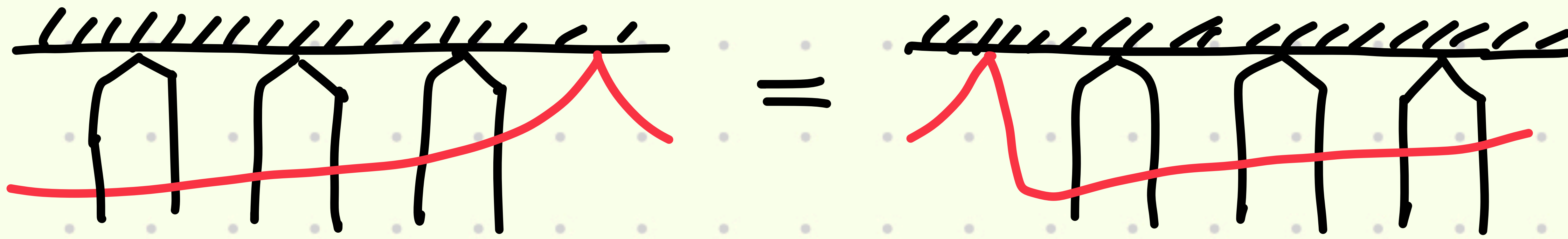


Compatibility with the RTT-relation $R_{12}(u-v)\,T_1(u)T_2(v) = T_2(v)T_1(u)R_{12}(u-v)$

$$\langle\Psi|[T_1(z_1)T_2(z_2)] = R^{-1}\langle\Psi|T_2(z_2)T_1(z_1)R = \ldots = (\ldots)\langle\Psi|T_1(-z_1)T_2(-z_2)(\ldots)$$

Properties of the KT-relation

$$K_0(z)\langle\Psi|T_0(z) = \langle\Psi|T_0(-z)K_0(z)$$



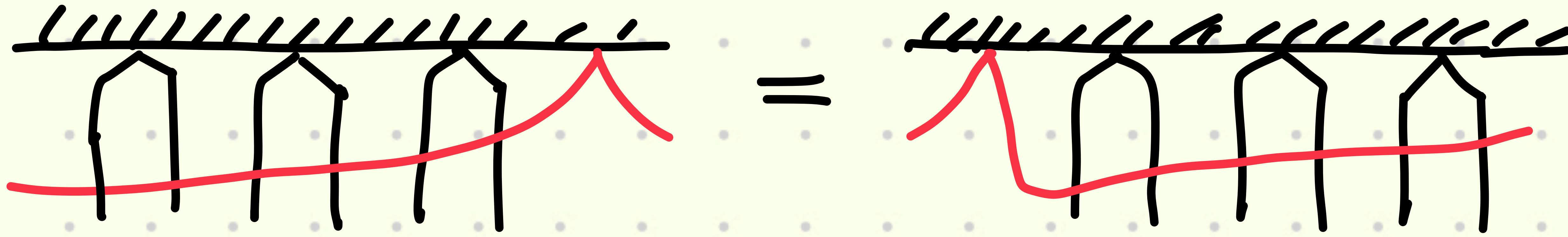
Compatibility with the RTT-relation $R_{12}(u-v)\,T_1(u)T_2(v) = T_2(v)T_1(u)R_{12}(u-v)$

$$\langle\Psi|[T_1(z_1)T_2(z_2)] = R^{-1}\langle\Psi|T_2(z_2)T_1(z_1)R = \ldots = (\ldots)\langle\Psi|T_1(-z_1)T_2(-z_2)(\ldots)$$

$$[\langle\Psi|T_1(z_1)]T_2(z_2) = K_1^{-1}\langle\Psi|T_1(-z_1)T_2(z_2)K_1 = \ldots = (\ldots)\langle\Psi|T_1(-z_1)T_2(-z_2)(\ldots)$$

Properties of the KT-relation

$$K_0(z)\langle\Psi|T_0(z) = \langle\Psi|T_0(-z)K_0(z)$$

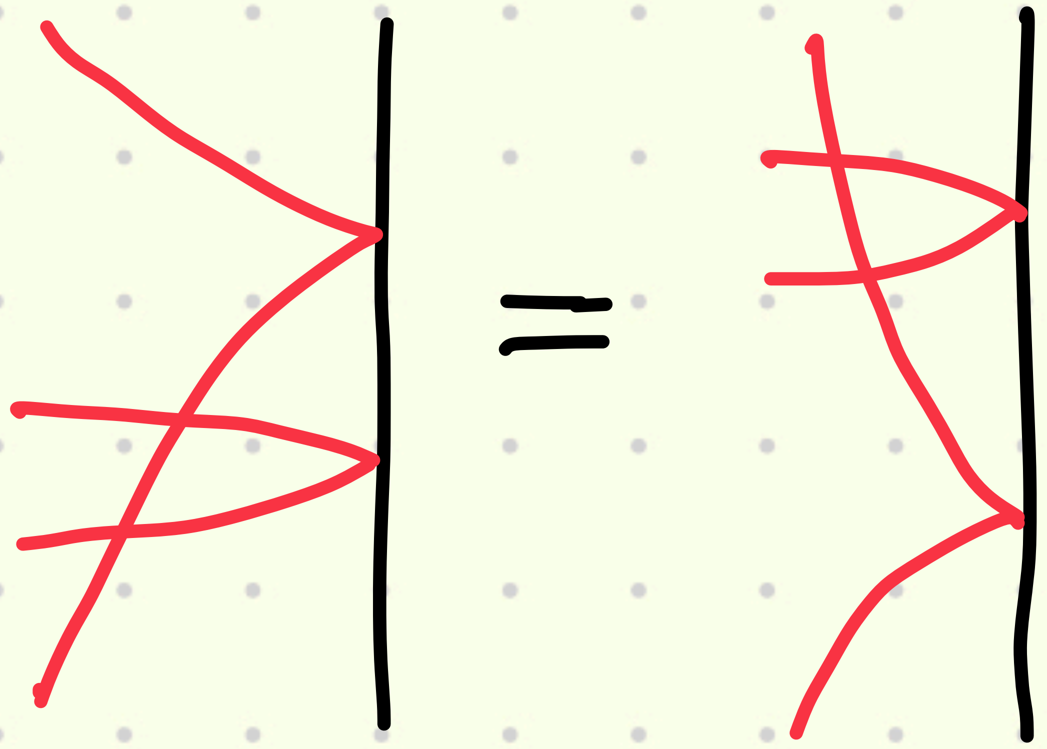


Compatibility with the RTT-relation $R_{12}(u-v)\,T_1(u)T_2(v) = T_2(v)T_1(u)R_{12}(u-v)$

$$\langle\Psi|\left[T_1(z_1)T_2(z_2)\right] = R^{-1}\langle\Psi|T_2(z_2)T_1(z_1)R = \ldots = (\ldots)\langle\Psi|T_1(-z_1)T_2(-z_2)(\ldots)$$

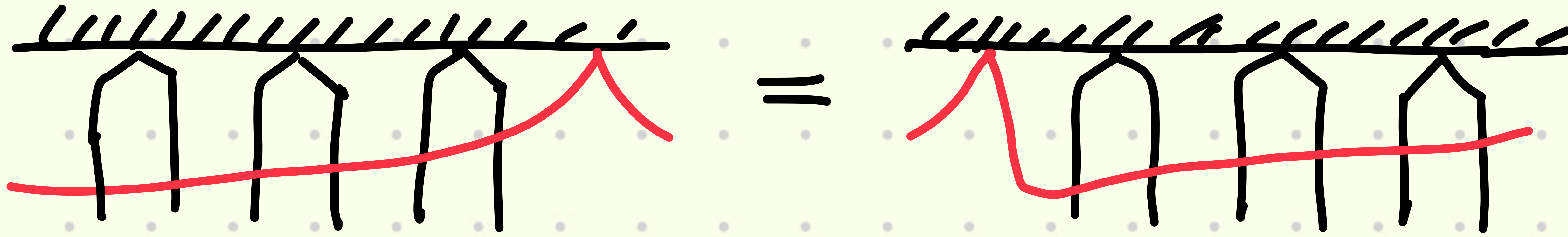
$$\left[\langle\Psi|T_1(z_1)\right]T_2(z_2) = K_1^{-1}\langle\Psi|T_1(-z_1)T_2(z_2)K_1 = \ldots = (\ldots)\langle\Psi|T_1(-z_1)T_2(-z_2)(\ldots)$$

$\Longrightarrow$ reflection equation $R_{12}(u-v)K_1(-u)R_{12}(u+v)K_2(-v) = K_2(-v)R_{12}(u+v)K_1(-u)R_{12}(u-v)$



Properties of the KT-relation

$$K_0(z)\langle\Psi|T_0(z) = \langle\Psi|T_0(-z)K_0(z)$$



Compatibility with the RTT-relation $R_{12}(u-v)\,T_1(u)T_2(v) = T_2(v)T_1(u)R_{12}(u-v)$

$$\langle\Psi|[T_1(z_1)T_2(z_2)] = R^{-1}\langle\Psi|T_2(z_2)T_1(z_1)R = \ldots = (\ldots)\langle\Psi|T_1(-z_1)T_2(-z_2)(\ldots)$$

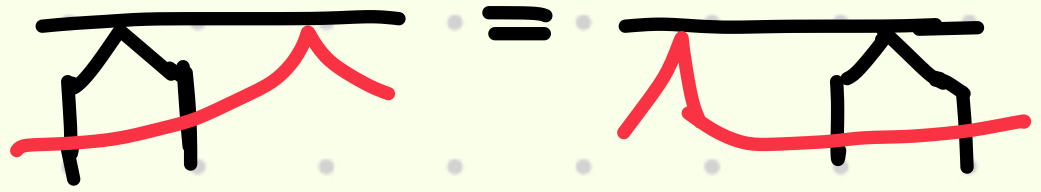
$$[\langle\Psi|T_1(z_1)]T_2(z_2) = K_1^{-1}\langle\Psi|T_1(-z_1)T_2(z_2)K_1 = \ldots = (\ldots)\langle\Psi|T_1(-z_1)T_2(-z_2)(\ldots)$$

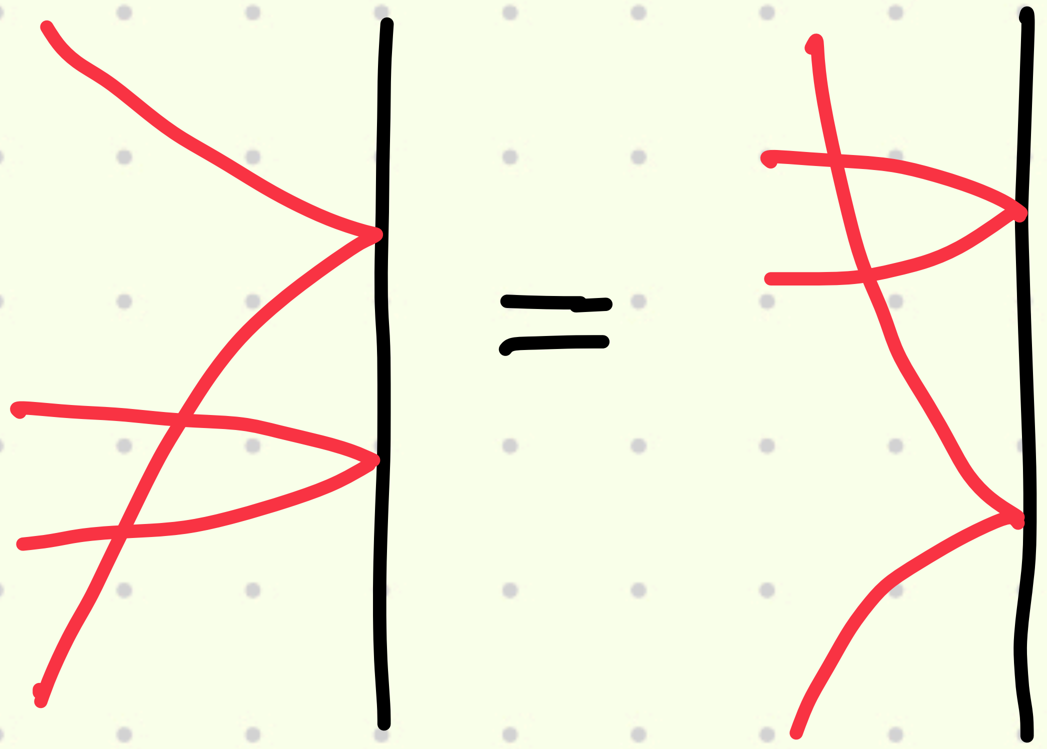
$\Longrightarrow$ reflection equation $R_{12}(u-v)K_1(-u)R_{12}(u+v)K_2(-v) = K_2(-v)R_{12}(u+v)K_1(-u)R_{12}(u-v)$

$= \langle\Psi| \rightarrow$

higher rep K-matrix

KT for $z=1$





Calculation of the Off-shell overlap

(2,2) component of the KT-relation

$$K_{2,1}(z)\langle\Psi|T_{1,2}(z)+K_{2,2}(z)\langle\Psi|T_{2,2}(z)=\langle\Psi|T_{2,1}(-z)K_{1,2}(z)+\langle\Psi|T_{2,2}(-z)K_{2,2}(z)$$

Calculation of the Off-shell overlap

(2,2) component of the KT-relation

$$K_{2,1}(z)\langle\Psi|T_{1,2}(z) + K_{2,2}(z)\langle\Psi|T_{2,2}(z) = \langle\Psi|T_{2,1}(-z)K_{1,2}(z)+\langle\Psi|T_{2,2}(-z)K_{2,2}(z)$$

Assuming $K_{2,1}\neq 0$ we can express $\langle\Psi|T_{1,2}$ with $\langle\Psi|T_{2,2}$ or $\langle\Psi|T_{2,1}$

Calculation of the Off-shell overlap

(2,2) component of the KT-relation

$$K_{2,1}(z)\langle\Psi|T_{1,2}(z) + K_{2,2}(z)\langle\Psi|T_{2,2}(z) = \langle\Psi|T_{2,1}(-z)K_{1,2}(z) + \langle\Psi|T_{2,2}(-z)K_{2,2}(z)$$

Assuming $K_{2,1} \neq 0$ we can express $\langle\Psi|T_{1,2}$ with $\langle\Psi|T_{2,2}$ or $\langle\Psi|T_{2,1}$

Creation diagonal annihilation

Calculation of the Off-shell overlap

(2,2) component of the KT-relation

$$K_{2,1}(z)\langle\Psi|T_{1,2}(z)+K_{2,2}(z)\langle\Psi|T_{2,2}(z)=\langle\Psi|T_{2,1}(-z)K_{1,2}(z)+\langle\Psi|T_{2,2}(-z)K_{2,2}(z)$$

Assuming $K_{2,1}\neq 0$ we can express $\langle\Psi|T_{1,2}$ with $\langle\Psi|T_{2,2}$ or $\langle\Psi|T_{2,1}$

Creation diagonal annihilation

off-shell overlap $S_{\lambda}(\bar{u})=\langle\Psi|\bar{u}\rangle$

Calculation of the Off-shell overlap

(2,2) component of the KT-relation

$$K_{2,1}(z)\langle\Psi|T_{1,2}(z) + K_{2,2}(z)\langle\Psi|T_{2,2}(z) = \langle\Psi|T_{2,1}(-z)K_{1,2}(z)+\langle\Psi|T_{2,2}(-z)K_{2,2}(z)$$

Assuming $K_{2,1}\neq 0$ we can express $\langle\Psi|T_{1,2}$ with $\langle\Psi|T_{2,2}$ or $\langle\Psi|T_{2,1}$

Creation diagonal annihilation

off-shell overlap $S_n(\bar{u}) = \langle\Psi|\bar{u}\rangle$ $|\bar{u}\rangle = \prod_{j=1}^{n} T_{1,2}(u_j)|0\rangle$

Calculation of the Off-shell overlap

(2,2) component of the KT-relation

$$K_{2,1}(z)\langle\Psi|T_{1,2}(z) + K_{2,2}(z)\langle\Psi|T_{2,2}(z) = \langle\Psi|T_{2,1}(-z)K_{1,2}(z) + \langle\Psi|T_{2,2}(-z)K_{2,2}(z)$$

Assuming $K_{2,1} \neq 0$ we can express $\langle\Psi|T_{1,2}$ with $\langle\Psi|T_{2,2}$ or $\langle\Psi|T_{2,1}$

Creation diagonal annihilation

off-shell overlap $S_n(\bar{u}) = \langle\Psi|\bar{u}\rangle$ $|\bar{u}\rangle = \prod_{j=1}^{n} T_{1,2}(u_j)|0\rangle$ $T_{i,i}(u)|0\rangle = \lambda_i(u)|0\rangle$

Calculation of the Off-shell overlap

(2,2) component of the KT-relation

$$K_{2,1}(z)\langle\Psi|T_{1,2}(z) + K_{2,2}(z)\langle\Psi|T_{2,2}(z) = \langle\Psi|T_{2,1}(-z)K_{1,2}(z)+\langle\Psi|T_{2,2}(-z)K_{2,2}(z)$$

Assuming $K_{2,1}\neq 0$ we can express $\langle\Psi|T_{1,2}$ with $\langle\Psi|T_{2,2}$ or $\langle\Psi|T_{2,1}$

Creation diagonal annihilation

off-shell overlap $S_n(\bar{u})=\langle\Psi|\bar{u}\rangle$ $|\bar{u}\rangle=\prod_{j=1}^{n} T_{1,2}(u_j)|0\rangle$ $T_{i,i}(u)|0\rangle=\lambda_i(u)|0\rangle$

$S_n(\{z,\bar{u}\})=\langle\Psi|T_{1,2}(z)|\bar{u}\rangle=$

Calculation of the Off-shell overlap

(2,2) component of the KT-relation

$$K_{2,1}(z)\langle\Psi|T_{1,2}(z)+K_{2,2}(z)\langle\Psi|T_{2,2}(z)=\langle\Psi|T_{2,1}(-z)K_{1,2}(z)+\langle\Psi|T_{2,2}(-z)K_{2,2}(z)$$

Assuming $K_{2,1}\neq 0$ we can express $\langle\Psi|T_{1,2}$ with $\langle\Psi|T_{2,2}$ or $\langle\Psi|T_{2,1}$

Creation diagonal annihilation

off-shell overlap $S_a(\bar{u})=\langle\Psi|\bar{u}\rangle$ $|\bar{u}\rangle=\prod_{j=1}^{n}T_{1,2}(u_j)|0\rangle$ $T_{i,i}(u)|0\rangle=\lambda_i(u)|0\rangle$

$$S_a(\{z,\bar{u}\})=\langle\Psi|T_{1,2}(z)|\bar{u}\rangle=\frac{K_{2,2}(z)}{K_{2,1}(z)}\left[\langle\Psi|T_{2,2}(-z)|\bar{u}\rangle-\langle\Psi|T_{2,2}(z)|\bar{u}\rangle\right]+\frac{K_{1,2}(z)}{K_{2,1}(z)}\langle\Psi|T_{2,1}(-z)|\bar{u}\rangle$$

