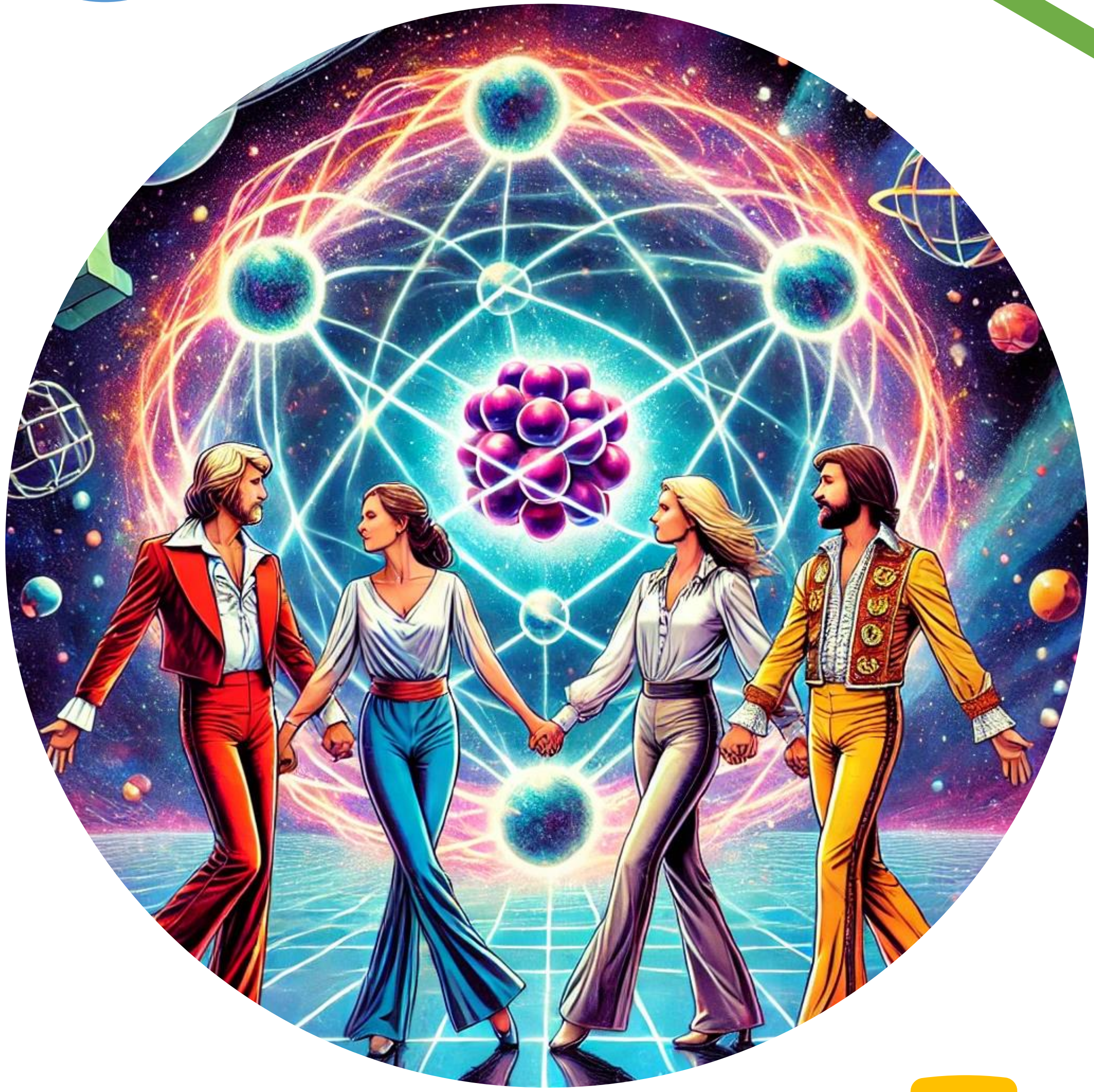


European Research Council

BFKL in N=4

Based on work with Ekhammar, Preti `24

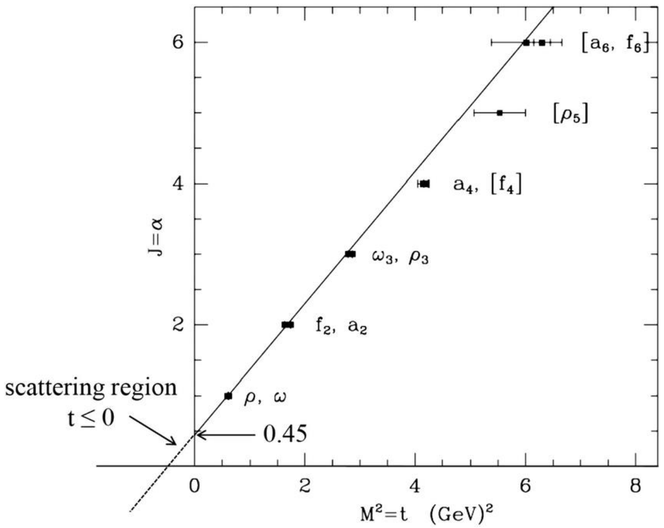
Integrability, Q-systems and Cluster Algebras

2024

Regge Trajectories

- Regge trajectories $J(m^2)$ key objects in the study of amplitudes.