Calculation of the Off-shell overlap

(2,2) component of the KT-relation

$$K_{2,1}(z)\langle\Psi|T_{1,2}(z)+K_{2,2}(z)\langle\Psi|T_{2,2}(z)=\langle\Psi|T_{2,1}(-z)K_{1,2}(z)+\langle\Psi|T_{2,2}(-z)K_{2,2}(z)$$

Assuming $K_{2,1}\neq 0$ we can express $\langle\Psi|T_{1,2}$ with $\langle\Psi|T_{2,2}$ or $\langle\Psi|T_{2,1}$

Creation diagonal annihilation

off-shell overlap $S_\lambda(\bar{u})=\langle\Psi|\bar{u}\rangle$ $|\bar{u}\rangle=\prod_{j=1}^{n}T_{1,2}(u_j)|0\rangle$ $T_{i,i}(u)|0\rangle=\lambda_i(u)|0\rangle$

$$S_\lambda(\{z,\bar{u}\})=\langle\Psi|T_{1,2}(z)|\bar{u}\rangle=\frac{K_{2,2}(z)}{K_{2,1}(z)}\left[\langle\Psi|T_{2,2}(-z)|\bar{u}\rangle-\langle\Psi|T_{2,2}(z)|\bar{u}\rangle\right]+\frac{K_{1,2}(z)}{K_{2,1}(z)}\langle\Psi|T_{2,1}(-z)|\bar{u}\rangle$$

$$S_\lambda(\{z,\bar{u}\})=\sum(\dots)S_\lambda(\bar{w})$$

$\#\bar{u}\geq\#\bar{w}$ $\bar{w}\subset\{z,-z\}\cup\bar{u}$

Properties of the off-shell overlaps

1) sum formula $S_{\bar{\lambda}}(\bar{u}) = \sum_{\bar{u}=\bar{u}_{I}\cup\bar{u}_{II}} W(\bar{u}_{I}|\bar{u}_{II})\,\lambda_1(\bar{u}_{I})\,\lambda_2(\bar{u}_{II})$

$$\lambda_k(\bar{u}) = \prod_{u_j\in\bar{u}} \lambda_k(u_j)$$

Properties of the off-shell overlaps

1) sum formula $S_{\bar{\lambda}}(\bar{u}) = \sum_{\bar{u} = \bar{u}_{I} \cup \bar{u}_{II}} W(\bar{u}_{I} | \bar{u}_{II}) \lambda_1(\bar{u}_{I}) \lambda_2(\bar{u}_{II})$

$\lambda_k(\bar{u}) = \prod_{u_j \in \bar{u}} \lambda_k(u_j)$

2) $W(\bar{u}_{I} | \bar{u}_{II}) = f(\bar{u}_{II}, \bar{u}_{I}) Z(\bar{u}_{I}) \bar{Z}(\bar{u}_{II})$

Properties of the off-shell overlaps

1) sum formula $S_{\bar\lambda}(\bar u) = \sum_{\bar u = \bar u_{\mathrm{I}} \cup \bar u_{\mathrm{II}}} W(\bar u_{\mathrm{I}} | \bar u_{\mathrm{II}}) \lambda_1(\bar u_{\mathrm{I}}) \lambda_2(\bar u_{\mathrm{II}})$

$\lambda_k(\bar u) = \prod_{u_j \in \bar u} \lambda_k(u_j)$

2) $W(\bar u_{\mathrm{I}} | \bar u_{\mathrm{II}}) = f(\bar u_{\mathrm{II}}, \bar u_{\mathrm{I}}) Z(\bar u_{\mathrm{I}}) \bar Z(\bar u_{\mathrm{II}})$

3) recursion for the HCs $Z, \bar Z$

Properties of the off-shell overlaps

1) sum formula $S_{\bar\lambda}(\bar u) = \sum_{\bar u = \bar u_I \cup \bar u_{II}} W(\bar u_I | \bar u_{II}) \lambda_1(\bar u_I) \lambda_2(\bar u_{II})$

$$\lambda_k(\bar u) = \prod_{u_j \in \bar u} \lambda_k(u_j)$$

2) $W(\bar u_I | \bar u_{II}) = f(\bar u_{II}, \bar u_I) Z(\bar u_I) \bar Z(\bar u_{II})$

3) recursion for the HCs $Z, \bar Z$

These can be derived from

Properties of the off-shell overlaps

1) sum formula $S_{\bar\lambda}(\bar u) = \sum_{\bar u = \bar u_I \cup \bar u_{II}} W(\bar u_I | \bar u_{II}) \lambda_1(\bar u_I) \lambda_2(\bar u_{II})$

$\lambda_k(\bar u) = \prod_{u_j \in \bar u} \lambda_k(u_j)$

2) $W(\bar u_I | \bar u_{II}) = f(\bar u_{II}, \bar u_I) Z(\bar u_I) \bar Z(\bar u_{II})$

3) recursion for the HCs $Z, \bar Z$

These can be derived from

1) KT-relation

Properties of the off-shell overlaps

1) sum formula $S_{\bar\lambda}(\bar u) = \sum_{\bar u = \bar u_I \cup \bar u_{II}} W(\bar u_I | \bar u_{II}) \lambda_1(\bar u_I) \lambda_2(\bar u_{II})$

$\lambda_k(\bar u) = \prod_{u_j \in \bar u} \lambda_k(u_j)$

2) $W(\bar u_I | \bar u_{II}) = f(\bar u_{II}, \bar u_I) Z(\bar u_I) \bar Z(\bar u_{II})$

3) recursion for the HCs $Z, \bar Z$

These can be derived from

1) KT-relation

2) recurrence equation

Properties of the off-shell overlaps

1) sum formula $S_{\bar{\lambda}}(\bar{u}) = \sum_{\bar{u}=\bar{u}_{I}\cup\bar{u}_{II}} W(\bar{u}_{I}|\bar{u}_{II})\lambda_1(\bar{u}_{I})\lambda_2(\bar{u}_{II})$

$\lambda_k(\bar{u}) = \prod_{u_j\in\bar{u}} \lambda_k(u_j)$

2) $W(\bar{u}_{I}|\bar{u}_{II}) = f(\bar{u}_{II},\bar{u}_{I})Z(\bar{u}_{I})\bar{Z}(\bar{u}_{II})$

3) recursion for the HCs $Z, \bar{Z}$

These can be derived from

1) KT-relation

2) recurrence equation

3) action formula

Properties of the off-shell overlaps

1) sum formula $S_{\bar\lambda}(\bar u) = \sum_{\bar u = \bar u_I \cup \bar u_{II}} W(\bar u_I | \bar u_{II}) \lambda_1(\bar u_I) \lambda_2(\bar u_{II})$

$$\lambda_k(\bar u) = \prod_{u_j \in \bar u} \lambda_k(u_j)$$

2) $W(\bar u_I | \bar u_{II}) = f(\bar u_{II}, \bar u_I) Z(\bar u_I) \tilde{Z}(\bar u_{II})$

3) recursion for the HCs $Z, \tilde{Z}$

These can be derived from

1) KT-relation

2) recurrence equation

3) action formula

4) co-product formula

Properties of the off-shell overlaps

1) sum formula $S_{\bar{\lambda}}(\bar{u}) = \sum_{\bar{u}=\bar{u}_{I}\cup\bar{u}_{II}} W(\bar{u}_{I}|\bar{u}_{II})\lambda_1(\bar{u}_{I})\lambda_2(\bar{u}_{II})$

$\lambda_k(\bar{u}) = \prod_{u_j\in\bar{u}} \lambda_k(u_j)$

2) $W(\bar{u}_{I}|\bar{u}_{II}) = f(\bar{u}_{II},\bar{u}_{I}) Z(\bar{u}_{I})\bar{Z}(\bar{u}_{II})$

3) recursion for the HCs $Z, \bar{Z}$

These can be derived from

1) KT-relation $\langle\Psi|T_{1,2} \to \langle\Psi|T_{2,2}$ & $\langle\Psi|T_{2,1}$

2) recurrence equation $|\{z,\bar{u}\}\rangle = T_{1,2}(z)|\bar{u}\rangle$

3) action formula $T_{i,j}(z)|\bar{u}\rangle = \sum(\ldots)|\bar{w}\rangle$

4) co-product formula $|\bar{u}\rangle = \sum(\ldots)|\bar{u}_{I}\rangle^{(1)}\otimes|\bar{u}_{II}\rangle^{(2)}$

On-shell limit

transfer matrix

$$\mathcal{T}(u) = \sum_{i=1}^{2} T_{i,j}\, G_{j,i}$$

twist matrix

$$G \in GL(2)$$

On-shell limit

transfer matrix

$$\Upsilon(u)=\sum_{i=1}^{2} T_{i,j}\, G_{j,i}$$

twist matrix

$$G \in GL(2)$$

$$\longrightarrow \quad \langle\Psi|\,\Upsilon(u)=\langle\Psi|\,\Upsilon(-u)$$

On-shell limit

transfer matrix

$$\mathcal{T}(u)=\sum_{i=1}^{2} T_{i,j}\, G_{j,i}$$

twist matrix

$$G \in GL(2)$$

$$\longrightarrow \quad \langle\Psi|\,\mathcal{T}(u)=\langle\Psi|\,\mathcal{T}(-u)$$

iff $[K,G]=0$

On-shell limit

transfer matrix

$$\mathcal{T}(u) = \sum_{i=1}^{2} T_{i,j}\, G_{j,i}$$

twist matrix

$$G \in GL(2)$$

$$\longrightarrow \langle \Psi | \, \mathcal{T}(u) = \langle \Psi | \, \mathcal{T}(-u)$$

iff $[K, G] = 0$

on-shell Bethe states

$$\mathcal{T}(z)|\bar{u}\rangle = \tau(z|\bar{u})|\bar{u}\rangle$$

On-shell limit

transfer matrix

$$\mathcal{T}(u)=\sum_{i=1}^{2} T_{i,j}\, G_{j,i}$$

twist matrix

$$G \in GL(2)$$

$$\longrightarrow \quad \langle\psi|\,\mathcal{T}(u)=\langle\psi|\,\mathcal{T}(-u)$$

iff $[K,G]=0$

on-shell Bethe states

$$\mathcal{T}(z)|\bar{u}\rangle=\tau(z|\bar{u})|\bar{u}\rangle$$

non-vanishing on-shell overlaps

$$\langle\psi|\bar{u}\rangle \neq 0$$

On-shell limit

transfer matrix

$$\mathcal{T}(u)=\sum_{i=1}^{2} T_{i,j}\, G_{j,i}$$

twist matrix

$$G \in GL(2)$$

$$\longrightarrow \quad \langle\Psi|\,\mathcal{T}(u)=\langle\Psi|\,\mathcal{T}(-u)$$

iff $[K,G]=0$

on-shell Bethe states

$$\mathcal{T}(z)|\bar{u}\rangle=\tau(z|\bar{u})|\bar{u}\rangle$$

non-vanishing

on-shell overlaps

$$\langle\Psi|\bar{u}\rangle\neq 0$$

$$\longrightarrow \quad \tau(z|\bar{u})=\tau(-z|\bar{u})$$

On-shell limit

transfer matrix

$$\mathcal{T}(u)=\sum_{i=1}^{2} T_{i,j}\, G_{j,i}$$

twist matrix

$$G\in GL(2)$$

$$\longrightarrow \langle\Psi|\,\mathcal{T}(u)=\langle\Psi|\,\mathcal{T}(-u)$$

iff $[K,G]=0$

on-shell Bethe states

$$\mathcal{T}(z)|\bar{u}\rangle=\tau(z|\bar{u})|\bar{u}\rangle$$

non-vanishing

on-shell overlaps

$$\langle\Psi|\bar{u}\rangle\neq 0$$

$$\longrightarrow \tau(z|\bar{u})=\tau(-z|\bar{u})$$

without twist $\longrightarrow$ pair structure

$$\{u_j\}_{j=1}^{r}=\{-u_j\}_{j=1}^{r}$$

$$Q(z)=(-1)^r Q(-z)$$

On-shell limit

transfer matrix

$$\mathcal{T}(u)=\sum_{i=1}^{2} T_{i,j}\, G_{j,i}$$

twist matrix

$$G\in GL(2)$$

$$\longrightarrow \quad \langle\Psi|\,\mathcal{T}(u)=\langle\Psi|\,\mathcal{T}(-u)$$

iff $[K,G]=0$

on-shell Bethe states

$$\mathcal{T}(z)|\bar{u}\rangle=\tau(z|\bar{u})|\bar{u}\rangle$$

non-vanishing on-shell overlaps

$$\langle\Psi|\bar{u}\rangle\neq 0$$

$$\longrightarrow \quad \tau(z|\bar{u})=\tau(-z|\bar{u})$$

without twist $\longrightarrow$ pair structure

$$\{u_j\}_{j=1}^{r}=\{-u_j\}_{j=1}^{r}$$

$$Q(z)=(-1)^r Q(-z)$$

with twist $\longrightarrow$ Two sets of roots

$$\{u_j\}_{j=1}^{r}=\{-v_j\}_{j=1}^{r}$$

$$Q_1(z)=(-1)^r Q_2(-z)$$

Untwisted on-shell limit

Bethe ansatz equations
$$e^{\Phi_j} := \frac{\lambda_1(u_j)}{\lambda_2(u_j)} \prod_{k\neq j} \frac{u_j-u_k-i}{u_j-u_k+i} = 1$$

Untwisted on-shell limit

Bethe ansatz equations
$$e^{\Phi_j} := \frac{\lambda_1(u_j)}{\lambda_2(u_j)} \prod_{k\neq j} \frac{u_j-u_k-i}{u_j-u_k+i} = 1$$

Gaudin matrix
$$G_{j,k} = \partial_{u_k} \log \Phi_j$$

Untwisted on-shell limit

Bethe ansatz equations $$e^{\Phi_j} := \frac{\lambda_1(u_j)}{\lambda_2(u_j)} \prod_{k\neq j} \frac{u_j - u_k - i}{u_j - u_k + i} = 1$$

Gaudin matrix $$G_{j,k} = \partial_{u_k} \log \Phi_j$$

on-shell norm $$\langle \bar{u} | \bar{u} \rangle \sim \det G$$

Untwisted on-shell limit

Bethe ansatz equations $$e^{\phi_j} := \frac{\lambda_1(u_j)}{\lambda_2(u_j)} \prod_{k\neq j} \frac{u_j-u_k-i}{u_j-u_k+i} = 1$$

Gaudin matrix $$G_{j,k} = \partial_{u_k} \log \phi_j$$

on-shell norm $$\langle \bar{u} | \bar{u} \rangle \sim \det G$$

proved by Korepin's criteria

Untwisted on-shell limit

Bethe ansatz equations $$e^{\phi_j} := \frac{\lambda_1(u_j)}{\lambda_2(u_j)} \prod_{k\neq j} \frac{u_j-u_k-i}{u_j-u_k+i} = 1$$

Gaudin matrix $$G_{j,k} = \partial_{u_k} \log \phi_j$$

on-shell norm $$\langle \bar{u} | \bar{u} \rangle \sim \det G$$

proved by Korepin's criteria

pair structure $$\bar{u} = \bar{u}^+ \cup \bar{u}^-$$

Untwisted on-shell limit

Bethe ansatz equations $$e^{\phi_j} := \frac{\lambda_1(u_j)}{\lambda_2(u_j)} \prod_{k\neq j} \frac{u_j-u_k-i}{u_j-u_k+i} = 1$$

Gaudin matrix $$G_{j,k} = \partial_{u_k} \log \phi_j$$

on-shell norm $$\langle \bar{u} | \bar{u} \rangle \sim \det G$$

proved by Korepin's criteria

pair structure $$\bar{u} = \bar{u}^+ \cup \bar{u}^- \qquad G^{\pm}_{j,k} = \left[\partial_{u_k^+} \pm \partial_{u_k^-}\right] \log \phi_j^+ \Big|_{\bar{u}^- = -\bar{u}^+}$$

Untwisted on-shell limit

Bethe ansatz equations $$e^{\Phi_j} := \frac{\lambda_1(u_j)}{\lambda_2(u_j)} \prod_{k\neq j} \frac{u_j-u_k-i}{u_j-u_k+i} = 1$$

Gaudin matrix $$G_{j,k} = \partial_{u_k} \log \Phi_j$$

on-shell norm $$\langle \bar{u} | \bar{u} \rangle \sim \det G$$

proved by Korepin's criteria

pair structure $\bar{u} = \bar{u}^+ \cup \bar{u}^-$ $$G^{\pm}_{j,k} = \left[\partial_{u^+_k} \pm \partial_{u^-_k}\right] \log \Phi^+_j \Big|_{\bar{u}^- = -\bar{u}^+}$$

factorisation $$\det G = \det G^+ \det G^-$$

Untwisted on-shell limit

Bethe ansatz equations $$e^{\phi_j} := \frac{\lambda_1(u_j)}{\lambda_2(u_j)} \prod_{k\neq j} \frac{u_j-u_k-i}{u_j-u_k+i} = 1$$

Gaudin matrix $$G_{j,k} = \partial_{u_k} \log \phi_j$$

on-shell norm $$\langle \bar{u} | \bar{u} \rangle \sim \det G$$

proved by Korepin's criteria

pair structure $\bar{u} = \bar{u}^+ \cup \bar{u}^-$ $$G^{\pm}_{j,k} = \left[\partial_{u^+_k} \pm \partial_{u^-_k}\right] \log \phi^+_j \Big|_{\bar{u}^- = -\bar{u}^+}$$

factorisation $$\det G = \det G^+ \det G^-$$

Korepin's criteria

Untwisted on-shell limit

Bethe ansatz equations $$e^{\phi_j} := \frac{\lambda_1(u_j)}{\lambda_2(u_j)} \prod_{k\neq j} \frac{u_j-u_k-i}{u_j-u_k+i} = 1$$

Gaudin matrix $$G_{j,k} = \partial_{u_k} \log \phi_j$$

on-shell norm $$\langle \bar{u} | \bar{u} \rangle \sim \det G$$

proved by Korepin's criteria

pair structure $\bar{u} = \bar{u}^+ \cup \bar{u}^-$ $$G^{\pm}_{j,k} = \left[\partial_{u^+_k} \pm \partial_{u^-_k}\right] \log \phi^+_j \Big|_{\bar{u}^- = -\bar{u}^+}$$

factorisation $$\det G = \det G^+ \det G^-$$

Korepin's criteria → $\langle \psi | \bar{u} \rangle \sim \det G^+$

Normalized on-shell overlaps

$$\frac{\langle \psi | \bar{u} \rangle}{\sqrt{\langle \bar{u} | \bar{u} \rangle}} = F(\bar{u}^{+}) \sqrt{\frac{\det G^{+}}{\det G^{-}}}$$

Normalized on-shell overlaps

$$\frac{\langle \psi | \bar{u} \rangle}{\sqrt{\langle \bar{u} | \bar{u} \rangle}} = \mp F(\bar{u}^{+}) \sqrt{\frac{\det G^{+}}{\det G^{-}}}$$

universal part $\sqrt{\frac{\det G^{+}}{\det G^{-}}}$ depends on the Bethe state

Normalized on-shell overlaps

$$\frac{\langle\Psi|\bar{u}\rangle}{\sqrt{\langle\bar{u}|\bar{u}\rangle}} = \mathcal{F}(\bar{u}^{+})\sqrt{\frac{\det G^{+}}{\det G^{-}}}$$

universal part $\sqrt{\frac{\det G^{+}}{\det G^{-}}}$ depends on the Bethe state

boundary part $\mathcal{F}(\bar{u}^{+}) = \prod_{j=1}^{r/2} \mathcal{F}(u_j^{+})$ depends on the b.s.

Normalized on-shell overlaps

$$\frac{\langle \Psi | \bar{u} \rangle}{\sqrt{\langle \bar{u} | \bar{u} \rangle}} = \mathcal{F}(\bar{u}^{+}) \sqrt{\frac{\det G^{+}}{\det G^{-}}}$$

universal part $\sqrt{\frac{\det G^{+}}{\det G^{-}}}$ depends on the Bethe state

boundary part $\mathcal{F}(\bar{u}^{+}) = \prod_{j=1}^{r/2} \mathcal{F}(u_j^{+})$ depends on the b.s.

or equivalently $K(u)$

Normalized on-shell overlaps

$$\frac{\langle\Psi|\bar{u}\rangle}{\sqrt{\langle\bar{u}|\bar{u}\rangle}} = F(\bar{u}^{+})\sqrt{\frac{\det G^{+}}{\det G^{-}}}$$

universal part $\sqrt{\frac{\det G^{+}}{\det G^{-}}}$ depends on the Bethe state

boundary part $F(\bar{u}^{+}) = \prod_{j=1}^{r/2} F(u_j^{+})$ depends on the b.s.

or equivalently $K(u)$

general solution of the reflection equation

$$K(u) = \frac{a}{u}\mathbb{1} + A = \begin{pmatrix} \frac{a}{u}+b_{11} & b_{12} \\ b_{21} & \frac{a}{u}-b_{11} \end{pmatrix}$$

$A^2 = \gamma\mathbb{1}$ $\gamma = -\det A$

Normalized on-shell overlaps

$$\frac{\langle \Psi | \bar{u} \rangle}{\sqrt{\langle \bar{u} | \bar{u} \rangle}} = \mathcal{F}(\bar{u}^+) \sqrt{\frac{\det G^+}{\det G^-}}$$

universal part $\sqrt{\frac{\det G^+}{\det G^-}}$ depends on the Bethe state

boundary part $\mathcal{F}(\bar{u}^+) = \prod_{j=1}^{r/2} \mathcal{F}(u_j^+)$ depends on the b.s.

or equivalently $K(u)$

general solution of the reflection equation

$$K(u) = \frac{\alpha}{u} \mathbb{1} + A = \begin{pmatrix} \frac{\alpha}{u} + b_{11} & b_{12} \\ b_{21} & \frac{\alpha}{u} - b_{11} \end{pmatrix} \longrightarrow \mathcal{F}(u) = \frac{\gamma}{b_{21}^2} \frac{u^2 - \alpha^2/\gamma}{u\sqrt{u^2 + 1/4}}$$

$$A^2 = \gamma \mathbb{1} \qquad \gamma = -\det A$$

twisted overlap

for spin 1/2 chain $Q_1(z) \sim Q_2(-z) \Rightarrow \#\bar{u} = \#\bar{v} = 3 = \#\bar{\theta}$

twisted overlap

for spin 1/2 chain $\quad Q_1(z) \sim Q_2(-z) \Rightarrow \#\bar{u} = \#\bar{v} = J = \#\bar{\theta}$

vandermonde matrix

$$V_{j,k} = (\theta_j + i/2)^{k-1}\, Q_1(\theta_j - i/2)\, Q_2(\theta_j - i/2) - (\theta_j - i/2)^{k-1}\, Q_1(\theta_j + i/2)\, Q_2(\theta_j + i/2)$$

twisted overlap

for spin 1/2 chain $Q_1(z) \sim Q_2(-z) \Rightarrow \#\bar{u} = \#\bar{v} = J = \#\bar{\theta}$

Vandermonde matrix

$$V_{j,k} = (\theta_j + i/2)^{k-1} Q_1(\theta_j - i/2) Q_2(\theta_j - i/2) -$$
$$(\theta_j - i/2)^{k-1} Q_1(\theta_j + i/2) Q_2(\theta_j + i/2)$$

Off-shell scalar product $\langle \bar{v} | \bar{u} \rangle \sim \det V$

twisted overlap

for spin 1/2 chain $Q_1(z) \sim Q_2(-z) \Rightarrow \# \bar{u} = \# \bar{v} = \mathcal{J} = \# \bar{\theta}$

vandermonde matrix
$$V_{j,k} = (\theta_j + i/2)^{k-1} Q_1(\theta_j - i/2) Q_2(\theta_j - i/2) - (\theta_j - i/2)^{k-1} Q_1(\theta_j + i/2) Q_2(\theta_j + i/2)$$

Off-shell scalar product $\langle \bar{v} | \bar{u} \rangle \sim \det V$

pair structure $\bar{v} = -\bar{u} \Rightarrow Q_1(z) = (-1)^{\mathcal{J}} Q_2(-z)$

twisted overlap

for spin 1/2 chain $Q_1(z) \sim Q_2(-z) \Rightarrow \#\bar{u} = \#\bar{v} = J = \#\bar{\theta}$

Vandermonde matrix

$$V_{j,k} = (\theta_j + i/2)^{k-1} Q_1(\theta_j - i/2) Q_2(\theta_j - i/2) - (\theta_j - i/2)^{k-1} Q_1(\theta_j + i/2) Q_2(\theta_j + i/2)$$

Off-shell scalar product $\langle \bar{v} | \bar{u} \rangle \sim \det V$

pair structure $\bar{v} = -\bar{u} \Rightarrow Q_1(z) = (-1)^J Q_2(-z)$

factorisation $\det V = \det V^+ \det V^-$

twisted overlap

for spin 1/2 chain $Q_1(z) \sim Q_2(-z) \Rightarrow \#\bar{u} = \#\bar{v} = J = \#\bar{\theta}$

vandermonde matrix

$$V_{j,k} = (\theta_j + i/2)^{k-1} Q_1(\theta_j - i/2) Q_2(\theta_j - i/2) - (\theta_j - i/2)^{k-1} Q_1(\theta_j + i/2) Q_2(\theta_j + i/2)$$

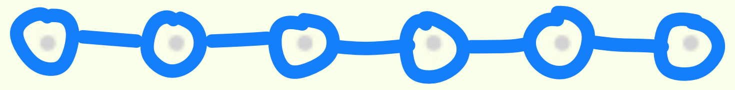
off-shell scalar product $\langle \bar{v} | \bar{u} \rangle \sim \det V$

pair structure $\bar{v} = -\bar{u} \Rightarrow Q_1(z) = (-1)^J Q_2(-z)$

factorisation $\det V = \det V^+ \det V^-$

$\rightarrow \langle \psi | \bar{u} \rangle \sim \det V^+$

$$gl(2) \longrightarrow gl(N)$$



$|\bar{u}\rangle$

$|\bar{u}\rangle \equiv |\bar{u}^1, \bar{u}^2, \ldots, \bar{u}^{N-1}\rangle$

Generalisation to gl(N) spin chains

two types of KT-relations

Generalisation to gl(N) spin chains

two types of KT-relations

non-crossed $K(u)\langle\psi|T(u) = \langle\psi|T(-u)K(u)$

crossed $K(u)\langle\psi|T(u) = \langle\psi|\hat{T}(-u)K(u)$

Generalisation to gl(N) spin chains

two types of KT-relations

non-crossed $K(u)\langle\psi|T(u) = \langle\psi|T(-u)K(u)$

crossed $K(u)\langle\psi|T(u) = \langle\psi|\hat{T}(-u)K(u)$

inverse monodromy matrix $\hat{T}^{t}(u)\,T(u) = \mathbb{1}$

Generalisation to gl(N) spin chains

two types of KT-relations

non-crossed $K(u)\langle\Psi|T(u) = \langle\Psi|T(-u)K(u)$

Crossed $K(u)\langle\Psi|T(u) = \langle\Psi|\hat{T}(-u)K(u)$

inverse monodromy matrix $\hat{T}^t(u)\,T(u) = \mathbb{1}$

compatibility conditions

$$R_{12}(u-v)K_1(-u)R_{12}(u+v)K_2(v) = K_2(-v)R_{12}(u+v)K_1(-u)R_{12}(u-v)$$

$$R_{12}(u-v)K_1(-u)\bar{R}_{12}(u+v)K_2(v) = K_2(-v)\bar{R}_{12}(u+v)K_1(-u)R_{12}(u-v)$$

Symmetries and pair structures

Non-crossed K-matrices

Symmetries and pair structures

Non-crossed K-matrices $K(u) = \frac{a}{u} \mathbb{1} + A$ $A^2 = \mathbb{1}$

Symmetries and pair structures

Non-crossed K-matrices

$K(u) = \frac{a}{u} \mathbb{1} + A \qquad A^2 = \mathbb{1}$

$A \sim \mathrm{diag}(\underbrace{+,+,\dots,+}_{N-M},\underbrace{-,-,\dots,-}_{M})$

Symmetries and pair structures

Non-crossed K-matrices $K(u) = \frac{a}{u}\mathbb{1} + A$ $A^2 = \mathbb{1}$ residual symmetry

$A \sim \mathrm{diag}(\underbrace{+,+,\dots,+}_{N-M},\underbrace{-,-,\dots,-}_{M})$ $gl(M)\oplus gl(N-M)$

Symmetries and pair structures

Non-crossed K-matrices

$$K(u) = \frac{a}{u}\mathbb{1} + A \qquad A^2 = \mathbb{1}$$

residual symmetry

$$A \sim \mathrm{diag}(\underbrace{+,+,\ldots,+}_{N-M},\underbrace{-,-,\ldots,-}_{M})$$

$$\mathfrak{gl}(M) \oplus \mathfrak{gl}(N-M)$$

Crossed K-matrices

Symmetries and pair structures

Non-crossed K-matrices $K(u) = \frac{a}{u}\mathbb{1} + A$ $A^2 = \mathbb{1}$ residual symmetry

$$A \sim \mathrm{diag}(\underbrace{+,+,\dots,+}_{N-M},\underbrace{-,-,\dots,-}_{M})$$

$$\mathfrak{gl}(M) \oplus \mathfrak{gl}(N-M)$$

Crossed K-matrices $K(u) = V$ $V^t = \pm V$

Symmetries and pair structures

Non-crossed K-matrices $K(u) = \frac{a}{u}\mathbb{1} + A$ $A^2 = \mathbb{1}$ residual symmetry

$A \sim \mathrm{diag}(\underbrace{+,+,\dots,+}_{N-M},\underbrace{-,-,\dots,-}_{M})$ $\mathfrak{gl}(M)\oplus\mathfrak{gl}(N-M)$

Crossed K-matrices $K(u) = V$ $V^t = \pm V$ $\mathfrak{o}(N)$
$\mathfrak{sp}(N)$

Symmetries and pair structures

Non-crossed K-matrices $K(u) = \frac{a}{u} \mathbb{1} + A$ $A^2 = \mathbb{1}$ residual symmetry

$A \sim \mathrm{diag}(\underbrace{+,+,\ldots,+}_{N-M},\underbrace{-,-,\ldots,-}_{M})$ $\mathfrak{gl}(M) \oplus \mathfrak{gl}(N-M)$

Crossed K-matrices $K(u) = V$ $V^t = \pm V$ $O(N)$ $Sp(N)$

Non-crossed KT $\langle \psi | \bar{u} \rangle \neq 0 \iff \tau(z|\bar{u}) = \tau(-z|\bar{u})$

Symmetries and pair structures

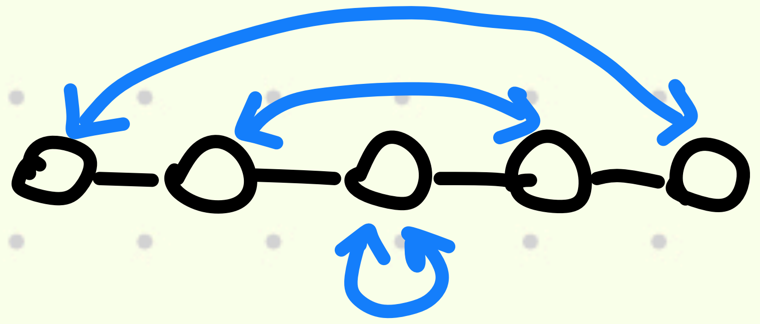
Non-crossed K-matrices $K(u) = \frac{a}{u}\mathbb{1} + A$ $A^2 = \mathbb{1}$ residual symmetry

$$A \sim \mathrm{diag}(\underbrace{+,+,\dots,+}_{N-M},\underbrace{-,-,\dots,-}_{M})$$ $\mathfrak{gl}(M)\oplus\mathfrak{gl}(N-M)$

Crossed K-matrices $K(u) = V$ $V^t = \pm V$ $O(N)$ $Sp(N)$

Non-crossed KT $\langle\psi|\bar{u}\rangle \neq 0 \iff \tau(z|\bar{u}) = \tau(-z|\bar{u})$

achiral pair structure $\bar{u}^{\nu} = -\bar{u}^{N-\nu}$



Symmetries and pair structures

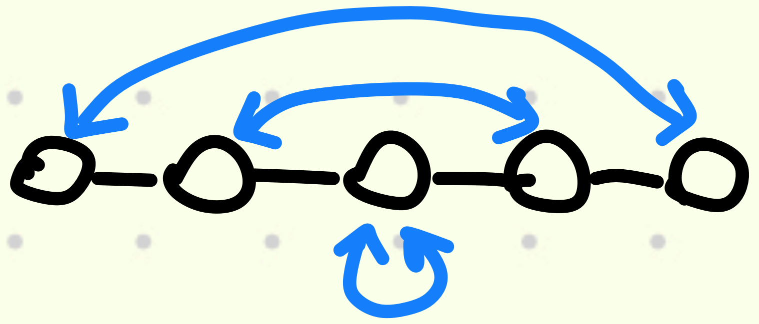
Non-crossed K-matrices $K(u) = \frac{a}{u}\mathbb{1} + A$ $A^2 = \mathbb{1}$ residual symmetry

$A \sim \mathrm{diag}(\underbrace{+,+,\dots,+}_{N-M},\underbrace{-,-,\dots,-}_{M})$ $\mathfrak{gl}(M)\oplus\mathfrak{gl}(N-M)$

Crossed K-matrices $K(u) = V$ $V^t = \pm V$ $O(N)$ $Sp(N)$

Non-crossed KT $\langle\psi|\bar{u}\rangle \neq 0 \iff \tau(z|\bar{u}) = \tau(-z|\bar{u})$

achiral pair structure $\bar{u}^{\nu} = -\bar{u}^{N-\nu}$



Crossed KT $\langle\psi|\bar{u}\rangle \neq 0 \iff \tau(z|\bar{u}) = \hat{\tau}(-z|\bar{u})$

Symmetries and pair structures

Non-crossed K-matrices $K(u) = \frac{a}{u}\mathbb{1} + A$ $A^2 = \mathbb{1}$ residual symmetry

$$A \sim \mathrm{diag}(\underbrace{+,+,\dots,+}_{N-M},\underbrace{-,-,\dots,-}_{M})$$

$\mathfrak{gl}(M) \oplus \mathfrak{gl}(N-M)$

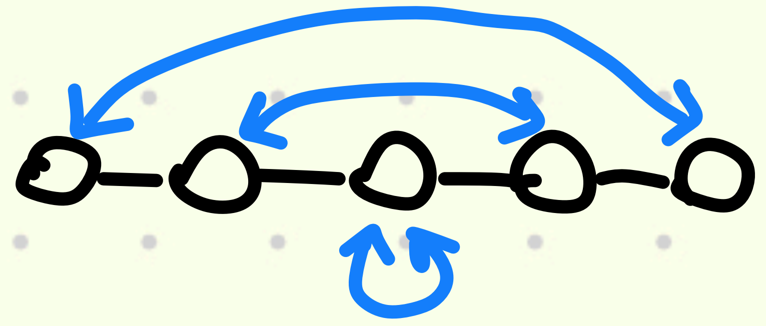
Crossed K-matrices $K(u) = V$ $V^t = \pm V$

$O(N)$

$Sp(N)$

Non-crossed KT $\langle\psi|\bar{u}\rangle \neq 0 \iff \tau(z|\bar{u}) = \tau(-z|\bar{u})$

achiral pair structure $\bar{u}^\nu = -\bar{u}^{N-\nu}$



Crossed KT $\langle\psi|\bar{u}\rangle \neq 0 \iff \tau(z|\bar{u}) = \hat{\tau}(-z|\bar{u})$

chiral pair structure $\bar{u}^\nu = -\bar{u}^\nu$

Symmetries and pair structures

Non-crossed K-matrices $K(u) = \frac{a}{u}\mathbb{1} + A$ $A^2 = \mathbb{1}$ residual symmetry

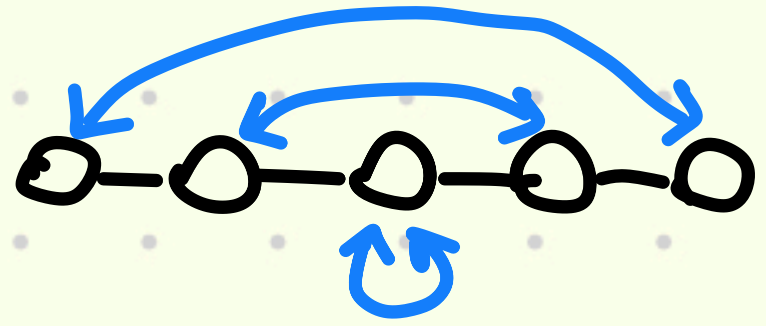
$A \sim \mathrm{diag}(\underbrace{+,+,\ldots,+}_{N-M},\underbrace{-,-,\ldots,-}_{M})$ $\mathfrak{gl}(M) \oplus \mathfrak{gl}(N-M)$

Crossed K-matrices $K(u) = V$ $V^t = \pm V$ $O(N)$ $Sp(N)$

with twist

Non-crossed KT $\langle \Psi | \bar{u} \rangle \neq 0 \iff \tau(z|\bar{u}) = \tau(-z|\bar{u})$ $[G, K] = 0$

achiral pair structure $\bar{u}^{\nu} = -\bar{u}^{N-\nu}$



$Q_a(z) \sim Q_{1\ldots\hat{a}\ldots N}(-z)$

$Q_{ab}(z) \sim Q_{1\ldots\hat{a}\ldots\hat{b}\ldots N}(-z)$
⋮

Crossed KT $\langle \Psi | \bar{u} \rangle \neq 0 \iff \tau(z|\bar{u}) = \hat{\tau}(-z|\bar{u})$

chiral pair structure $\bar{u}^{\nu} = -\bar{u}^{\nu}$

$Q_a(z) \sim Q_{\bar{a}}(-z)$

$Q_{ab}(z) \sim Q_{\bar{a}\bar{b}}(-z)$

Off-shell overlaps

List of criteria for off-shell overlaps

$$S_{\bar{\lambda}}(\bar{u}) = \sum W(\bar{u}_{I} | \bar{u}_{II}) \prod_{\nu=1}^{N-1} \lambda_{\nu}(\bar{u}^{\nu}_{I}) \lambda_{\nu+1}(\bar{u}^{\nu}_{II})$$