- They link the spectrum of particles with the high energy asymptotic of amplitudes



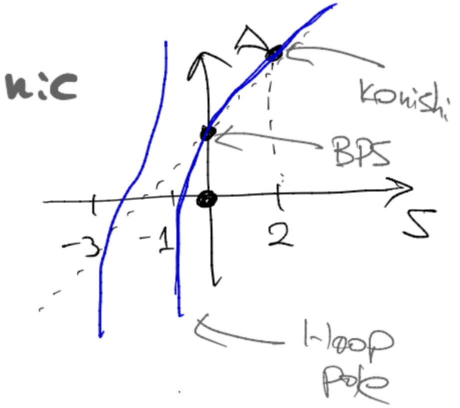
Regge Trajectories in $\mathcal{N}=4$ SYM

- $\mathcal{N}=4$ is a CFT $\quad m^2 \longrightarrow \Delta$

  $J \longrightarrow S$

- example! $\mathrm{tr}\, Z D_+^S Z = O_S \qquad \langle O_S \bar{O}_S \rangle = \frac{1}{x^{2\Delta(S)}}$
- tree level $\quad \Delta = S + 2$
- 1-loop $\quad \Delta = S + 2 + g^2 H_1(S)$

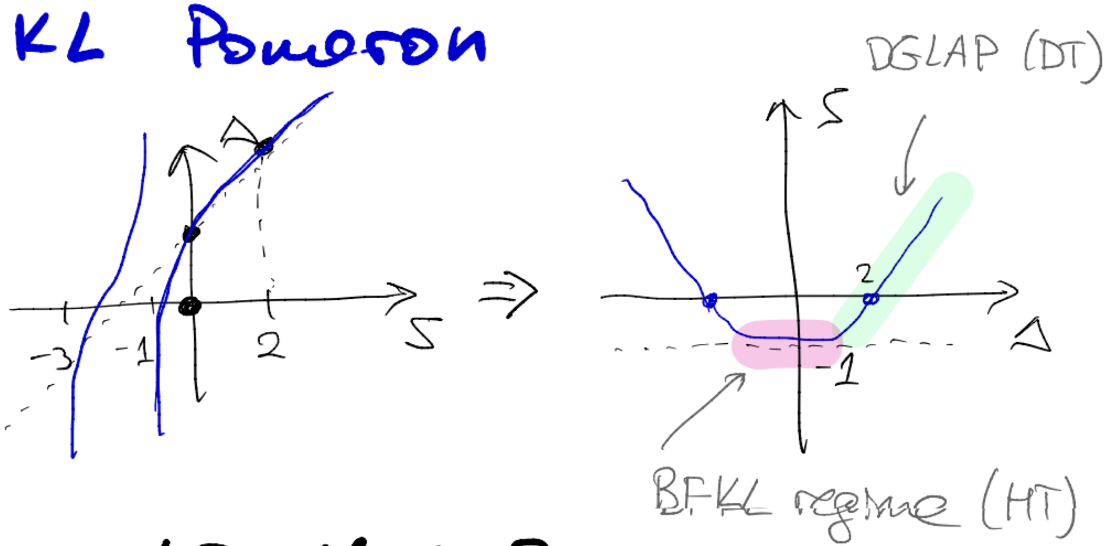
Harmonic sum



- Near the pole $\Delta = \# + g^2 \frac{\#}{\omega} + g^4 \frac{\#}{\omega^2} + \dots =$ BFKL Pomeron
- $\frac{1}{\omega} \sim \ln S$ leading logs resummation

BFKL Pomeron

- $\Delta(S) \to S(\Delta)$



- LO QCD Pomeron = LO $\mathcal{N}=4$ Pomeron

$$S(\Delta) = -1 + g^2\left(\psi\left(\frac{\Delta+1}{2}\right) + \psi\left(\frac{1-\Delta}{2}\right) + 2\gamma\right)$$

↑polygamma function $\sum_{n=0}^{\infty}\frac{1}{x+n}=\psi(x)$

- Only NLO g^4 known in QCD (8 years to comput

BFKL Pomeron

Jaroszewicz, 1982
Lipatov 1986
Kotikov,Lipatov 2002

- At the LO:

$$\chi(\gamma) = \boxed{2\Psi(1) - \Psi(\gamma) - \Psi(1-\gamma)}\,, \quad \Psi(\gamma) = \Gamma'(\gamma)/\Gamma(\gamma)$$

Fadin, Lipatov 1998
Kotikov,Lipatov 2002
Kotikov,Lipatov 2000

- At NLO:

$$\begin{aligned}\delta(\gamma) \;=\; & -\Big[\Big(\frac{11}{3} - \frac{2n_f}{3N_c}\Big)\frac{1}{2}\left(\chi^2(\gamma) - \Psi'(\gamma) + \Psi'(1-\gamma)\right) - \Big(\frac{67}{9} \boxed{-\frac{\pi^2}{3}} - \frac{10}{9}\frac{n_f}{N_c}\Big)\chi(\gamma \\ & \boxed{-6\zeta(3)} + \frac{\pi^2\cos(\pi\gamma)}{\sin^2(\pi\gamma)(1-2\gamma)}\Big(3 + \Big(1 + \frac{n_f}{N_c^3}\Big)\frac{2+3\gamma(1-\gamma)}{(3-2\gamma)(1+2\gamma)}\Big) \\ & \boxed{-\Psi''(\gamma) - \Psi''(1-\gamma)} - \frac{\pi^3}{\sin(\pi\gamma)} + \boxed{4\phi(\gamma)}\Big]\,.\end{aligned}$$

$$\begin{aligned}\phi(\gamma) \;&=\; -\int_0^1 \frac{dx}{1+x}\left(x^{\gamma-1} + x^{-\gamma}\right)\int_x^1 \frac{dt}{t}\ln(1-t) \\ &=\; \sum_{n=0}^{\infty}(-1)^n\left[\frac{\Psi(n+1+\gamma)-\Psi(1)}{(n+\gamma)^2} + \frac{\Psi(n+2-\gamma)-\Psi(1)}{(n+1-\gamma)^2}\right]\end{aligned}$$

N=4 SYM

BFKL at NNLO

$$S = -1 + \sum_{n=1}^{\infty} g^{2n} \left[F_n\left(\tfrac{\Delta-1}{2}\right) + F_n\left(\tfrac{-\Delta-1}{2}\right) \right]$$