Off-shell overlaps

List of criteria for off-shell overlaps $\quad S_{\bar\lambda}(\bar u) = \sum W(\bar u_{I} | \bar u_{II}) \prod_{\nu=1}^{N-1} \lambda_{\nu}(\bar u^{\nu}_{I}) \lambda_{\nu+1}(\bar u^{\nu}_{II})$

1) KT- relation: creation to annihilation $\quad \langle\psi| T_{1,j} \mapsto \langle\psi| T_{k,\ell} \quad k>1$ ✓

Off-shell overlaps

List of criteria for off-shell overlaps

$$S_{\bar\lambda}(\bar u) = \sum W(\bar u_{I} | \bar u_{II}) \prod_{\nu=1}^{N-1} \lambda_\nu(\bar u^\nu_{I})\, \lambda_{\nu+1}(\bar u^\nu_{II})$$

1) KT- relation: creation to annihilation $\langle\psi|T_{1,j} \mapsto \langle\psi|T_{k,\ell}$ $k>1$ ✓

2) recurrence formula $|\{z,\bar u^1\},\bar u^2,\dots\rangle = \sum(\dots)T_{1,j}(z)|\bar u^1,\bar u^2,\dots\rangle$

Off-shell overlaps

List of criteria for off-shell overlaps $\quad S_\lambda(\bar{u}) = \sum W(\bar{u}_I | \bar{u}_{II}) \prod_{\nu=1}^{N-1} \lambda_\nu(\bar{u}^\nu_I)\lambda_{\nu+1}(\bar{u}^\nu_{II})$

1) KT- relation: creation to annihilation $\quad \langle\psi|T_{1,j} \mapsto \langle\psi|T_{k,l} \quad k>1$ ✓

2) recurrence formula $\quad |\{z,\bar{u}^1\},\bar{u}^2,\ldots\rangle = \sum(\ldots)T_{1,j}(z)|\bar{u}^1,\bar{u}^2,\ldots\rangle$

3) action formula $\quad T_{i,j}(z)|\bar{u}\rangle = \sum(\ldots)|\bar{w}\rangle$

Off-shell overlaps

List of criteria for off-shell overlaps $\qquad S_{\lambda}(\bar{u}) = \sum W(\bar{u}_{I} | \bar{u}_{II}) \prod_{\nu=1}^{N-1} \lambda_{\nu}(\bar{u}^{\nu}_{I}) \lambda_{\nu+1}(\bar{u}^{\nu}_{II})$

1) KT- relation: creation to annihilation $\quad \langle\Psi| T_{1,j} \mapsto \langle\Psi| T_{k,l} \quad k>1$ ✓

2) recurrence formula $\quad |\{z, \bar{u}^1\}, \bar{u}^2, \dots\rangle = \sum (\dots) T_{1,j}(z) |\bar{u}^1, \bar{u}^2, \dots\rangle$

3) action formula $\quad T_{i,j}(z)|\bar{u}\rangle = \sum(\dots)|\bar{w}\rangle$

4) co-product formula $\quad |\bar{u}\rangle = \sum(\dots)|\bar{u}_{I}\rangle^{(1)} \otimes |\bar{u}_{II}\rangle^{(2)}$

Off-shell overlaps

List of criteria for off-shell overlaps $$S_\lambda(\bar{u}) = \sum W(\bar{u}_{I} | \bar{u}_{II}) \prod_{\nu=1}^{N-1} \lambda_\nu(\bar{u}^\nu_{I}) \lambda_{\nu+1}(\bar{u}^\nu_{II})$$

1) KT- relation: creation to annihilation $\langle\psi| T_{1,j} \mapsto \langle\psi| T_{k,\ell}$ $k>1$ ✓

2) recurrence formula $|\{z, \bar{u}^1\}, \bar{u}^2, \dots\rangle = \sum(\dots) T_{1,j}(z) |\bar{u}^1, \bar{u}^2, \dots\rangle$

3) action formula $T_{i,j}(z)|\bar{u}\rangle = \sum(\dots)|\bar{w}\rangle$

4) co-product formula $|\bar{u}\rangle = \sum(\dots)|\bar{u}_{I}\rangle^{(1)} \otimes |\bar{u}_{II}\rangle^{(2)}$

$\Longrightarrow$ **Hutsalyuk, Liashyk, Pakuliak, Ragoucy, Slavnov ’16, ’17, ‘20**

Off-shell overlaps

List of criteria for off-shell overlaps $\quad S_{\lambda}(\bar{u}) = \sum W(\bar{u}_{I} | \bar{u}_{II}) \prod_{\nu=1}^{N-1} \lambda_{\nu}(\bar{u}^{\nu}_{I}) \lambda_{\nu+1}(\bar{u}^{\nu}_{II})$

1) KT- relation: creation to annihilation $\quad \langle\Psi| T_{1,j} \mapsto \langle\Psi| T_{k,l} \quad k>1$ ✓

2) recurrence formula $\quad |\{z,\bar{u}^1\},\bar{u}^2,\dots\rangle = \sum(\dots) T_{1,j}(z)|\bar{u}^1,\bar{u}^2,\dots\rangle$

3) action formula $\quad T_{i,j}(z)|\bar{u}\rangle = \sum(\dots)|\bar{w}\rangle$

4) co-product formula $\quad |\bar{u}\rangle = \sum(\dots)|\bar{u}_{I}\rangle^{(1)} \otimes |\bar{u}_{II}\rangle^{(2)}$

$\Longrightarrow$ **Hutsalyuk, Liashyk, Pakuliak, Ragoucy, Slavnov ’16, ’17, ‘20**

Recursion: $\quad \langle\Psi|\{z,\bar{u}^1\},\bar{u}^2,\dots\rangle = \sum_j(\dots)\langle\Psi|T_{1,j}|\bar{u}^1,\bar{u}^2,\dots\rangle = \sum_j(\dots)\langle\Psi|\bar{w}^1,\bar{w}^2,\dots\rangle$

$\#\bar{w}^1 \leq \bar{u}^1$

Off-shell overlaps

List of criteria for off-shell overlaps $\quad S_{\lambda}(\bar{u}) = \sum W(\bar{u}_{I} | \bar{u}_{II}) \prod_{\nu=1}^{N-1} \lambda_{\nu}(\bar{u}^{\nu}_{I}) \lambda_{\nu+1}(\bar{u}^{\nu}_{II})$

1) KT- relation: creation to annihilation $\quad \langle\Psi| T_{1,j} \mapsto \langle\Psi| T_{k,l} \quad k>1$ ✓

2) recurrence formula $\quad |\{z,\bar{u}^1\},\bar{u}^2,\dots\rangle = \sum(\dots) T_{1,j}(z) |\bar{u}^1,\bar{v}^2,\dots\rangle$

3) action formula $\quad T_{i,j}(z)|\bar{u}\rangle = \sum(\dots)|\bar{w}\rangle$

4) co-product formula $\quad |\bar{u}\rangle = \sum(\dots)|\bar{u}_{I}\rangle^{(1)} \otimes |\bar{u}_{II}\rangle^{(2)}$

$\Longrightarrow$ **Hutsalyuk, Liashyk, Pakuliak, Ragoucy, Slavnov '16, '17, '20**

Recursion: $\quad \langle\Psi|\{z,\bar{u}^1\},\bar{u}^2,\dots\rangle = \sum_{j}(\dots)\langle\Psi|T_{1,j}|\bar{u}^1,\bar{v}^2,\dots\rangle = \sum_{j}(\dots)\langle\Psi|\bar{w}^1,\bar{w}^2,\dots\rangle$

$\#\bar{w}^1 \leq \bar{u}^1 \quad \Longrightarrow$ We can eliminate $\bar{u}^1$

On-shell overlaps without twists

Korepin's criteria $\longrightarrow$

$$\frac{\langle\psi|\bar{u}\rangle}{\sqrt{\langle\bar{u}|\bar{u}\rangle}} = \prod_{\nu} F_{\nu}(\bar{u}^{+\nu})\sqrt{\frac{\det G^{+}}{\det G^{-}}}$$

On-shell overlaps without twists

Korepin's criteria $\longrightarrow$

$$\frac{\langle \psi | \bar{u} \rangle}{\sqrt{\langle \bar{u} | \bar{u} \rangle}} = \prod_{\nu} F_{\nu}(\bar{u}^{+\nu}) \sqrt{\frac{\det G^{+}}{\det G^{-}}}$$

$G^{\pm}$ depends on the pair structure

On-shell overlaps without twists

Korepin's criteria →

$$\frac{\langle \psi | \bar{u} \rangle}{\sqrt{\langle \bar{u} | \bar{u} \rangle}} = \prod_{\nu} F_{\nu}(\bar{u}^{+\nu}) \sqrt{\frac{\det G^{+}}{\det G^{-}}}$$

$G^{\pm}$ depends on the pair structure

$F_{\nu}(u)$ given by the K-matrix

On-shell overlaps without twists

Korepin's criteria → $$\frac{\langle \psi | \bar{u} \rangle}{\sqrt{\langle \bar{u} | \bar{u} \rangle}} = \prod_{\nu} F_{\nu}(\bar{u}^{+\nu}) \sqrt{\frac{\det G^{+}}{\det G^{-}}}$$

$G^{\pm}$ depends on the pair structure

$F_{\nu}(u)$ given by the K-matrix

- $gl(M) \oplus gl(N-M)$ $$\frac{Q_M(a)}{\sqrt{Q_n(0) Q_n(\frac{i}{2})}} \sqrt{\frac{\det G^{+}}{\det G^{-}}}$$ $n = \frac{N}{2}$

On-shell overlaps without twists

Korepin's criteria $\longrightarrow$ $$\frac{\langle\psi|\bar{u}\rangle}{\sqrt{\langle\bar{u}|\bar{u}\rangle}} = \prod_{\nu} F_\nu(\bar{u}^{+\nu})\sqrt{\frac{\det G^+}{\det G^-}}$$

$G^\pm$ depends on the pair structure

$F_\nu(u)$ given by the K-matrix

$$Q_\nu(z) = \prod_{j=1}^{r_\nu}(z - u_j^\nu)$$

- $gl(M)\oplus gl(N-M)$ $$\frac{Q_M(a)}{\sqrt{Q_n(0)Q_n(\frac{i}{2})}}\sqrt{\frac{\det G^+}{\det G^-}}$$ $n = \frac{N}{2}$

On-shell overlaps without twists

Korepin's criteria → $$\frac{\langle\psi|\bar{u}\rangle}{\sqrt{\langle\bar{u}|\bar{u}\rangle}} = \prod_{\nu} F_\nu(\bar{u}^{+\nu}) \sqrt{\frac{\det G^+}{\det G^-}}$$

$G^\pm$ depends on the pair structure

$F_\nu(u)$ given by the K-matrix

$$Q_\nu(z) = \prod_{j=1}^{r_\nu} (z - u_j^\nu)$$

- $gl(M) \oplus gl(N-M)$ $$\frac{Q_M(a)}{\sqrt{Q_n(0)\,Q_n(\frac{i}{2})}} \sqrt{\frac{\det G^+}{\det G^-}}$$ $n = \frac{N}{2}$ achiral $\bar{u}^\nu = -\bar{u}^{N-\nu}$

On-shell overlaps without twists

Korepin's criteria →

$$\frac{\langle\psi|\bar{u}\rangle}{\sqrt{\langle\bar{u}|\bar{u}\rangle}} = \prod_{\nu} F_\nu(\bar{u}^{+\nu})\sqrt{\frac{\det G^+}{\det G^-}}$$

$G^\pm$ depends on the pair structure

$F_\nu(u)$ given by the K-matrix

$$Q_\nu(z)=\prod_{j=1}^{r_\nu}(z-u_j^\nu)$$

- $gl(M)\oplus gl(N-M)$ $\quad \dfrac{Q_M(a)}{\sqrt{Q_n(0)Q_n(\frac{i}{2})}}\sqrt{\dfrac{\det G^+}{\det G^-}}$ $\quad n=\frac{N}{2}$ $\quad$ achiral $\quad \bar{u}^\nu=-\bar{u}^{N-\nu}$

- $o(N)$ $\quad \displaystyle\prod_{\nu=1}^{N-1}\sqrt{\frac{Q_\nu(0)}{Q_\nu(\frac{i}{2})}}\sqrt{\frac{\det G^+}{\det G^-}}$

- $sp(N)$ $\quad \displaystyle\prod_{\nu}\sqrt{\frac{Q_{2\nu}(0)Q_{2\nu}(\frac{i}{2})}{Q_{2\nu-1}(0)Q_{2\nu-1}(\frac{i}{2})}}\sqrt{\frac{\det G^+}{\det G^-}}$

On-shell overlaps without twists

Korepin's criteria →
$$\frac{\langle\psi|\bar{u}\rangle}{\sqrt{\langle\bar{u}|\bar{u}\rangle}} = \prod_{\nu} F_{\nu}(\bar{u}^{+\nu})\sqrt{\frac{\det G^{+}}{\det G^{-}}}$$

$G^{\pm}$ depends on the pair structure

$F_{\nu}(u)$ given by the K-matrix

$$Q_{\nu}(z)=\prod_{j=1}^{r_{\nu}}(z-u_j^{\nu})$$

- $gl(M)\oplus gl(N-M)$ $$\frac{Q_M(a)}{\sqrt{Q_n(0)Q_n(\frac{i}{2})}}\sqrt{\frac{\det G^{+}}{\det G^{-}}}$$ $n=\frac{N}{2}$ achiral $\bar{u}^{\nu}=-\bar{u}^{N-\nu}$

- $o(N)$ $$\prod_{\nu=1}^{N-1}\sqrt{\frac{Q_{\nu}(0)}{Q_{\nu}(\frac{i}{2})}}\sqrt{\frac{\det G^{+}}{\det G^{-}}}$$

- $sp(N)$ $$\prod_{\nu}\sqrt{\frac{Q_{2\nu}(0)Q_{2\nu}(\frac{i}{2})}{Q_{2\nu-1}(0)Q_{2\nu-1}(\frac{i}{2})}}\sqrt{\frac{\det G^{+}}{\det G^{-}}}$$

$\Big\}$ chiral $\bar{u}^{\nu}=-\bar{u}^{\nu}$

On-shell overlaps with twist

diagonal twist $G = \mathrm{diag}(z_1, z_2, \ldots, z_N)$

On-shell overlaps with twist

diagonal twist $G = \mathrm{diag}(z_1, z_2, \dots, z_N)$

compatibility $[G, K] = 0 \Rightarrow$ K-matrix is diagonal

On-shell overlaps with twist

diagonal twist $G = \mathrm{diag}(z_1, z_2, \dots, z_N)$

compatibility $[G,K]=0 \Rightarrow$ K-matrix is diagonal

non-crossed case $K(u) = \mathrm{diag}\Big(\underbrace{\frac{a-z}{z},\dots,\frac{a-z}{z}}_{M},\underbrace{\frac{a+z}{z},\dots,\frac{a+z}{z}}_{N-M}\Big)$

$\mathfrak{gl}(M) \oplus \mathfrak{gl}(N-M)$
symmetry

On-shell overlaps with twist

diagonal twist $G = \text{diag}(z_1, z_2, \ldots, z_N)$

compatibility $[G, K] = 0 \Rightarrow$ K-matrix is diagonal

non-crossed case $K(u) = \text{diag}\Big(\underbrace{\frac{a-z}{z}, \ldots, \frac{a-z}{z}}_{M}, \underbrace{\frac{a+z}{z}, \ldots, \frac{a+z}{z}}_{N-M}\Big)$ $\quad gl(M) \oplus gl(N-M)$ symmetry

$M = 0 \qquad K(u) \approx 1\!\!1$

On-shell overlaps with twist

diagonal twist $G = \mathrm{diag}(z_1, z_2, \ldots, z_N)$

compatibility $[G, K] = 0 \Rightarrow$ K-matrix is diagonal

non-crossed case $K(u) = \mathrm{diag}\Big(\underbrace{\frac{a-z}{z}, \ldots, \frac{a-z}{z}}_{M}, \underbrace{\frac{a+z}{z}, \ldots, \frac{a+z}{z}}_{N-M}\Big)$ $\mathfrak{gl}(M) \oplus \mathfrak{gl}(N-M)$ symmetry

$M = 0$ $K(u) \simeq \mathbb{1}$ $\langle \psi_0 | \bar{u} \rangle$ is given by the functional SoV

Ekhammar, Gromov, Ryan '24

On-shell overlaps with twist

diagonal twist $G = \mathrm{diag}(z_1, z_2, \dots, z_N)$

compatibility $[G, K] = 0 \Rightarrow$ K-matrix is diagonal

non-crossed case $K(u) = \mathrm{diag}\Big(\underbrace{\frac{a-z}{z}, \dots, \frac{a-z}{z}}_{M}, \underbrace{\frac{a+z}{z}, \dots, \frac{a+z}{z}}_{N-M}\Big)$ $\mathfrak{gl}(M) \oplus \mathfrak{gl}(N-M)$ symmetry

$M=0$ $K(u) \approx \mathbb{1}$ $\langle \Psi_0 | \bar{u} \rangle$ is given by the functional SoV

Ekhammar, Gromov, Ryan '24

general M $\langle \Psi_M | \bar{u} \rangle = Q_M(a) \langle \Psi_0 | \bar{u} \rangle$ ← my off-shell sum formula

On-shell overlaps with twist

diagonal twist $G = \mathrm{diag}(z_1, z_2, \dots, z_N)$

compatibility $[G, K] = 0 \Rightarrow$ K-matrix is diagonal

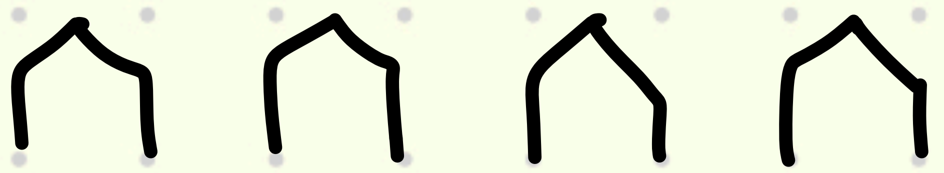
non-crossed case $K(u) = \mathrm{diag}\Big(\underbrace{\frac{a-z}{z}, \dots, \frac{a-z}{z}}_{M}, \underbrace{\frac{a+z}{z}, \dots, \frac{a+z}{z}}_{N-M}\Big)$ $\quad gl(M) \oplus gl(N-M)$ symmetry

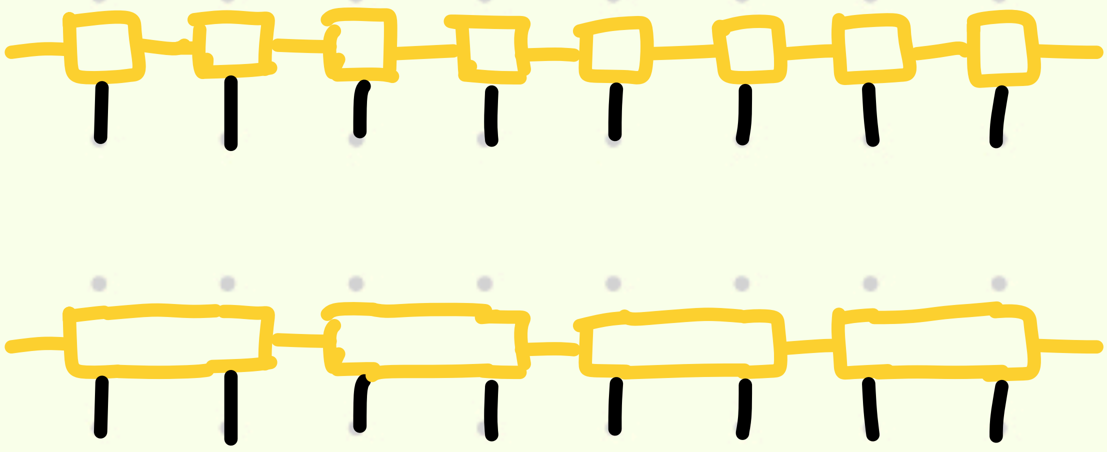
$M=0$ $\quad K(u) \simeq \mathbb{1}$ $\quad \langle \Psi_0 | \bar{u} \rangle$ is given by the functional SoV

Ekhammar, Gromov, Ryan '24

general M $\quad \langle \Psi_M | \bar{u} \rangle = Q_M(a) \langle \Psi_0 | \bar{u} \rangle$ ← my off-shell sum formula

crossed case? $\quad K(z) = \mathrm{diag}(\underbrace{1, \dots, 1}_{\frac{N}{2}}, \underbrace{\pm 1, \dots, \pm 1}_{\frac{N}{2}})$ $\quad o(N)$ or $sp(N)$ symmetric cases two-site states → matrix product states





Generalisation to MPS

two-site state ⇒ matrix product state

Generalisation to MPS

two-site state ⇒ matrix product state

$$\langle \Psi_{\alpha\beta}| = \sum \left(M_{i_{2J-1}, i_{2J}}(\theta_J) \dots M_{i_1, i_2}(\theta_1) \right)_{\alpha\beta} \langle i_1, i_2, \dots, i_{2J-1}, i_J|$$

Generalisation to MPs

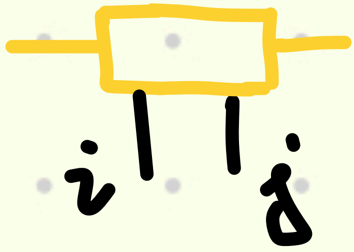
two-site state matrix product state

$$\langle\Psi_{\alpha\beta}| = \sum \left(M_{i_{2J-1},i_{2J}}(\theta_J)\dots M_{i_1,i_2}(\theta_1)\right)_{\alpha\beta} \langle i_1, i_2, \dots, i_{2J-1}, i_J|$$

$M_{ij}(\theta) \in \mathrm{End}(\mathcal{H}_B)$

$i = 1, \dots, d$

$\mathcal{H} = [\mathbb{C}^d]^{\otimes 2J}$



Generalisation to MPs

two-site state

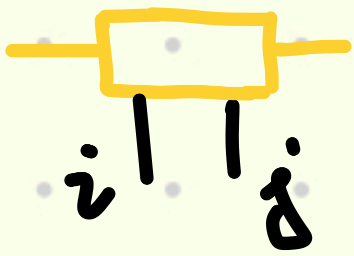
matrix product state

$$\langle \psi_{\alpha\beta}| = \sum \left(M_{i_{2z-1},i_{2z}}(\theta_z)\dots M_{i_1,i_2}(\theta_1)\right)_{\alpha\beta} \langle i_1, i_2, \dots, i_{2z-1}, i_z|$$

$M_{ij}(\theta) \in \mathrm{End}(\mathcal{H}_B)$

$i = 1, \dots, d$

$\mathcal{H} = [\mathbb{C}^d]^{\otimes 2z}$



$$\langle \Psi| = \sum_{\alpha,\beta} \langle \psi_{\alpha,\beta}| \otimes e_{\alpha,\beta} \in \mathcal{H}^* \otimes \mathrm{End}(\mathcal{H}_B)$$

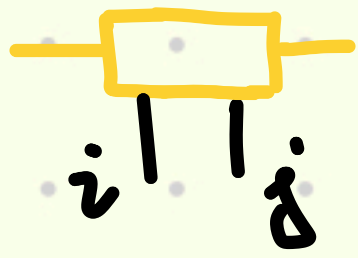
Generalisation to MPs

two-site state ⇒ matrix product state

$$\langle \psi_{\alpha\beta}| = \sum \left(M_{i_{2J-1}, i_{2J}}(\theta_J) \dots M_{i_1, i_2}(\theta_1) \right)_{\alpha\beta} \langle i_1, i_2, \dots, i_{2J-1}, i_J|$$

$M_{ij}(\theta) \in \mathrm{End}(\mathcal{H}_B)$

$i = 1, \dots, d$



$\mathcal{H} = [\mathbb{C}^d]^{\otimes 2J}$

$$\langle \Psi| = \sum_{\alpha,\beta} \langle \psi_{\alpha,\beta}| \otimes e_{\alpha,\beta} \in \mathcal{H}^* \otimes \mathrm{End}(\mathcal{H}_B)$$

$$K(\omega) \in \mathrm{End}(\mathbb{C}^N \otimes \mathcal{H}_B)$$

Generalisation to MPs

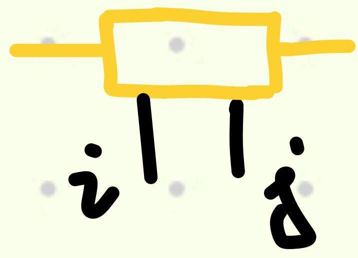
two-site state

matrix product state

$$\langle \psi_{\alpha\beta}| = \sum \left(M_{i_{2J-1}, i_{2J}}(\theta_J) \dots M_{i_1, i_2}(\theta_1) \right)_{\alpha\beta} \langle i_1, i_2, \dots, i_{2J-1}, i_J| \quad M_{ij}(\theta) \in \mathrm{End}(\mathcal{H}_B)$$

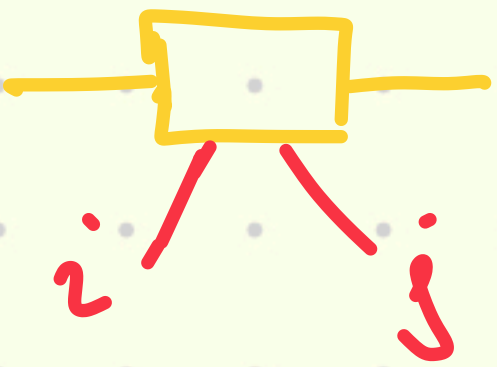
$i = 1, \dots, d$

$\mathcal{H} = [\mathbb{C}^d]^{\otimes 2J}$



$$\langle \Psi| = \sum_{\alpha,\beta} \langle \psi_{\alpha,\beta}| \otimes e_{\alpha,\beta} \in \mathcal{H}^* \otimes \mathrm{End}(\mathcal{H}_B)$$

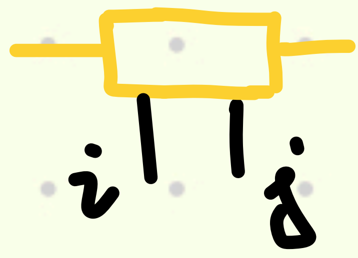
$$K(u) \in \mathrm{End}(\mathbb{C}^N \otimes \mathcal{H}_B) \qquad K_{ij}(u) \in \mathrm{End}(\mathcal{H}_B)$$



Generalisation to MPs

two-site state matrix product state

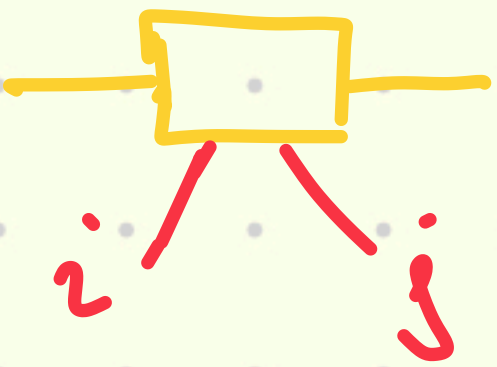
$$\langle \psi_{\alpha\beta}| = \sum \left(M_{i_{2J-1}, i_{2J}}(\theta_J) \dots M_{i_1, i_2}(\theta_1) \right)_{\alpha\beta} \langle i_1, i_2, \dots, i_{2J-1}, i_J| \qquad M_{ij}(\theta) \in \mathrm{End}(\mathcal{H}_B) \qquad i = 1, \dots, d$$



$$\mathcal{H} = [\mathbb{C}^d]^{\otimes 2J}$$

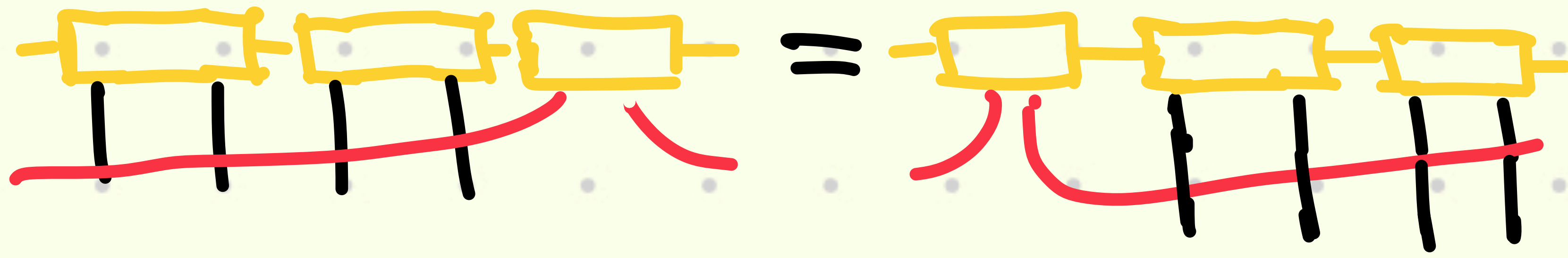
$$\langle \Psi| = \sum_{\alpha,\beta} \langle \psi_{\alpha,\beta}| \otimes e_{\alpha,\beta} \in \mathcal{H}^* \otimes \mathrm{End}(\mathcal{H}_B)$$

$$K(u) \in \mathrm{End}(\mathbb{C}^N \otimes \mathcal{H}_B) \qquad K_{ij}(u) \in \mathrm{End}(\mathcal{H}_B)$$



KT- relation

$$\sum_{j=1}^{N} K_{ij}(z) \langle \Psi| T_{j,\ell}(z) = \sum_{j=1}^{N} \langle \Psi| T_{i,j}(-z) K_{j,\ell}(z)$$



Integrability condition

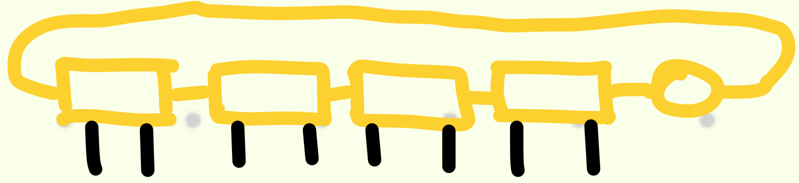
if the twist is symmetry of the K-matrix

Integrability condition

if the twist is symmetry of the K-matrix $[G\otimes g, K]=0$

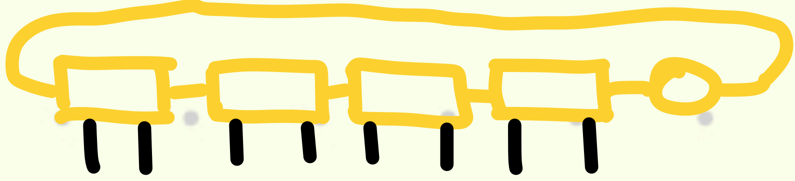
Integrability condition

if the twist is symmetry of the K-matrix $[G\otimes\mathcal{G}, K]=0$

$\Rightarrow$ $\langle MPS| =$  $= \sum_{\alpha\beta} \langle\psi_{\alpha\beta}| \mathcal{G}_{\beta\alpha}$

Integrability condition

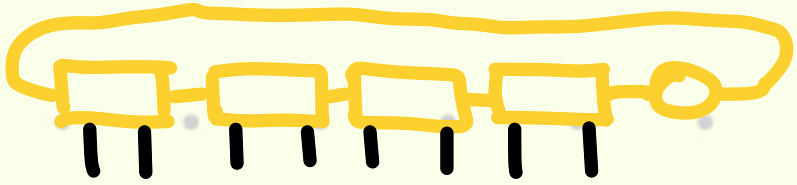
if the twist is symmetry of the K-matrix $\qquad [G\otimes \mathcal{G}, K]=0$

$\Rightarrow \quad \langle MPS| =$  $= \sum_{\alpha\beta} \langle \psi_{\alpha\beta}| \mathcal{G}_{\beta\alpha}$

$\Rightarrow \quad \langle MPS|T(u) = \langle MPS|T(-u) \quad$ or $\quad \langle MPS|T(u) = \langle MPS|\widehat{T}(-u)$

Integrability condition

if the twist is symmetry of the K-matrix $[G\otimes \mathcal{G}, K]=0$

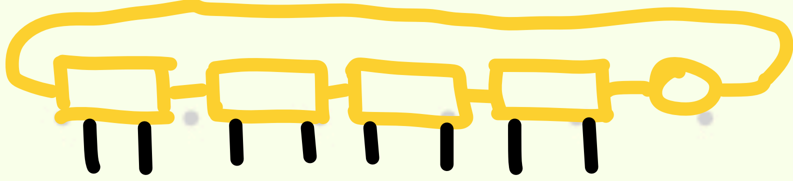
$\Rightarrow$ $\langle MPS| =$  $= \sum_{\alpha\beta} \langle \Psi_{\alpha\beta}| \mathcal{G}_{\beta\alpha}$

$\Rightarrow$ $\langle MPS|T(u) = \langle MPS|T(-u)$ or $\langle MPS|T(u) = \langle MPS|\widehat{T}(-u)$

homogeneous limit

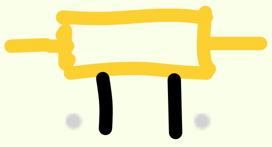
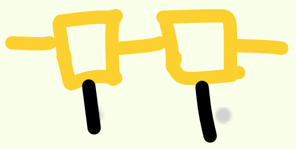
Integrability condition

if the twist is symmetry of the K-matrix $[G\otimes\mathcal{G}, K]=0$

$\Rightarrow$ $\langle MPS| =$  $= \sum_{\alpha\beta}\langle\psi_{\alpha\beta}|\mathcal{G}_{\beta\alpha}$

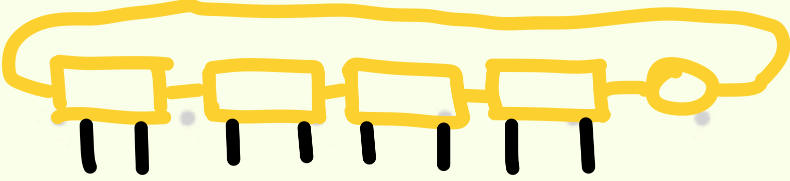
$\Rightarrow$ $\langle MPS|T(u)=\langle MPS|T(-u)$ or $\langle MPS|T(u)=\langle MPS|\widehat{T}(-u)$

homogeneous limit

If  $\xrightarrow[\theta=0]{}$ 

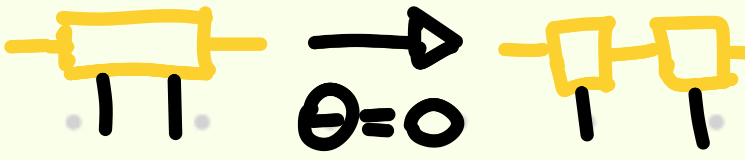
Integrability condition

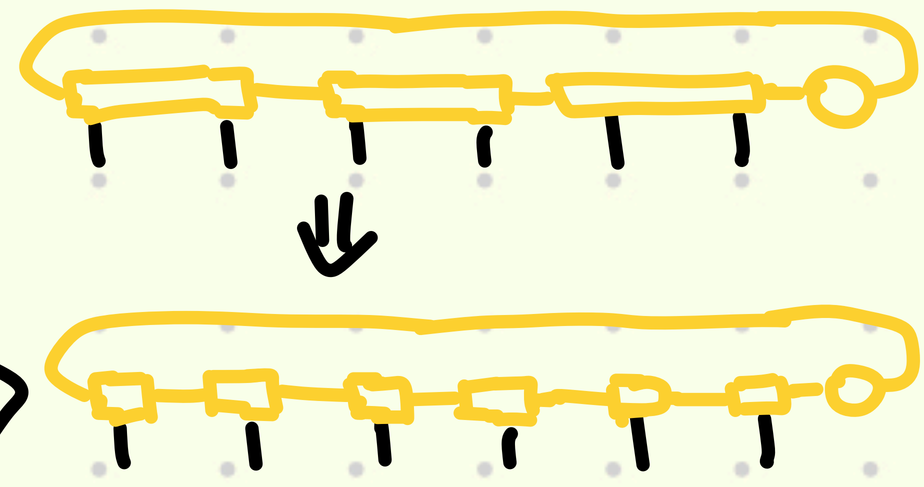
if the twist is symmetry of the K-matrix $[G \otimes g, K] = 0$

⇒ $\langle MPS| =$  $= \sum_{\alpha\beta} \langle \Psi_{\alpha\beta}| G_{\beta\alpha}$

⇒ $\langle MPS|T(u) = \langle MPS|T(-u)$ or $\langle MPS|T(u) = \langle MPS|\hat{T}(-u)$

homogeneous limit

If 

⇒ 

Classification of the K-matrices

compatibility of the KT with RTT →

Classification of the K-matrices

compatibility of the KT with RTT → Reflection equation
(bYBe)

Classification of the K-matrices

compatibility of the KT with RTT → Reflection equation (bYBe)

$$R K \underset{\bar{R}}{R} K = K \underset{\bar{R}}{R} K R$$

Classification of the K-matrices

compatibility of the KT with RTT → Reflection equation (bYBe)

$$R K \underset{\bar{R}}{R} K = K \underset{\bar{R}}{R} K R$$

↓

K-matrices are representations of reflection algebras

Classification of the K-matrices

compatibility of the KT with RTT → Reflection equation

(bYBe)

$$R K \underset{\bar{R}}{R} K = K \underset{\bar{R}}{R} K R$$

↓

K-matrices are representations of reflection algebras

non-crossed bYBe $\longrightarrow$ $\mathcal{B}(N,M)$ algebra

Classification of the K-matrices

compatibility of the KT with RTT → Reflection equation

$$R K \underset{\bar{R}}{R} K = K \underset{\bar{R}}{R} K R$$

(bYBe)