NG,Levkovich-Maslyk,Sizov
Phys.Rev.Lett. 115 (2015)

$$\begin{aligned}
&\frac{1}{256} F_3 = \\
&-\frac{5S_{-5}}{8} - \frac{S_{-4,1}}{2} + \frac{S_1 S_{-3,1}}{2} + \frac{S_{-3,2}}{2} - \frac{5S_2 S_{-2,1}}{4} \\
&+\frac{S_{-4}S_1}{4} + \frac{S_{-3}S_2}{8} + \frac{3S_{3,-2}}{4} - \frac{3S_{-3,1,1}}{2} - S_1 S_{-2,1,1} \\
&+S_{2,-2,1} + 3S_{-2,1,1,1} - \frac{3S_{-2}S_3}{4} - \frac{S_5}{8} + \frac{S_{-2}S_1S_2}{4} \\
&+\pi^2 \left[\frac{S_{-2,1}}{8} - \frac{7S_{-3}}{48} - \frac{S_{-2}S_1}{12} + \frac{S_1 S_2}{48} \right] \\
&+\zeta_3 \left[-\frac{7S_{-1,1}}{4} + \frac{7S_{-2}}{8} + \frac{7S_{-1}S_1}{4} - \frac{S_2}{16} \right] \\
&+\left[2\mathrm{Li}_4 - \frac{\pi^2 \log^2 2}{12} + \frac{\log^4 2}{12} \right] (S_{-1} - S_1) - \pi^4 \left[\frac{2S_{-1}}{45} - \frac{S_1}{96} \right] \\
&+\frac{\log^5 2}{60} - \frac{\pi^2 \log^3 2}{36} - \frac{2\pi^4 \log 2}{45} - \frac{\pi^2 \zeta_3}{24} + \frac{49\zeta_5}{32} - 2\mathrm{Li}_5
\end{aligned}$$

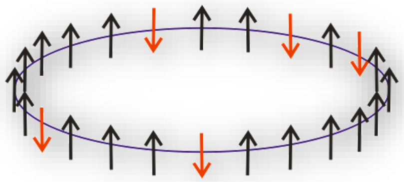
Found from integrability

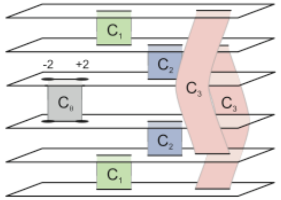
Confirmed by an independent calculation by Caron-Huot, Herran

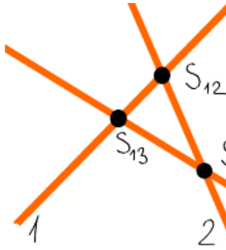
- This talk $\underbrace{N \ldots N}_{L/2 \text{ times}}LO$ for $\mathrm{tr}\, D_+^S Z^{\boxed{L}} \longleftarrow \text{R-charge}$

We of course need integrability to do this

Historical path for N=4 SYM







Origins of YM integrability: Lipatov's BFKL Hamiltonian

Perturbative integrability in N=4 SYM

Classical integrability of string σ-model on AdS5×S5, quasiclassics

S-matri
asympto
Bethe Ans
Cusp
dimensi

1993

2002

2003-2008

2004-20

Lipatov
Faddeev,Korchemsky

Minahan,Zarembo,
Beisert,Kristijanssen,Staudacher

Bena,Polchinski,Roiban
Kazakov,Marshakov, Minahan,
Zarembo, Frolov, Tseytlin
Schafer-Nameki
Beisert,Kazakov,Sakai,Zarembo
NG,Vieira

Arutyunov,Frolov
Staudacher,
Janik
Hernandez,
Roiban, Ts
Beisert,Eden,S

$\mathcal{N}=4$ Integrability / Strings in AdS_5

- Integrability => $\Psi(r,\theta,\varphi)=\Psi_\theta(\theta)\,\Psi_\varphi(\varphi)\,\Psi_r(r)$
- String theory $\Psi($  $)=\prod_{i=0}^{\infty} Q_{A_i}(u_i)$

  string configuration ↗ ↖ polarization

- 2^8 different Q-functions, 4+4 elementary

  $Q_i(u)$ $i=1\ldots4$ ⟷ $su(2,2)$ AdS polarizations

  $P_a(u)$ $a=1\ldots4$ ⟷ $su(4)$ S^5 R-symmetry

  $Q_1\sim u^{\Delta+S}$ $P_a\sim u^{\pm L}$ at $u\to\infty$

  $Q_2\sim u^{\Delta-S}$

  $\vdots$

- How do we find the Q-functions?

XXX warm up

$$(*)\ Q(u+i)(u-i/2)^L + T(u)\,Q(u) + Q(u-i)(u+i/2)^L = 0$$

- Has 2-solutions Q_1 Q_2
- Wronskian is $Q_1(u+i/2)\,Q_2(u-i/2) - \text{c.c.} = u^L$
- $(*) \Leftrightarrow$ $$0 = \begin{vmatrix} Q_1(u+i) & Q_1(u) & Q_1(u-i) \\ Q_1(u+i) & Q_1(u) & Q_1(u-i) \\ Q_2(u+i) & Q_2(u) & Q_2(u-i) \end{vmatrix} = Q_1(u+i)\ldots + Q_1(u)$$
- states 1 to 1 with a) Wronskian
  b) Q_a are polynomials
- su(N): Q_i $i=1\ldots N$ Baxter: $(N+1)\times(N+1)$ det
- Nested BA: zeroes of $Q_1, Q_{12}, Q_{123}\ldots$

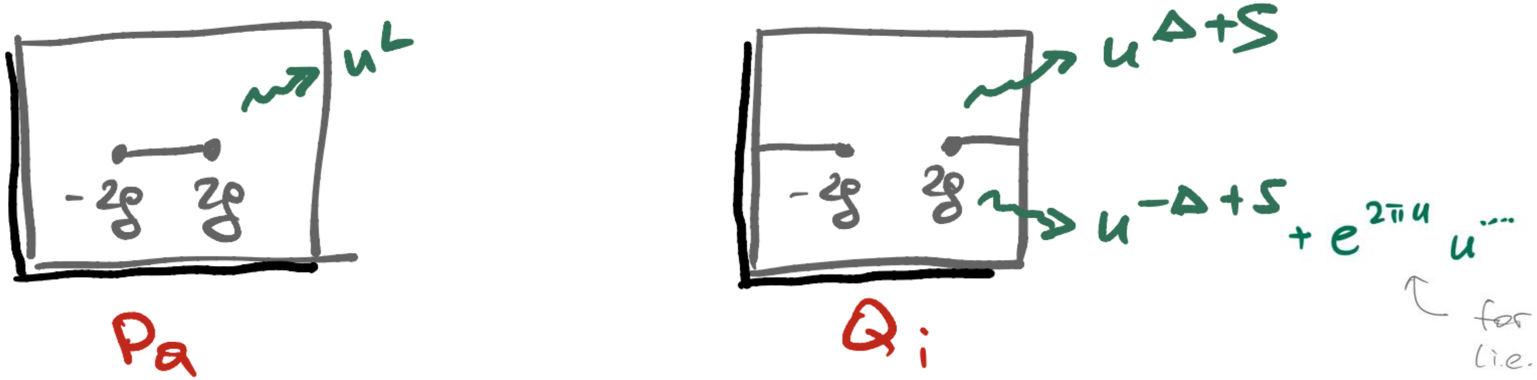
e.g. $Q_{12}(u+i/2)\,Q_{13}(u-i/2) - Q_{12}(u-i/2)\,Q_{13}(u+i/2) = Q_{123}$