↓

K-matrices are representations of reflection algebras

non-crossed bYBe → $\mathcal{B}(N,M)$ algebra → residual symmetry:

$$gl(M) \oplus gl(N-M)$$

Classification of the K-matrices

compatibility of the KT with RTT → Reflection equation (bYBe)

$$R K \underset{\bar{R}}{R} K = K \underset{\bar{R}}{R} K R$$

↓

K-matrices are representations of reflection algebras

non-crossed bYBe ⟶ $\mathcal{B}(N,M)$ algebra ⟶ residual symmetry: $gl(M) \oplus gl(N-M)$

Crossed bYBe ↗ $Y^{+}(N)$ algebras

Crossed bYBe ↘ $Y^{-}(N)$ algebras

Classification of the K-matrices

compatibility of the KT with RTT → Reflection equation (bYBe)

$$R K \underset{\bar{R}}{R} K = K \underset{\bar{R}}{R} K R$$

↓

K-matrices are representations of reflection algebras

non-crossed bYBe → $\mathcal{B}(N,M)$ algebra → residual symmetry: $gl(M) \oplus gl(N-M)$

Crossed bYBe ↗ $Y^+(N)$ ↘ $Y^-(N)$ algebras → residual symmetries ↗ $o(N)$ ↘ $sp(N)$

Examples for $Y^+(N)$

Examples for $Y^+(N)$

algebra homomorphisms between

$$\begin{tikzcd} & Y^+(N) & \\ \mathfrak{o}_N \arrow[ur, leftrightarrow] & & Y(N) \arrow[ul] \\ & \mathfrak{gl}_N \arrow[ul] \arrow[ur, leftrightarrow] & \end{tikzcd}$$

Examples for $Y^+(N)$

algebra homomorphisms between

$$\begin{tikzcd}
 & Y^+(N) \arrow[dl, leftrightarrow] & \\
O_N & & Y(N) \arrow[ul] \arrow[dl, leftrightarrow] \\
 & \mathfrak{gl}_N \arrow[ul] &
\end{tikzcd}$$

evaluation homomorphism

Examples for $Y^+(N)$

algebra homomorphisms between

$$\begin{tikzcd}
 & Y^+(N) \arrow[dl, leftrightarrow] & \\
O_N & & Y(N) \arrow[ul] \\
 & \mathfrak{gl}_N \arrow[ul] \arrow[ur, leftrightarrow] &
\end{tikzcd}$$

evaluation homomorphism $\mathfrak{gl}_N \to Y(N)$ $T_{i,j}(u) = \delta_{i,j} + \frac{1}{u-\theta} E_{j,i}$

Examples for $Y^+(N)$

algebra homomorphisms between

$$\begin{tikzcd}
 & Y^+(N) \arrow[dl, leftrightarrow] & \\
\mathfrak{o}_N & & Y(N) \arrow[ul] \\
 & \mathfrak{gl}_N \arrow[ul] \arrow[ur, leftrightarrow] &
\end{tikzcd}$$

evaluation homomorphism

$\mathfrak{gl}_N \to Y(N) \quad T_{i,j}(u) = \delta_{i,j} + \frac{1}{u-\theta} E_{j,i}$

$\mathfrak{o}_N \to Y^+(N) \quad K_{i,j}(u) = \delta_{i,j} + \frac{1}{u+1/2} F_{i,j}$

Examples for $Y^+(N)$

algebra homomorphisms between

$$\begin{tikzcd}
& Y^+(N) & \\
O_N \arrow[ur, leftrightarrow] & & Y(N) \arrow[ul] \\
& \mathfrak{gl}_N \arrow[ul] \arrow[ur, leftrightarrow] &
\end{tikzcd}$$

evaluation homomorphism

$\mathfrak{gl}_N \to Y(N)$ $\quad T_{i,j}(u) = \delta_{i,j} + \frac{1}{u-\theta} E_{j,i}$ $\quad L(\lambda|\theta)$

$O_N \to Y^+(N)$ $\quad K_{i,j}(u) = \delta_{i,j} + \frac{1}{u+1/2} F_{i,j}$ $\quad V(\mu)$

Examples for $Y^+(N)$

algebra homomorphisms between

$$\begin{tikzcd} & Y^+(N) \arrow[dl, leftrightarrow] & \\ O_N & & Y(N) \arrow[ul] \\ & \mathfrak{gl}_N \arrow[ul] \arrow[ur, leftrightarrow] & \end{tikzcd}$$

evaluation homomorphism

$\mathfrak{gl}_N \to Y(N)$ $\quad T_{i,j}(u) = \delta_{i,j} + \frac{1}{u-\theta} E_{j,i}$ $\quad L(\lambda|\theta)$

$O_N \to Y^+(N)$ $\quad K_{i,j}(u) = \delta_{i,j} + \frac{1}{u+1/2} F_{i,j}$ $\quad V(\mu)$

Coproduct

Examples for $Y^+(N)$

algebra homomorphisms between

$$\begin{tikzcd}
 & Y^+(N) & \\
O_N \arrow[ur, leftrightarrow] & & Y(N) \arrow[ul] \\
 & \mathfrak{gl}_N \arrow[ul] \arrow[ur, leftrightarrow] &
\end{tikzcd}$$

evaluation homomorphism

$\mathfrak{gl}_N \to Y(N)$ $\quad T_{i,j}(u) = \delta_{i,j} + \frac{1}{u-\Theta} E_{j,i}$ $\quad L(\lambda|\Theta)$

$O_N \to Y^+(N)$ $\quad K_{i,j}(u) = \delta_{i,j} + \frac{1}{u+1/2} F_{i,j}$ $\quad V(\mu)$

Coproduct

$Y(N) \to Y(N) \otimes Y(N)$ $\quad T_{i,j}(u) \mapsto \sum_{k=1}^{N} T_{k,j}(u) \otimes T_{i,k}(u)$

Examples for $Y^+(N)$

algebra homomorphisms between

$$\begin{tikzcd}
& Y^+(N) & \\
O_N \arrow[ur, leftrightarrow] & & Y(N) \arrow[ul] \\
& \mathfrak{gl}_N \arrow[ul] \arrow[ur, leftrightarrow] &
\end{tikzcd}$$

evaluation homomorphism

$\mathfrak{gl}_N \rightarrow Y(N)$ $\quad T_{i,j}(u) = \delta_{i,j} + \frac{1}{u-\theta} E_{j,i}$ $\quad L(\lambda|\theta)$

$O_N \rightarrow Y^+(N)$ $\quad K_{i,j}(u) = \delta_{i,j} + \frac{1}{u+1/2} F_{i,j}$ $\quad V(\mu)$

Coproduct

$Y(N) \rightarrow Y(N) \otimes Y(N)$ $\quad T_{i,j}(u) \mapsto \sum_{k=1}^{N} T_{k,j}(u) \otimes T_{i,k}(u)$

$Y^+(N) \rightarrow Y(N) \otimes Y^+(N)$ $\quad K_{i,j}(u) \mapsto \sum_{k,l} T_{i,k}(u) T_{j,l}(-u) \otimes K_{k,l}(u)$

Examples for $Y^{+}(N)$

algebra homomorphisms between

$$\begin{tikzcd}
 & Y^{+}(N) & \\
O_N \arrow[ur, leftrightarrow] & & Y(N) \arrow[ul] \\
 & \mathfrak{gl}_N \arrow[ul] \arrow[ur, leftrightarrow] &
\end{tikzcd}$$

evaluation homomorphism

$\mathfrak{gl}_N \to Y(N)$ $\quad T_{i,j}(u) = \delta_{i,j} + \frac{1}{u-\theta} E_{j,i}$ $\quad L(\lambda|\theta)$

$O_N \to Y^{+}(N)$ $\quad K_{i,j}(u) = \delta_{i,j} + \frac{1}{u+1/2} F_{i,j}$ $\quad V(\mu)$

Coproduct

$Y(N) \to Y(N) \otimes Y(N)$ $\quad T_{i,j}(u) \mapsto \sum_{k=1}^{N} T_{k,j}(u) \otimes T_{i,k}(u)$ $\quad L_1 \otimes L_2$

$Y^{+}(N) \to Y(N) \otimes Y^{+}(N)$ $\quad K_{i,j}(u) \mapsto \sum_{k,l} T_{i,k}(u) T_{j,l}(-u) \otimes K_{k,l}(u)$ $\quad L \otimes V$

Examples for $Y^+(N)$

algebra homomorphisms between

$$\begin{tikzcd} & Y^+(N) & \\ O_N \arrow[ur, leftrightarrow] & & Y(N) \arrow[ul] \\ & \mathfrak{gl}_N \arrow[ul] \arrow[ur, leftrightarrow] & \end{tikzcd}$$

evaluation homomorphism

$\mathfrak{gl}_N \to Y(N)$ $\quad T_{i,j}(u) = \delta_{i,j} + \frac{1}{u-\theta} E_{j,i}$ $\quad L(\lambda|\theta)$

$O_N \to Y^+(N)$ $\quad K_{i,j}(u) = \delta_{i,j} + \frac{1}{u+1/2} F_{i,j}$ $\quad V(\mu)$

Coproduct

$Y(N) \to Y(N) \otimes Y(N)$ $\quad T_{i,j}(u) \mapsto \sum_{k=1}^{N} T_{k,j}(u) \otimes T_{i,k}(u)$ $\quad L_1 \otimes L_2$

$Y^+(N) \to Y(N) \otimes Y^+(N)$ $\quad K_{i,j}(u) \mapsto \sum_{k,l} T_{i,k}(u) T_{j,l}(-u) \otimes K_{k,l}(u)$ $\quad L \otimes V$

$\mathfrak{gl}_N$ irreps λ^j and O_N irrep μ $\longmapsto$

$Y^+(N)$ rep. $L(\lambda^1|\zeta_1) \otimes \ldots \otimes L(\lambda^k|\zeta_k) \otimes V(\mu)$

Examples for $Y^+(N)$

algebra homomorphisms between

$$\begin{tikzcd} & Y^+(N) \arrow[dl, leftrightarrow] & \\ O_N & & Y(N) \arrow[ul] \\ & \mathfrak{gl}_N \arrow[ul] \arrow[ur, leftrightarrow] & \end{tikzcd}$$

evaluation homomorphism

$\mathfrak{gl}_N \to Y(N)$ $\quad T_{i,j}(u) = \delta_{i,j} + \frac{1}{u-\theta} E_{j,i}$ $\quad L(\lambda|\theta)$

$O_N \to Y^+(N)$ $\quad K_{i,j}(u) = \delta_{i,j} + \frac{1}{u+1/2} F_{i,j}$ $\quad V(\mu)$

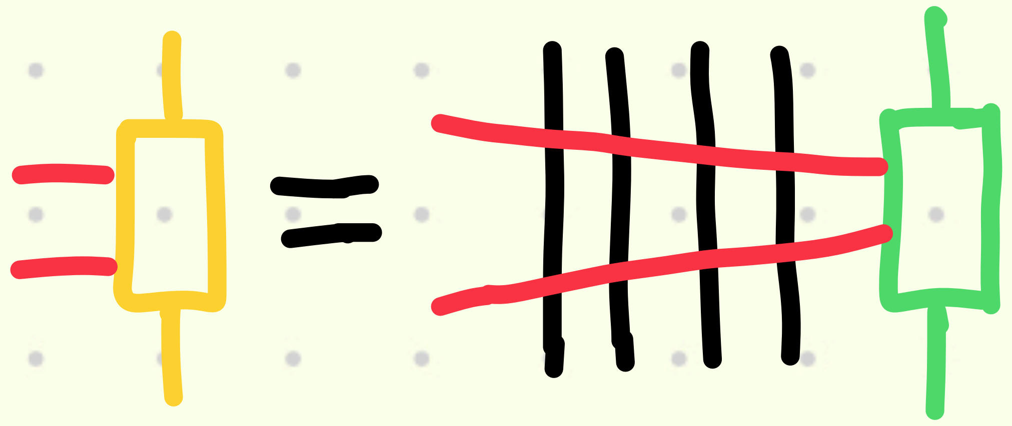
Coproduct

$Y(N) \to Y(N) \otimes Y(N)$ $\quad T_{i,j}(u) \mapsto \sum_{k=1}^{N} T_{k,j}(u) \otimes T_{i,k}(u)$ $\quad L_1 \otimes L_2$

$Y^+(N) \to Y(N) \otimes Y^+(N)$ $\quad K_{i,j}(u) \mapsto \sum_{k,l} T_{i,k}(u) T_{j,l}(-u) \otimes K_{k,l}(u)$ $\quad L \otimes V$

$\mathfrak{gl}_N$ irreps λ^j and O_N irrep μ $\longmapsto$

$Y^+(N)$ rep. $L(\lambda^1|\varsigma_1) \otimes \ldots \otimes L(\lambda^k|\varsigma_k) \otimes V(\mu)$



Examples for $Y^+(N)$

algebra homomorphisms between

$$\begin{tikzcd} & Y^+(N) \arrow[dl, leftrightarrow] & \\ O_N & & Y(N) \arrow[ul] \\ & \mathfrak{gl}_N \arrow[ul] \arrow[ur, leftrightarrow] & \end{tikzcd}$$

evaluation homomorphism

$\mathfrak{gl}_N \to Y(N)$ $\quad T_{i,j}(u) = \delta_{i,j} + \frac{1}{u-\theta} E_{j,i}$ $\quad L(\lambda|\theta)$

$O_N \to Y^+(N)$ $\quad K_{i,j}(u) = \delta_{i,j} + \frac{1}{u+1/2} F_{i,j}$ $\quad V(\mu)$

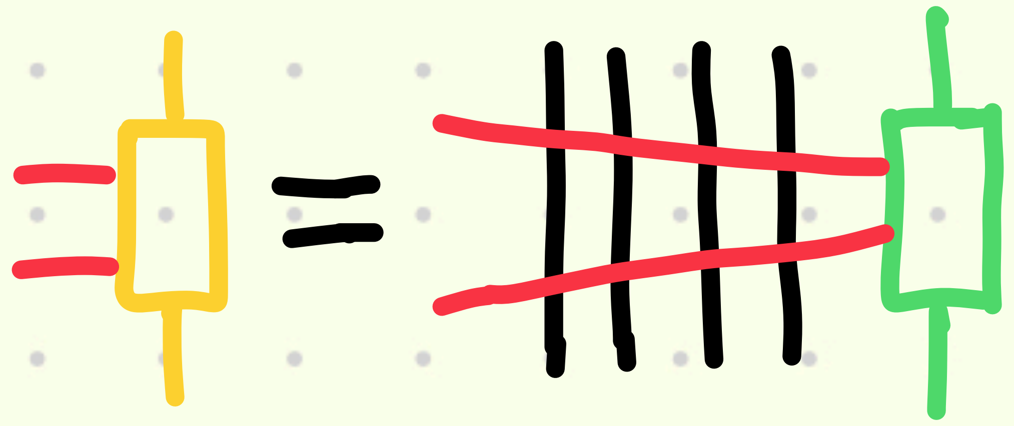
Coproduct

$Y(N) \to Y(N) \otimes Y(N)$ $\quad T_{i,j}(u) \mapsto \sum_{k=1}^{N} T_{k,j}(u) \otimes T_{i,k}(u)$ $\quad L_1 \otimes L_2$

$Y^+(N) \to Y(N) \otimes Y^+(N)$ $\quad K_{i,j}(u) \mapsto \sum_{k,l} T_{i,k}(u) T_{j,l}(-u) \otimes K_{k,l}(u)$ $\quad L \otimes V$

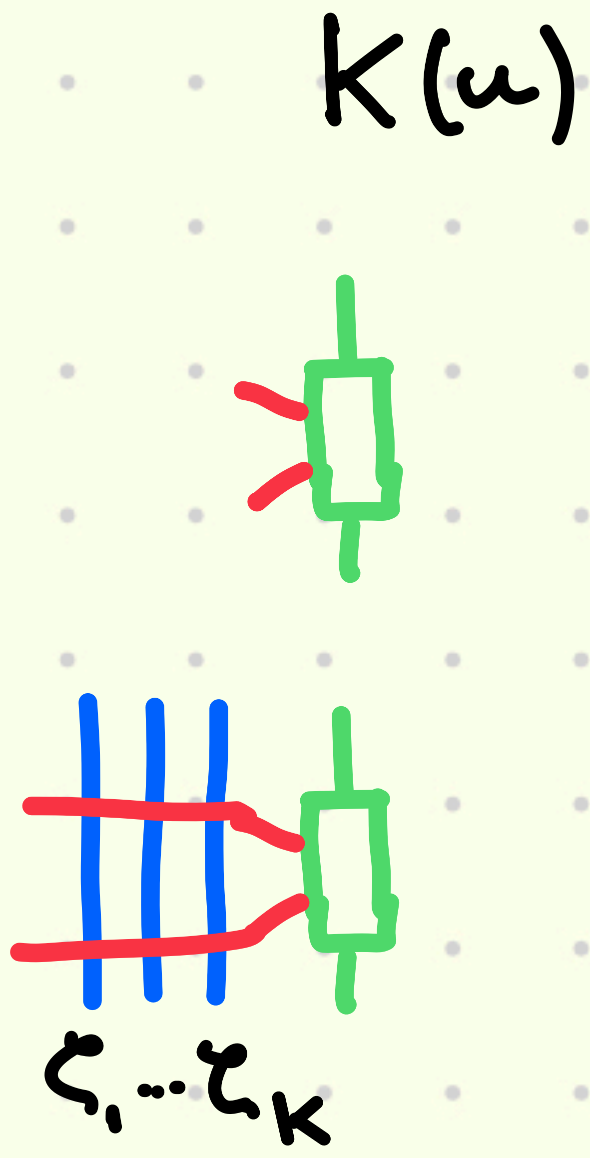
$\mathfrak{gl}_N$ irreps λ^j and O_N irrep μ $\quad\longmapsto$

$Y^+(N)$ rep. $\quad L(\lambda^1|\zeta_1) \otimes \ldots \otimes L(\lambda^k|\zeta_k) \otimes V(\mu)$