Spectrum of int. models

a) QQ-wronskian relations

b) Analyticity

For N=4:



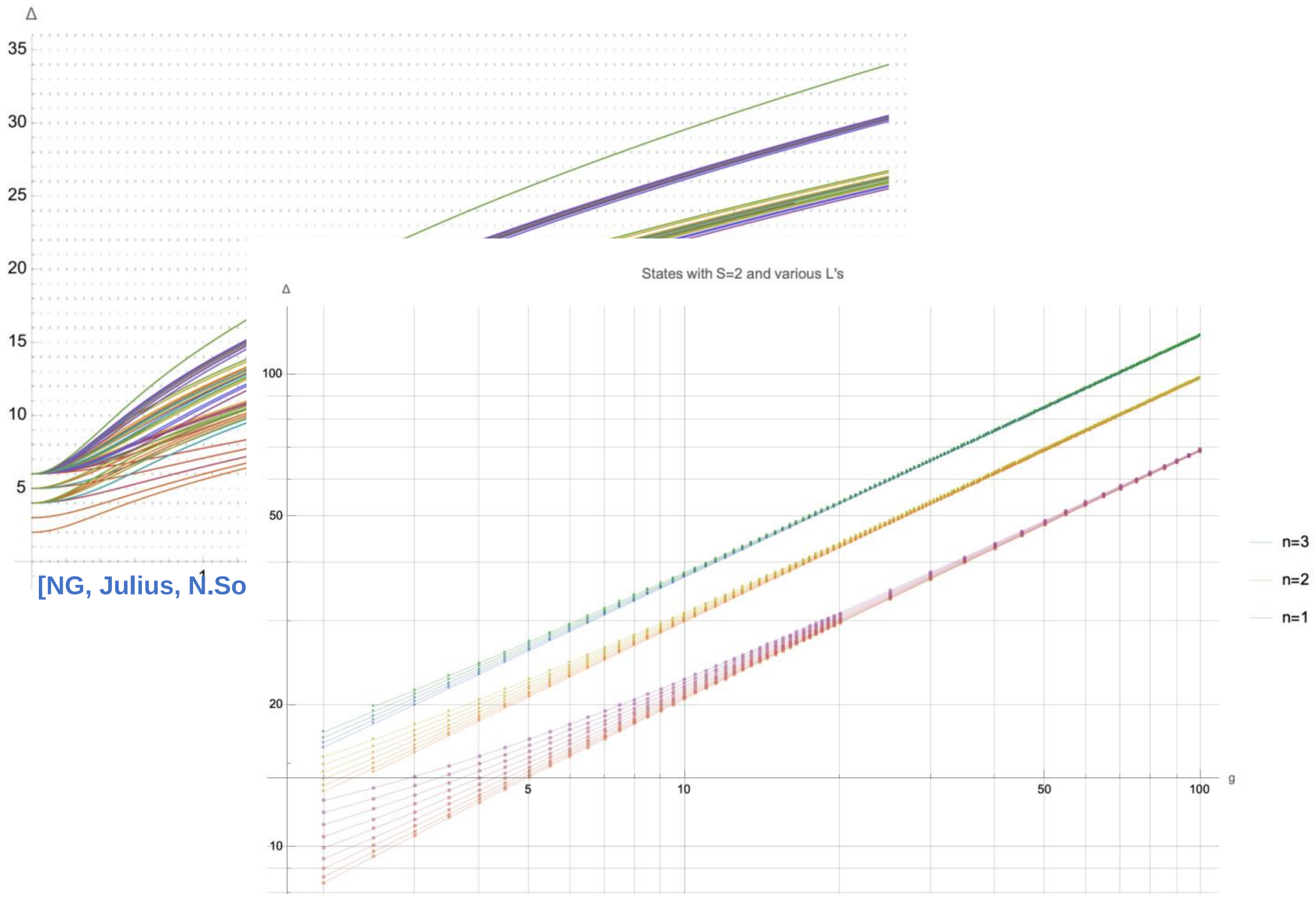
for non-integer S
(i.e. light-ray operators)

claim: a) + b) is in 1-to-1 with local operators

Generalization to: ABJM, β-def, AdS_3 ...

Some results of QSC

Spectrum of local operators:

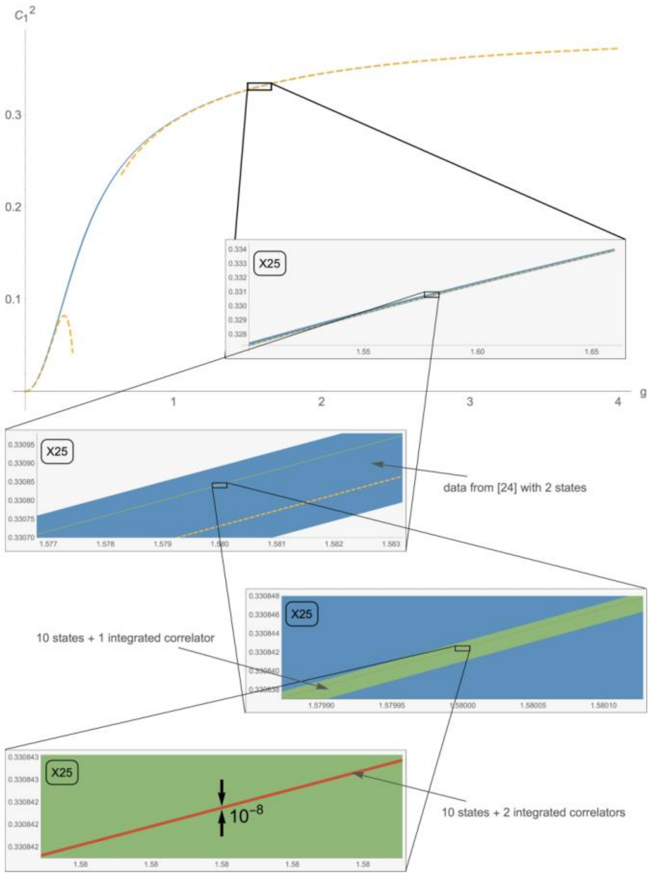


[NG, Julius, N.So

[Ekhammar, NG, Ryan ‘24] <- See Simon’s talk

Structure constants:

[NG, Kazakov, Leurent, Volin ‘13]



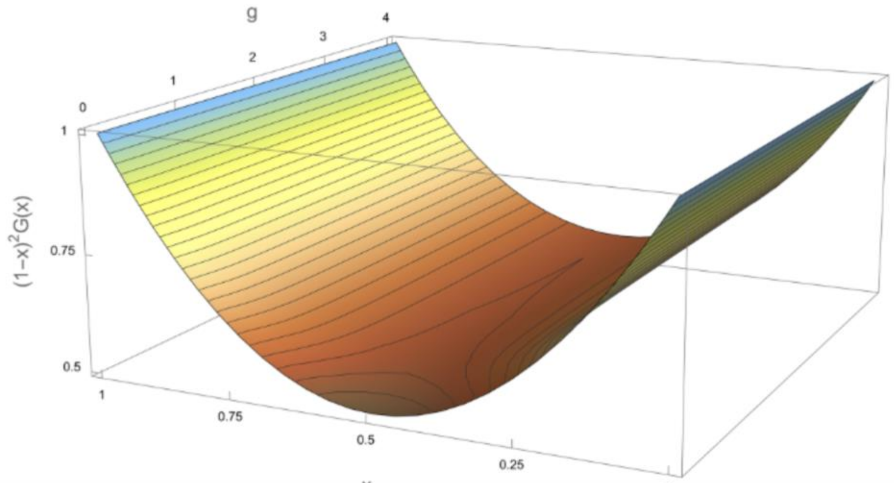
[Cavaglia, NG, Julius, Preti ‘21]

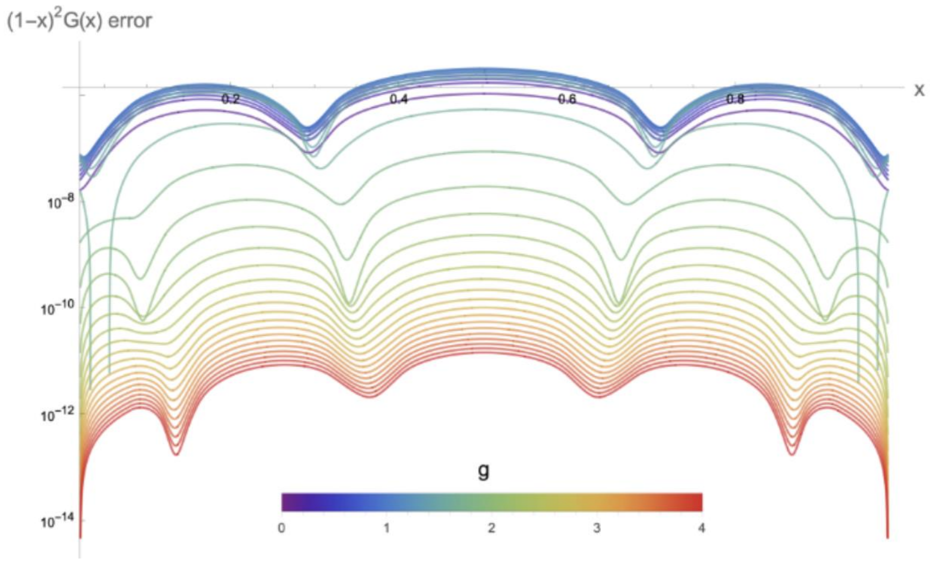
And beyond…

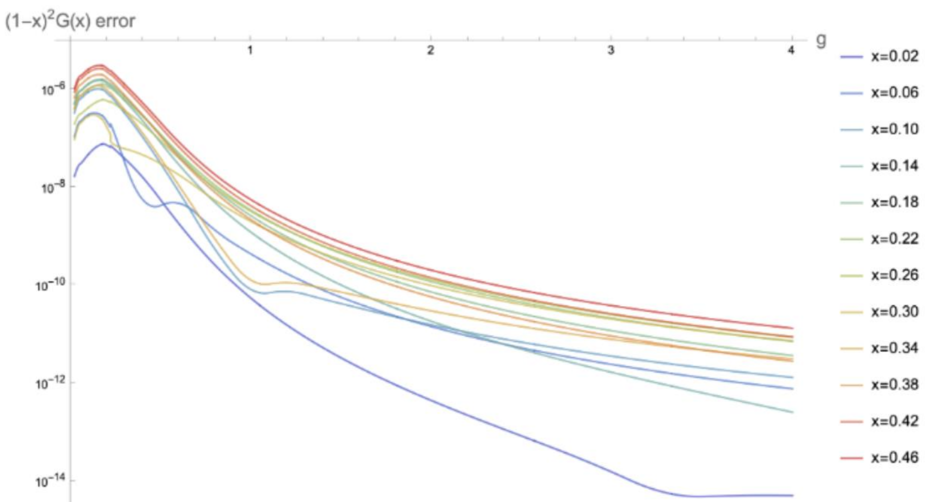
[Basso Georgoudis Sueiro ’22]

4 pt correlation functions in N=4

[Cavaglia, NG, Julius, Preti ‘23]