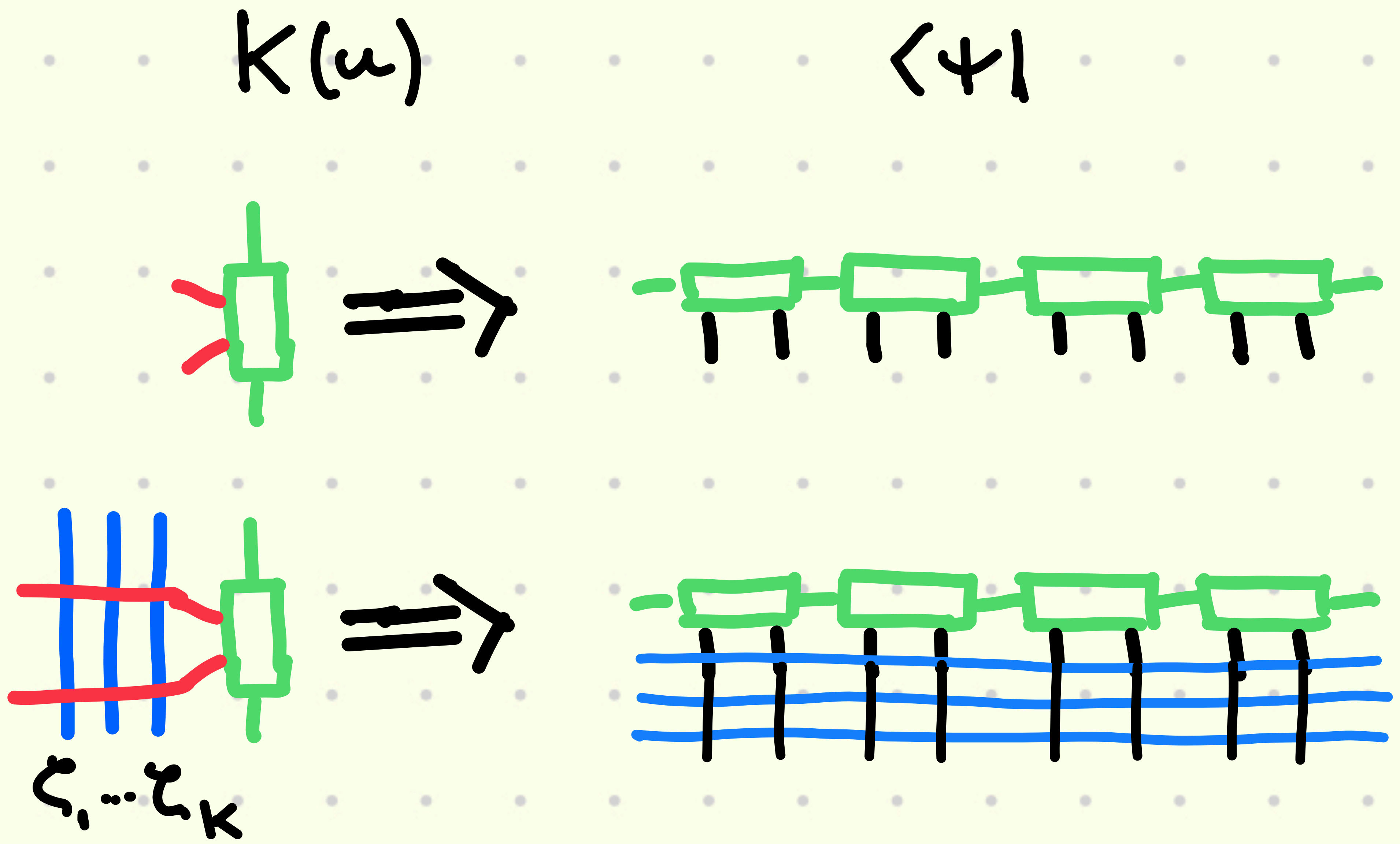


"dressing"

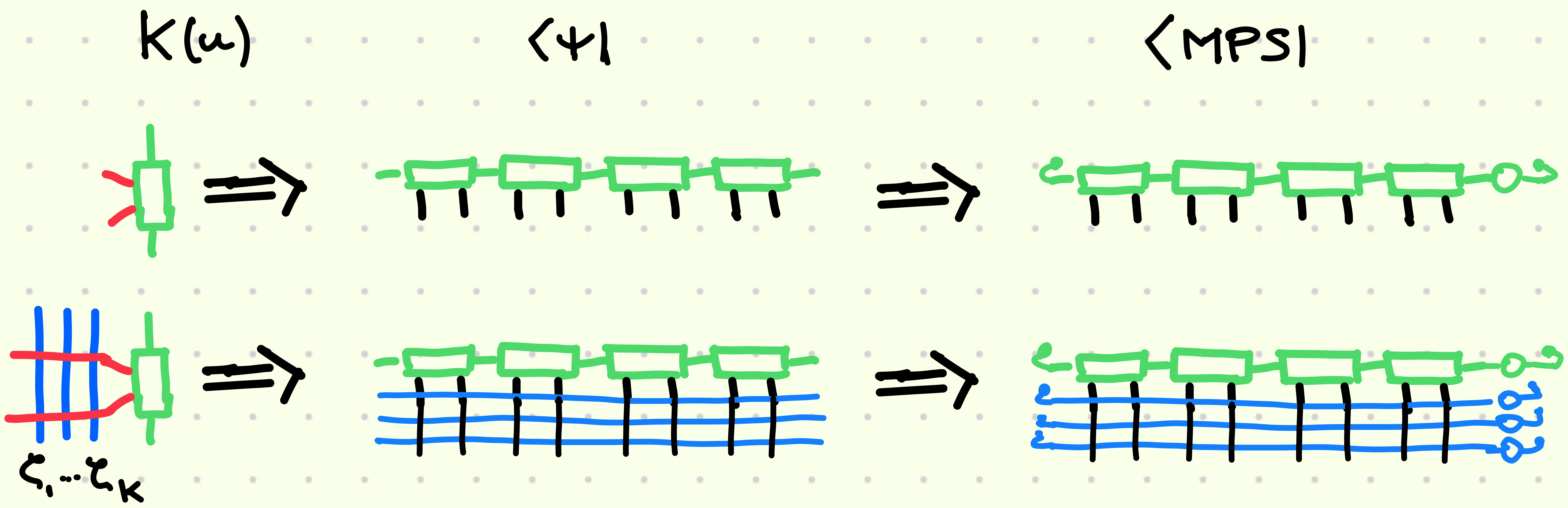
Dressed boundary states



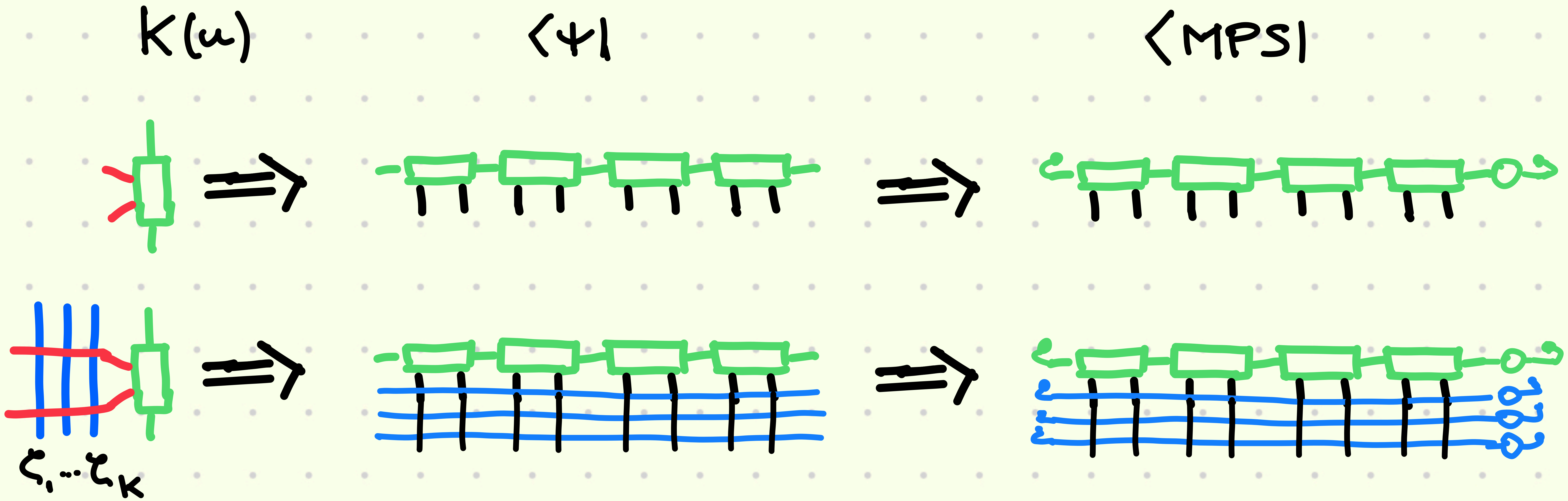
Dressed boundary states



Dressed boundary states

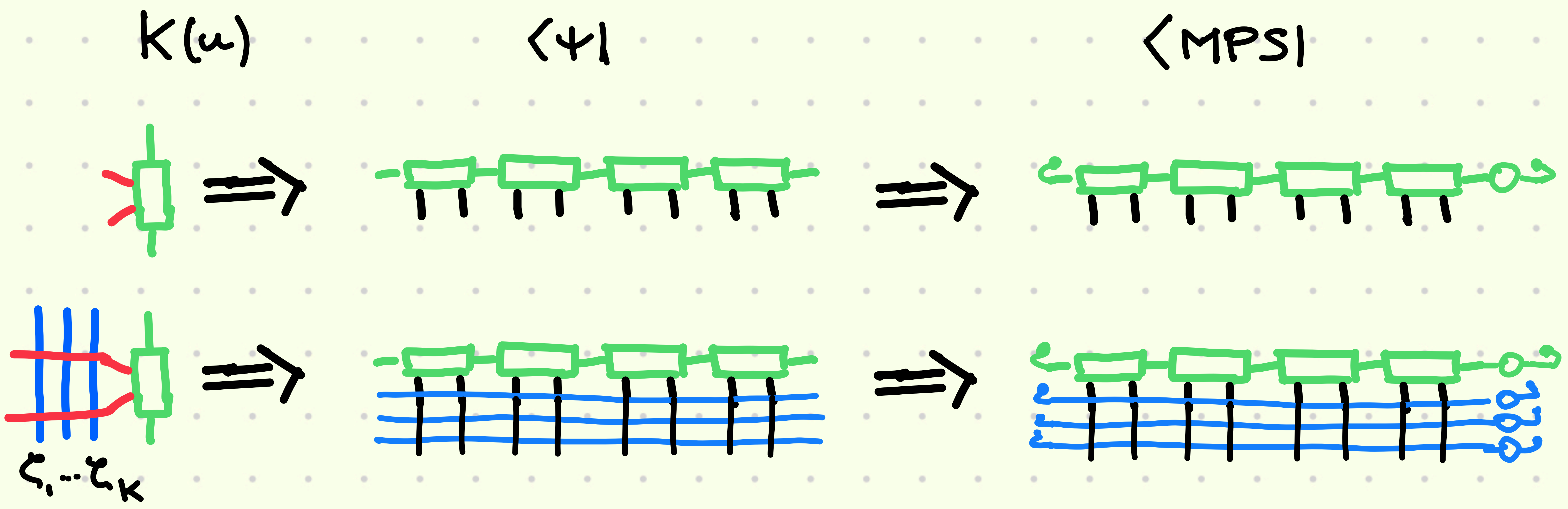


Dressed boundary states



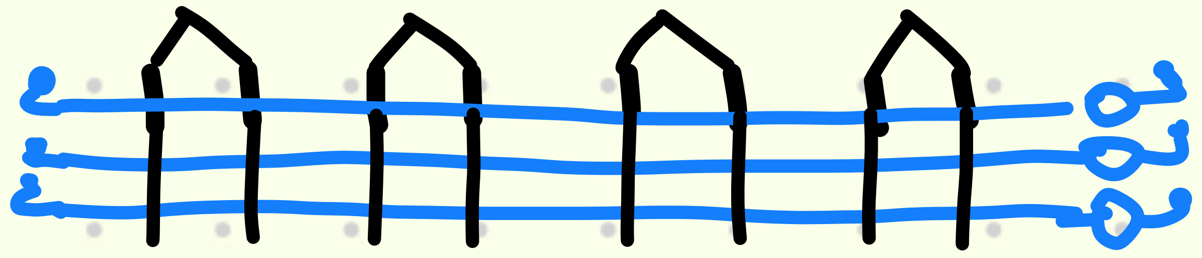
dressed MPS $\langle \mathrm{MPS}| T_{\lambda^1}(\zeta_1) \ldots T_{\lambda^K}(\zeta_K)$

Dressed boundary states



dressed MPs $\langle MPS|T_{\lambda^1}(\zeta_1)\dots T_{\lambda^K}(\zeta_K)$

$\Rightarrow$ the overlaps for the reps $L(\lambda^1|\zeta_1)\otimes\dots\otimes L(\lambda^K|\zeta_K) \longrightarrow \dfrac{\langle MPS|\bar{u}\rangle}{\langle 2\text{-site}|\bar{u}\rangle} = \tilde{\tau}_{\lambda^1}(\zeta_1|\bar{u})\dots\tilde{\tau}_{\lambda^K}(\zeta_K|\bar{u})$



Recursion for off-shell overlaps

for crossed KT

Recursion for off-shell overlaps

for crossed KT →

Recursion for off-shell overlaps

for crossed KT → assuming $K_{1,1}$ is invertible

Recursion for off-shell overlaps

for crossed KT → assuming $K_{1,1}$ is invertible → recursion for $\bar{u}^1$ roots

Recursion for off-shell overlaps

for crossed KT → assuming $K_{1,1}$ is invertible → recursion for $\bar{u}^1$ roots

$$\langle \psi | \bar{u}^1, \bar{w}^2, \ldots \rangle = \sum (\ldots) \langle \psi | \emptyset, \bar{w}^2, \ldots \rangle$$

Recursion for off-shell overlaps

for crossed KT → assuming $K_{1,1}$ is invertible → recursion for $\bar{u}^1$ roots

$\langle \psi | \bar{u}^1, \bar{w}^2, \dots \rangle = \sum (\dots) \langle \psi | \emptyset, \tilde{w}^2, \dots \rangle$ → $\mathfrak{gl}(N-1)$ overlaps

Recursion for off-shell overlaps

for crossed KT → assuming $K_{1,1}$ is invertible → recursion for $\bar{u}^1$ roots

$$\langle \Psi | \bar{u}^1, \bar{u}^2, \ldots \rangle = \sum (\ldots) \langle \Psi | \emptyset, \bar{u}^2, \ldots \rangle \longrightarrow gl(N-1) \text{ overlaps}$$

gl(N-1) KT- relation?

Recursion for off-shell overlaps

for crossed KT $\rightarrow$ assuming $K_{1,1}$ is invertible $\rightarrow$ recursion for $\bar{u}^1$ roots

$$\langle\Psi|\bar{u}^1,\bar{w}^2,\ldots\rangle=\sum(\ldots)\langle\Psi|\emptyset,\bar{w}^2,\ldots\rangle \longrightarrow \mathfrak{gl}(N-1) \text{ overlaps}$$

gl(N-1) KT- relation?

$$K=\left(\begin{array}{c|ccc} K_{11} & K_{12} & \ldots & K_{1N} \\ \hline K_{21} & K_{22} & \ldots & K_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ K_{N1} & K_{N2} & & K_{NN} \end{array}\right)$$

Recursion for off-shell overlaps

for crossed KT → assuming $K_{1,1}$ is invertible → recursion for $\bar{u}^1$ roots

$$\langle \psi | \bar{u}^1, \bar{w}^2, \dots \rangle = \sum (\dots) \langle \psi | \emptyset, \bar{w}^2, \dots \rangle \longrightarrow \mathfrak{gl}(N-1) \text{ overlaps}$$

gl(N-1) KT- relation?

$$K = \left(\begin{array}{c:cccc} K_{11} & K_{12} & \dots & K_{1N} \\ \hdashline K_{21} & K_{22} & \dots & K_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ K_{N1} & K_{N2} & & K_{NN} \end{array}\right) \qquad K^{(2)}_{a,b} = K_{a,b} - K_{a,1} K^{-1}_{1,1} K_{1,b}$$

Recursion for off-shell overlaps

for crossed KT → assuming $K_{1,1}$ is invertible → recursion for $\bar{u}^1$ roots

$$\langle \Psi | \bar{u}^1, \bar{w}^2, \dots \rangle = \sum (\dots) \langle \Psi | \emptyset, \bar{w}^2, \dots \rangle \longrightarrow \mathfrak{gl}(N-1) \text{ overlaps}$$

gl(N-1) KT- relation?

$$K=\left(\begin{array}{c:cccc} K_{11} & K_{12} & & \dots & K_{1N} \\ \hdashline K_{21} & K_{22} & & \dots & K_{2N} \\ \vdots & \vdots & & \ddots & \vdots \\ K_{N1} & K_{N2} & & & K_{NN} \end{array}\right)$$

$$K^{(2)}_{a,b} = K_{a,b} - K_{a,1} K^{-1}_{1,1} K_{1,b}$$

↳ gl(N-1) K-matrix

Recursion for off-shell overlaps

for crossed KT → assuming $K_{1,1}$ is invertible → recursion for $\bar{u}^1$ roots

$$\langle\psi|\bar{u}^1, \bar{w}^2, \dots\rangle = \sum(\dots)\langle\psi|\emptyset, \bar{w}^2, \dots\rangle \longrightarrow \mathfrak{gl}(N-1) \text{ overlaps}$$

gl(N-1) KT- relation?

$$K = \left(\begin{array}{c|cccc} K_{11} & K_{12} & \dots & K_{1N} \\ \hline K_{21} & K_{22} & \dots & K_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ K_{N1} & K_{N2} & & K_{NN} \end{array}\right)$$

$$K^{(2)}_{a,b} = K_{a,b} - K_{a,1} K_{1,1}^{-1} K_{1,b}$$

↳ gl(N-1) K-matrix

$$[K_{1,1}(u), K^{(2)}_{a,b}(v)] = 0$$

Recursion for off-shell overlaps

for crossed KT → assuming $K_{1,1}$ is invertible → recursion for $\bar{u}^1$ roots

$$\langle \psi | \bar{u}^1, \bar{w}^2, \dots \rangle = \sum (\dots) \langle \psi | \emptyset, \bar{w}^2, \dots \rangle \longrightarrow \mathfrak{gl}(N-1) \text{ overlaps}$$

gl(N-1) KT- relation?

$$K = \left(\begin{array}{c|cccc} K_{11} & K_{12} & \dots & K_{1N} \\ \hline K_{21} & K_{22} & \dots & K_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ K_{N1} & K_{N2} & & K_{NN} \end{array} \right)$$

$$K^{(2)}_{a,b} = K_{a,b} - K_{a,1} K^{-1}_{1,1} K_{1,b}$$

↳ gl(N-1) K-matrix

$$[K_{1,1}(u), K^{(2)}_{a,b}(v)] = 0$$

nesting:

Recursion for off-shell overlaps

for crossed KT → assuming $K_{1,1}$ is invertible → recursion for $\bar{u}^1$ roots

$$\langle\Psi|\bar{u}^1,\bar{w}^2,\dots\rangle=\sum(\dots)\langle\Psi|\emptyset,\tilde{w}^2,\dots\rangle \longrightarrow \mathfrak{gl}(N-1) \text{ overlaps}$$

gl(N-1) KT- relation?

$$K=\left(\begin{array}{c|ccc} K_{11} & K_{12} & \dots & K_{1N} \\ \hline K_{21} & K_{22} & \dots & K_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ K_{N1} & K_{N2} & & K_{NN} \end{array}\right)$$

$$K^{(2)}_{a,b}=K_{a,b}-K_{a,1}K^{-1}_{1,1}K_{1,b}$$

↳ gl(N-1) K-matrix

$$[K_{1,1}(u),K^{(2)}_{a,b}(v)]=0$$

nesting: $\mathfrak{gl}(N)\longrightarrow\mathfrak{gl}(N-1)\rightarrow\dots\rightarrow\mathfrak{gl}(1)$

$$K^{(1)}\equiv K \rightarrow K^{(2)}\rightarrow\dots\rightarrow K^{(N)}$$

Recursion for off-shell overlaps

for crossed KT → assuming $K_{1,1}$ is invertible → recursion for $\bar{u}^1$ roots

$$\langle \Psi | \bar{u}^1, \bar{w}^2, \ldots \rangle = \sum (\ldots) \langle \Psi | \emptyset, \bar{w}^2, \ldots \rangle \longrightarrow gl(N-1) \text{ overlaps}$$

gl(N-1) KT- relation?

$$K = \left(\begin{array}{c|cccc} K_{11} & K_{12} & \ldots & K_{1N} \\ \hline K_{21} & K_{22} & \ldots & K_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ K_{N1} & K_{N2} & & K_{NN} \end{array} \right)$$

$$K^{(2)}_{a,b} = K_{a,b} - K_{a,1} K^{-1}_{1,1} K_{1,b}$$

$\hookrightarrow$ gl(N-1) K-matrix

$$[K_{1,1}(u), K^{(2)}_{a,b}(v)] = 0$$

nesting: $gl(N) \longrightarrow gl(N-1) \rightarrow \ldots \rightarrow gl(1)$

$$K^{(1)} \equiv K \rightarrow K^{(2)} \rightarrow \ldots \rightarrow K^{(N)}$$

$$G^{(k)} \equiv K^{(k)}_{k,k}$$

Recursion for off-shell overlaps

for crossed KT → assuming $K_{1,1}$ is invertible → recursion for $\bar{u}^1$ roots

$$\langle \psi | \bar{u}^1, \bar{w}^2, \ldots \rangle = \sum (\ldots) \langle \psi | \emptyset, \bar{w}^2, \ldots \rangle \longrightarrow \mathfrak{gl}(N-1) \text{ overlaps}$$

gl(N-1) KT- relation?

$$K = \left(\begin{array}{c|ccc} K_{11} & K_{12} & \cdots & K_{1N} \\ \hline K_{21} & K_{22} & \cdots & K_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ K_{N1} & K_{N2} & & K_{NN} \end{array}\right)$$

$$K^{(2)}_{a,b} = K_{a,b} - K_{a,1} K^{-1}_{1,1} K_{1,b}$$

↳ $\mathfrak{gl}(N-1)$ K-matrix

$$[K_{1,1}(u), K^{(2)}_{a,b}(v)] = 0$$

nesting: $\mathfrak{gl}(N) \longrightarrow \mathfrak{gl}(N-1) \to \ldots \to \mathfrak{gl}(1)$

$$K^{(1)} \equiv K \to K^{(2)} \to \ldots \to K^{(N)}$$

$$G^{(k)} \equiv K^{(k)}_{k,k} \Rightarrow [G^{(k)}(u), G^{(l)}(v)] = 0$$

Other recursions

$K_{1,1}^{-1}$ exists for $Y^+(N)$

Other recursions

$K_{1,1}^{-1}$ exists for $Y^{+}(N)$

$Y^{-}(N) \longrightarrow$ other recursion

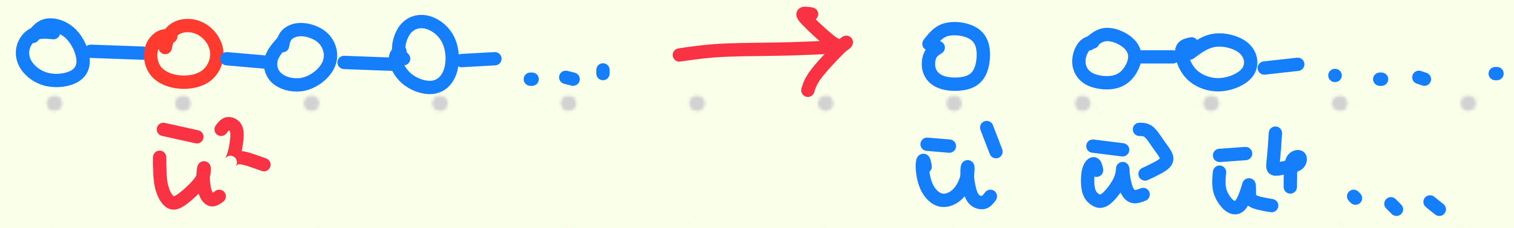
$\bar{w}^{2} -$

Other recursions

$K_{1,1}^{-1}$ exists for $Y^+(N)$

$Y^-(N) \longrightarrow$ other recursion

$\bar{u}^2$



Other recursions

$K_{1,1}^{-1}$ exists for $Y^+(N)$

$Y^-(N) \longrightarrow$ other recursion

$\bar{u}^2$

it also gives operators $G^{(k)}$ for $k=1,\dots,N$

Other recursions

$K_{1,1}^{-1}$ exists for $Y^+(N)$

$Y^-(N) \longrightarrow$ other recursion

$\bar{u}^2$

it also gives operators $G^{(k)}$ for $k=1,\dots,N$ $\quad [G^{(k)}(u), G^{(\ell)}(v)] = 0$

Other recursions

$K_{1,1}^{-1}$ exists for $Y^{+}(N)$

$Y^{-}(N) \longrightarrow$ other recursion

$\bar{u}^{2}$

it also gives operators $G^{(k)}$ for $k = 1, \dots, N$ $\quad [G^{(k)}(u), G^{(\ell)}(v)] = 0$

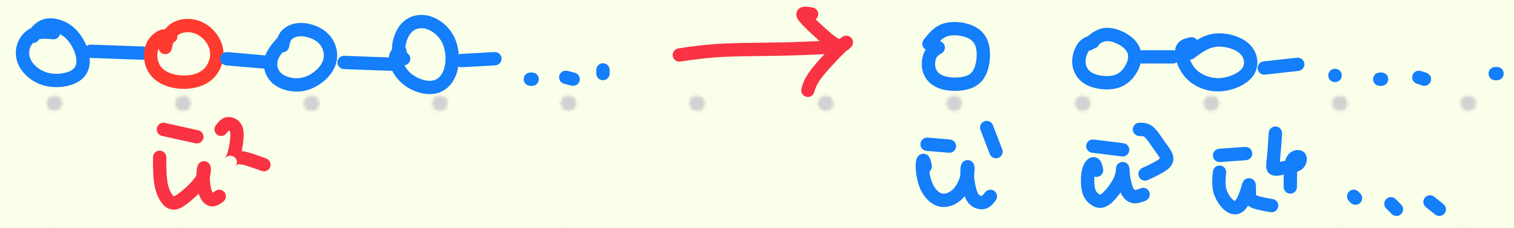
untwisted bYBe

Other recursions

$K_{1,1}^{-1}$ exists for $Y^+(N)$

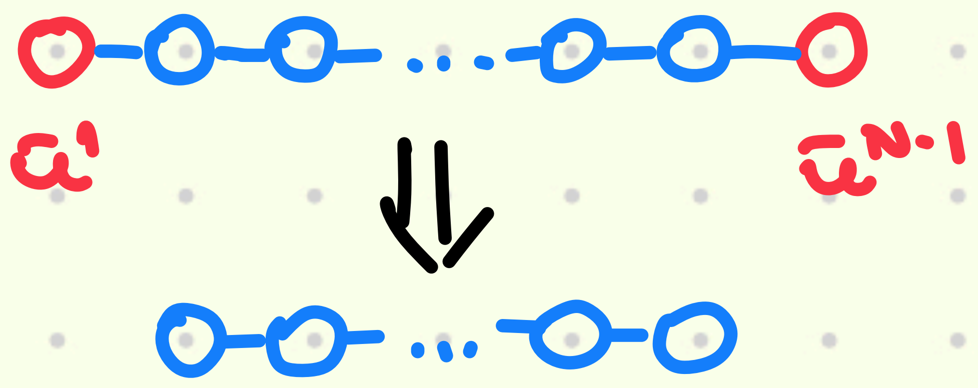
$Y^-(N) \longrightarrow$ other recursion

$\bar{u}^2$



it also gives operators $G^{(k)}$ for $k = 1, \dots, N$ $[G^{(k)}(u), G^{(\ell)}(v)] = 0$

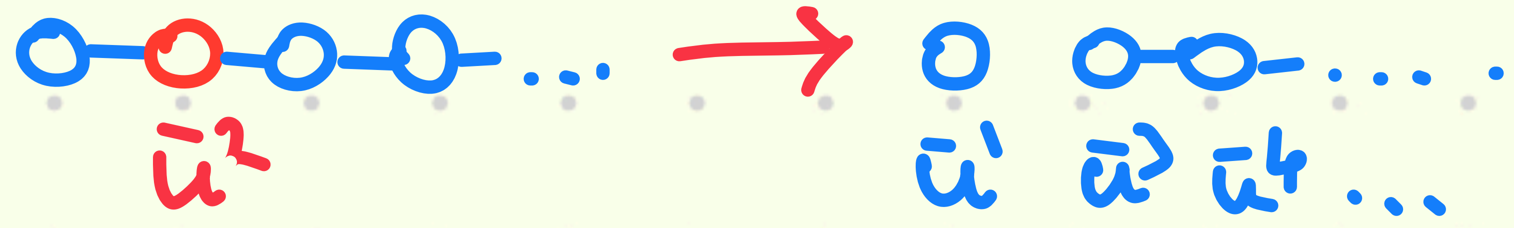
untwisted bYBe



Other recursions

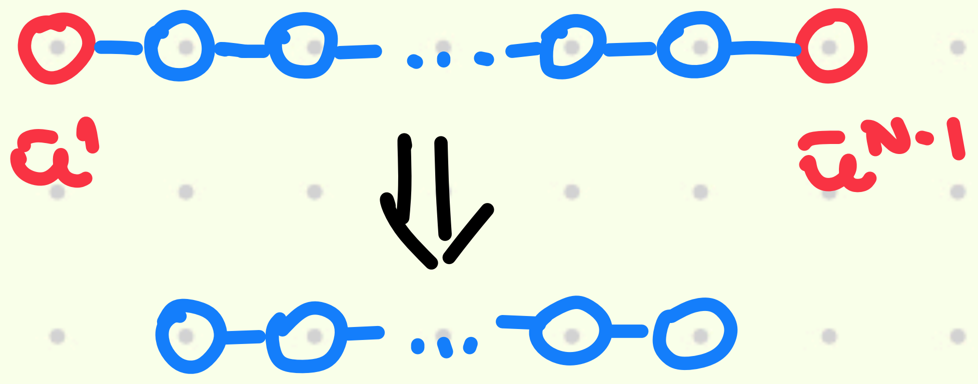
$K_{1,1}^{-1}$ exists for $Y^+(N)$

$Y^-(N) \longrightarrow$ other recursion $\bar{u}^2$



it also gives operators $G^{(k)}$ for $k=1,\dots,N$ $\quad [G^{(k)}(u), G^{(\ell)}(v)] = 0$

untwisted bYBe



$$\mathfrak{gl}(N) \to \mathfrak{gl}(N-2) \to \dots \to \begin{matrix}\mathfrak{gl}(2)\\ \mathfrak{gl}(3)\end{matrix} \to \mathfrak{gl}(1)$$

$K^{(1)} \equiv K \qquad K^{(2)} \qquad K^{(n)} \qquad K^{(n+1)}$

$G^{(1)} \qquad G^{(2)} \qquad G^{(n)} \qquad G^{(n+1)}$

$n = \lfloor N/2 \rfloor$

On-shell overlaps without twist

$$\mathcal{F}^{(k)} = \left[G^{(k)}\right]^{-1} G^{(k+1} \longrightarrow \left[\mathcal{F}^{(k)}(u), \mathcal{F}^{(\ell)}(v)\right] = 0$$

On-shell overlaps without twist

$$F^{(k)} = \left[G^{(k)}\right]^{-1} G^{(k+1)} \longrightarrow \left[F^{(k)}(u), F^{(\ell)}(v)\right] = 0$$

choosing diagonal basis $F^{(k)} = \mathrm{diag}\left(F_1^{(k)}, \ldots, F_{d_B}^{(k)}\right)$

On-shell overlaps without twist

$$F^{(k)} = \left[G^{(k)}\right]^{-1} G^{(k+1)} \longrightarrow \left[F^{(k)}(u), F^{(\ell)}(v)\right] = 0$$

choosing diagonal basis $F^{(k)} = \mathrm{diag}\left(F_1^{(k)}, \dots, F_{d_B}^{(k)}\right)$

$$\frac{\langle \mathrm{MPS}|\bar{u}\rangle}{\sqrt{\langle\bar{u}|\bar{u}\rangle}} = \left[\sum_{\alpha=1}^{d_B} \prod_s \tilde{F}_\alpha^{(s)}(\bar{u}^{+s})\right] \times \sqrt{\frac{\det G^+}{\det G^-}}$$

On-shell overlaps without twist

$$\mathcal{F}^{(k)} = \left[\mathcal{G}^{(k)}\right]^{-1} \mathcal{G}^{(k+1)} \longrightarrow \left[\mathcal{F}^{(k)}(u), \mathcal{F}^{(\ell)}(v)\right] = 0$$

choosing diagonal basis $\mathcal{F}^{(k)} = \mathrm{diag}\left(\mathcal{F}_1^{(k)}, \ldots, \mathcal{F}_{d_B}^{(k)}\right)$

$$\frac{\langle \mathrm{MPS} | \bar{u} \rangle}{\sqrt{\langle \bar{u} | \bar{u} \rangle}} = \left[\sum_{\alpha=1}^{d_B} \prod_s \tilde{\mathcal{F}}_\alpha^{(s)}(\bar{u}^{+,s})\right] \times \sqrt{\frac{\det G^+}{\det G^-}}$$

Crossed

$$\tilde{\mathcal{F}}_\alpha^{(s)}(u) = \mathcal{F}_\alpha^{(s)}\left(u - i\frac{s}{2}\right)\sqrt{\frac{u^2}{u^2+1/4}}$$

Non-crossed

$$\tilde{\mathcal{F}}_\alpha^{(s)}(u) = \begin{cases} \mathcal{F}_\alpha^{(s)}(u - c_s) \\ \mathcal{F}_\alpha^{(\frac{N-1}{2})}\left(u - c_{\frac{N-1}{2}}\right)\sqrt{\frac{u^2}{u^2+1/4}} \end{cases}$$

Other spin chains

The on-shell overlaps are extended to other rational spin chains without proofs

Symmetry of the spin chain	Type of refl.	Residual symmetry	Pair structure
$gl(N)$	AI	$o(N)$	Chiral
	AII	$sp(N)$	Chiral
	AIII	$gl(M)+gl(N-M)$	Achiral
$o(2n+1)$	BI	$o(M)+o(N-M)$	Chiral
$sp(2n)$	CI	$sp(2m)+sp(2n-2m)$	Chiral
	CII	$gl(n)$	Chiral
$o(2n)$	DI	$o(M)+O(2n-M)$	$n-M=0$ (mod 2) chiral
			$n-M=1$ (mod 2) achrial
	DII	$gl(n)$	$n=0$ (mod 2) chiral
			$n=1$ (mod 2) achrial

Other spin chains

The on-shell overlaps are extended to other rational spin chains without proofs

Symmetry of the spin chain	Type of refl.	Residual symmetry	Pair structure
$gl(N)$	AI	$o(N)$	Chiral
	AII	$sp(N)$	Chiral
	AIII	$gl(M)+gl(N-M)$	Achiral
$o(2n+1)$	BI	$o(M)+o(N-M)$	Chiral
$sp(2n)$	CI	$sp(2m)+sp(2n-2m)$	Chiral
	CII	$gl(n)$	Chiral
$o(2n)$	DI	$o(M)+O(2n-M)$	$n-M=0$ (mod 2) chiral
			$n-M=1$ (mod 2) achrial
	DII	$gl(n)$	$n=0$ (mod 2) chiral
			$n=1$ (mod 2) achrial

The overlaps are also conjectured for graded spin chains,
including $gl(m|n)$ and $osp(m|2n)$

Conclusions

MPS ⟷ K-matrix ⟷ Representation of a given reflection algebra

Conclusions

MPS ⟷ K-matrix ⟷ Representation of a given reflection algebra

Cartan subalgebra $\mathcal{F}^{(s)}(u)$

Conclusions

MPS ⟷ K-matrix ⟷ Representation of a given reflection algebra

Cartan subalgebra $\mathcal{F}^{(s)}(u)$

On-shell overlaps for integrable MPS

$$\frac{\langle \mathrm{MPS} | \bar{u} \rangle}{\sqrt{\langle \bar{u} | \bar{u} \rangle}} = \left[\sum_{\alpha=1}^{d_B} \prod_s \tilde{\mathcal{F}}_\alpha^{(s)}(\bar{u}^{+,s}) \right] \times \sqrt{\frac{\det G^+}{\det G^-}}$$

Conclusions

MPS ⟷ K-matrix ⟷ Representation of a given reflection algebra

Cartan subalgebra $\mathcal{F}^{(s)}(u)$

On-shell overlaps for integrable MPS
$$\frac{\langle \mathrm{MPS} | \bar{u} \rangle}{\sqrt{\langle \bar{u} | \bar{u} \rangle}} = \left[\sum_{\alpha=1}^{d_B} \prod_{s} \tilde{\mathcal{F}}_{\alpha}^{(s)}(\bar{u}^{+,s}) \right] \times \sqrt{\frac{\det G^{+}}{\det G^{-}}}$$

where $\tilde{\mathcal{F}}_{\alpha}^{(s)}(u)$ are the eigenvalues of the operators $\mathcal{F}^{(s)}(u)$

Conclusions

MPS $\longleftrightarrow$ K-matrix $\longleftrightarrow$ Representation of a given reflection algebra

Cartan subalgebra $F^{(s)}(u)$

On-shell overlaps for integrable MPS $$\frac{\langle \mathrm{MPS}|\bar{u}\rangle}{\sqrt{\langle\bar{u}|\bar{u}\rangle}} = \left[\sum_{\alpha=1}^{d_B}\prod_{s}\tilde{F}^{(s)}_{\alpha}(\bar{u}^{+,s})\right]\times\sqrt{\frac{\det G^{+}}{\det G^{-}}}$$

where $\tilde{F}^{(s)}_{\alpha}(u)$ are the eigenvalues of the operators $F^{(s)}(u)$

Proof is possible if 1) KT- relation: creation to annihilation

2) recurrence formula $|\{z,\bar{u}^1\},\bar{u}^2,\dots\rangle = \sum(..)T_{1,j}(z)|\bar{u}^1,\bar{u}^2,\dots\rangle$

3) action formula $T_{i,j}(z)|\bar{u}\rangle = \sum(\dots)|\bar{w}\rangle$

4) co-product formula $|\bar{u}\rangle = \sum(\dots)|\bar{u}_{I}\rangle^{(1)}\otimes|\bar{u}_{II}\rangle^{(2)}$