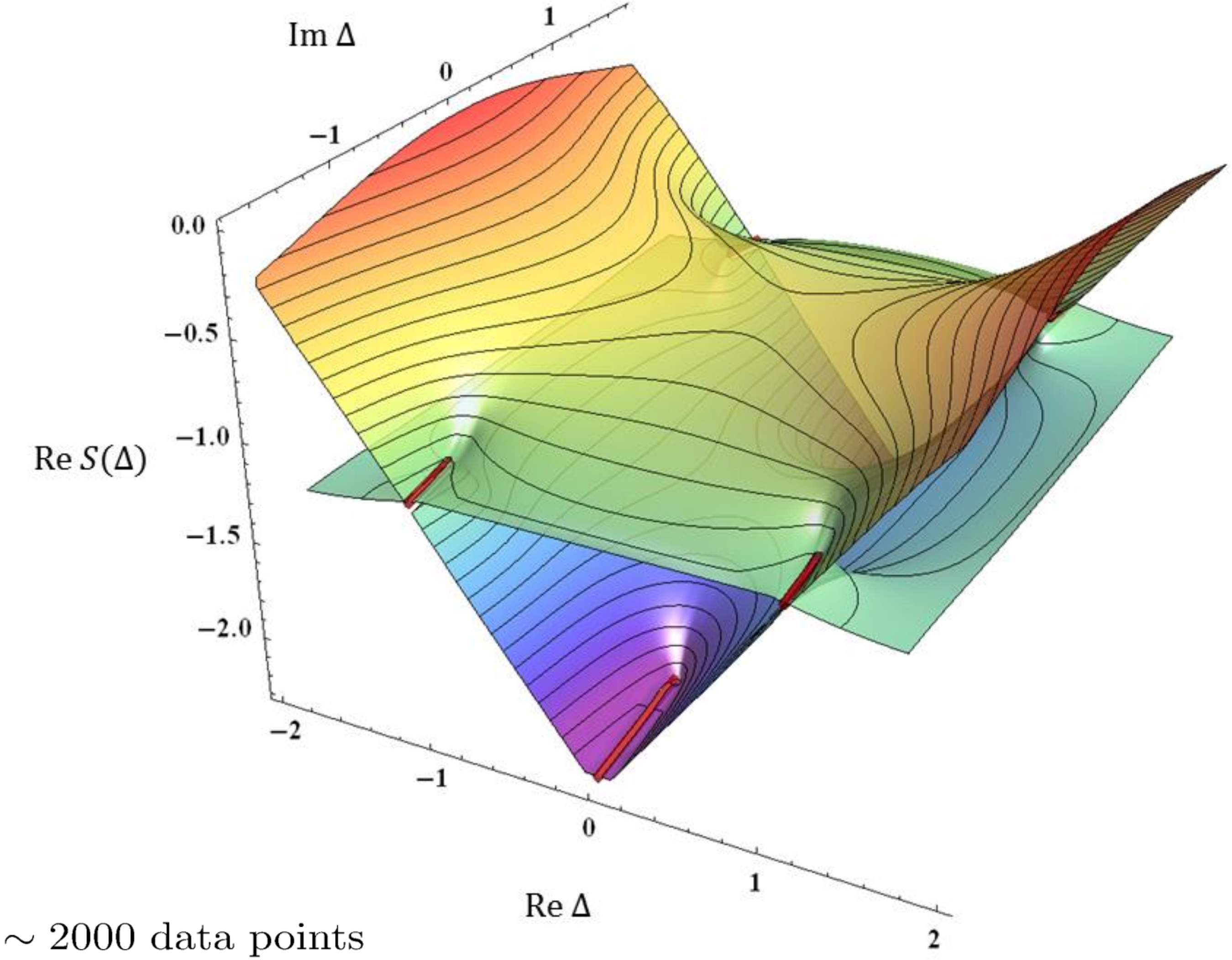




Back to BFKL

Twist 2 – N=4 SYM Pomeron Trajectory, non-perturbative:

Gromov,FLM,Sizov 2015

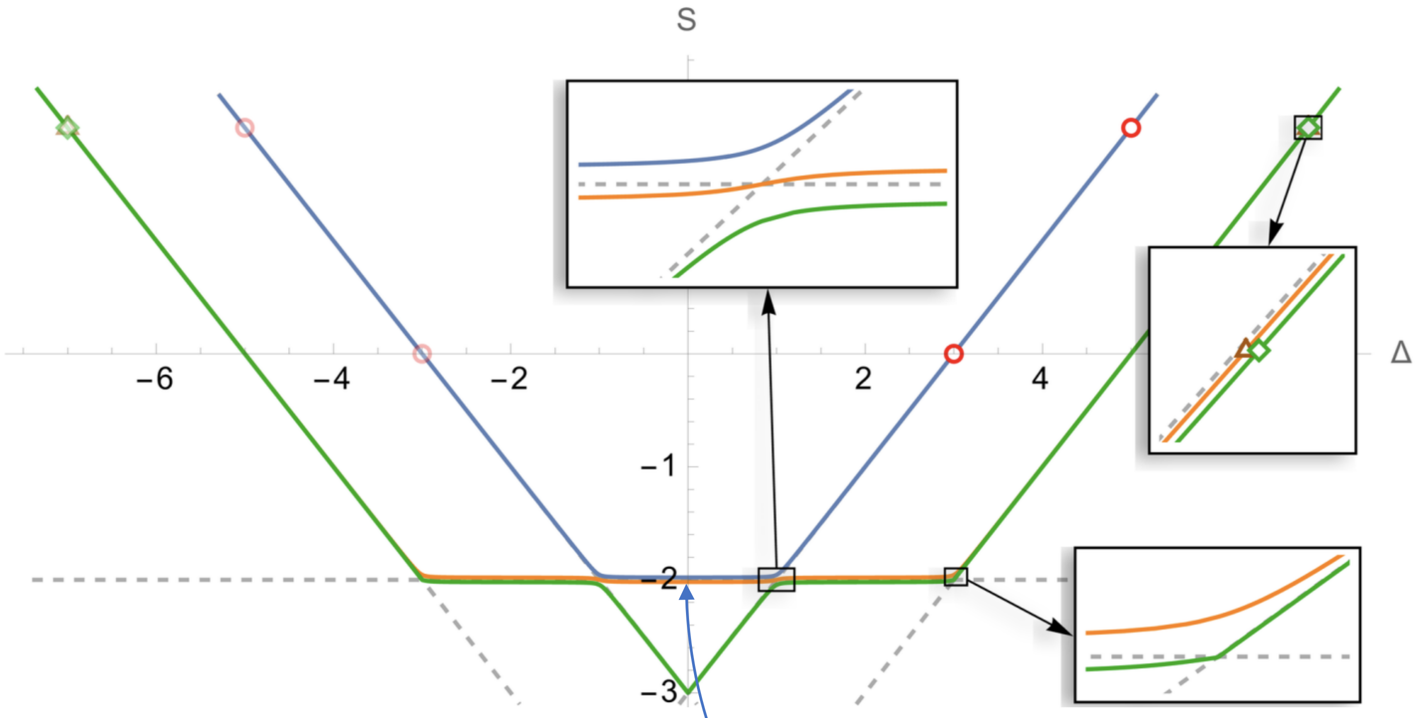


$\sim$ 2000 data points

Higher R-charge – new physics?

[Klabbers, Preti, Szecsenyi `23]

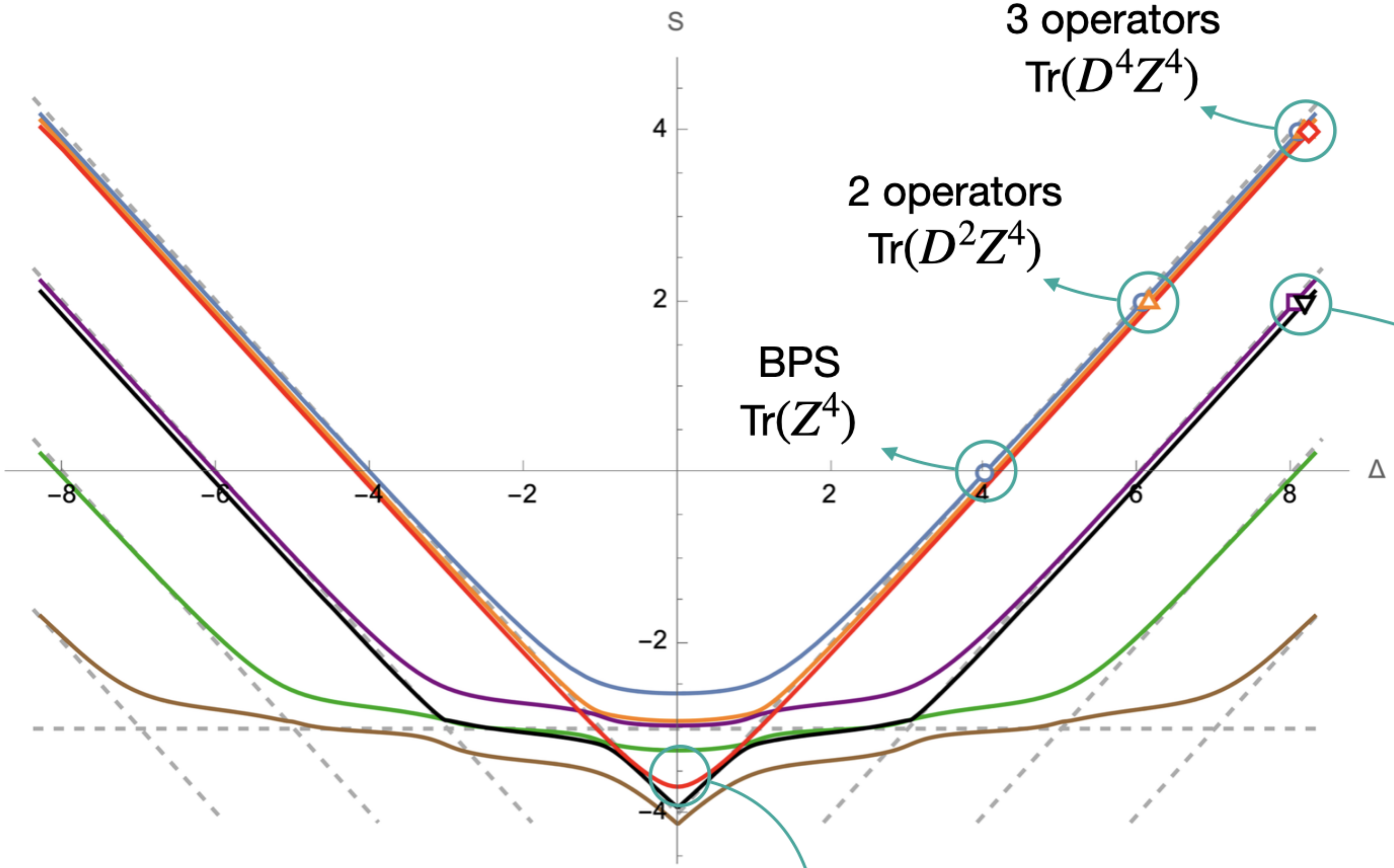
Twist 3



$$\alpha = -2 + 2g + 16 \log 2\ g^2 - \frac{2\pi^2}{3} g^3$$

Twist 4

Coupling
$g = 0.1$



3 operators
$\mathrm{Tr}(D^4Z^4)$

2 operators
$\mathrm{Tr}(D^2Z^4)$

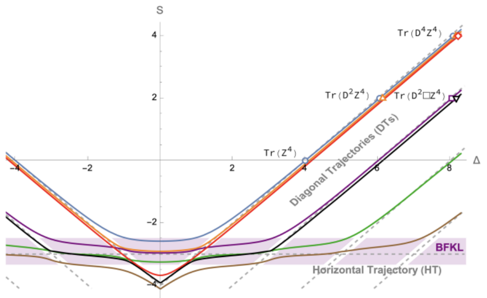
BPS
$\mathrm{Tr}(Z^4)$

Twist 6
operators

Red trajectory is one of an infinite set of trajectories
that doesn't bend at the branch point

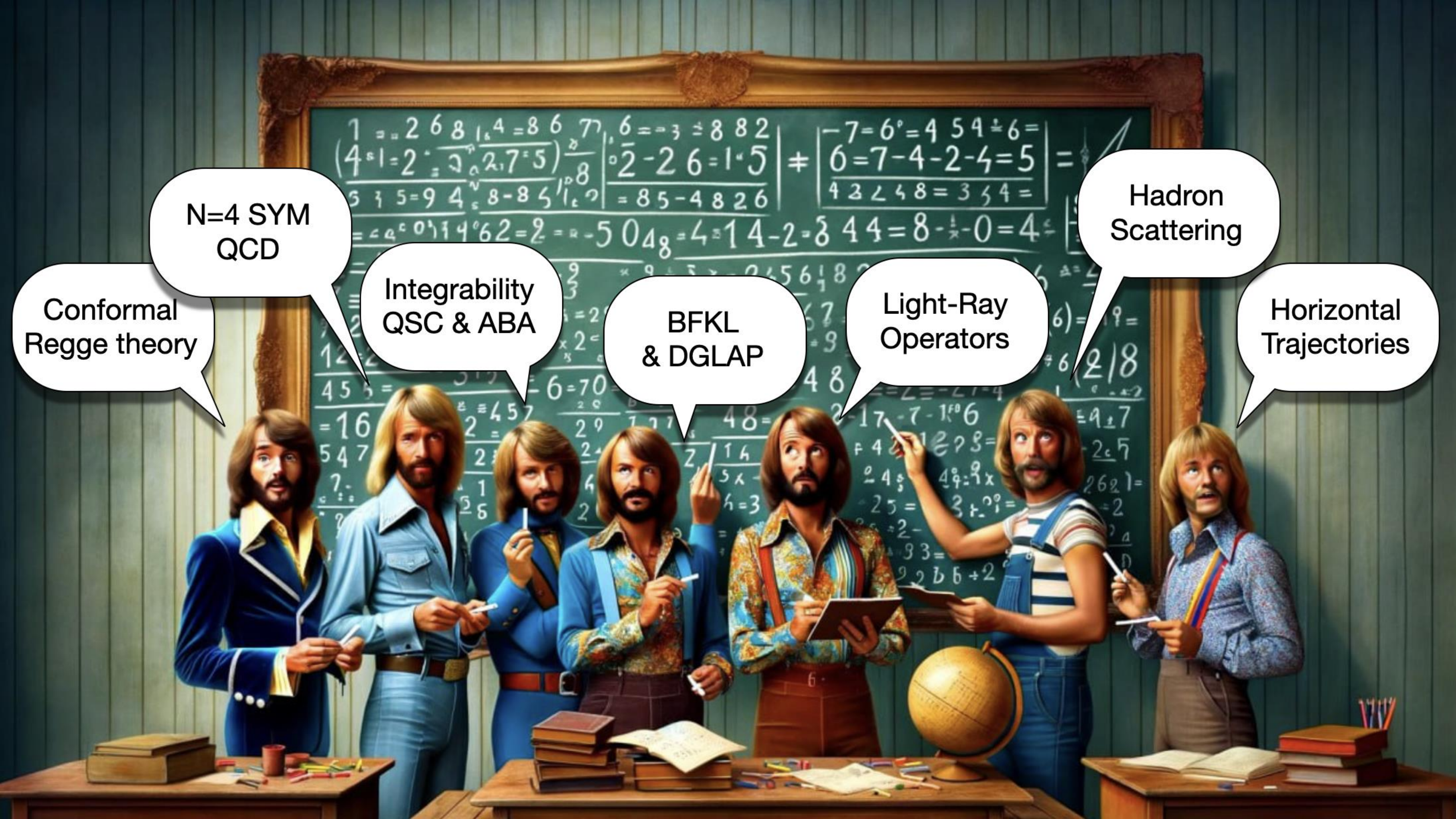
Asymptotic Baxter-Bethe Ansatz

- for local operators Asymptotic Bethe Ansatz (ABA) gives us great control over Δ up to g^{2L+4} order
- It enables weak coupling Form-factors for structure constants (Hexagon)



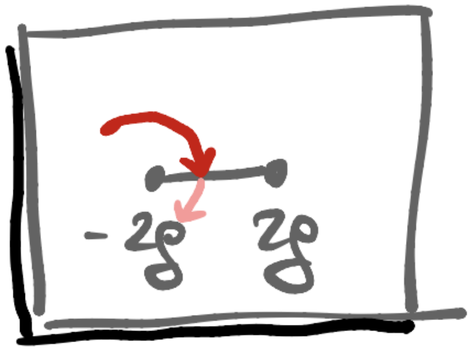
- We found that in BFKL regime a similar equations can be written valid to g^{L+2} order, but with a different particle content
- New feature: massless magnons

$$E = g \sin P/2 \quad \text{vs} \quad \sqrt{1+16g^2 \sin^2 P/2}$$

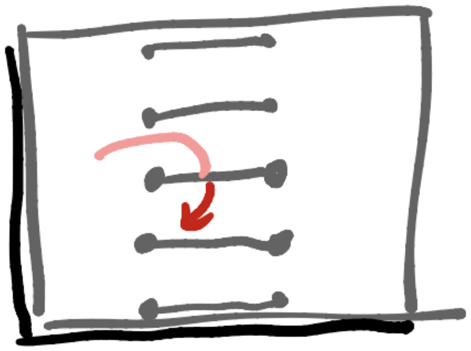

Conformal Regge theory
N=4 SYM QCD
Integrability QSC & ABA
BFKL & DGLAP
Light-Ray Operators
Hadron Scattering
Horizontal Trajectories

ABA from QSC

Pμ - system



P_a



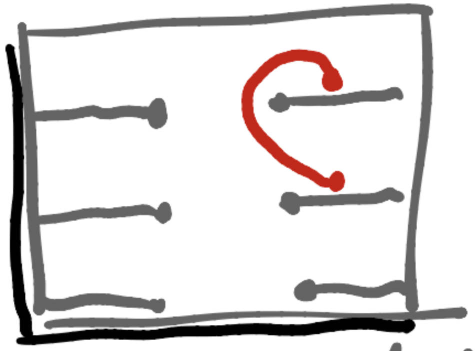
$\tilde{P}_a$

monodromy

$$\tilde{P}_a = \mu_{ab} P^b$$

monodromy

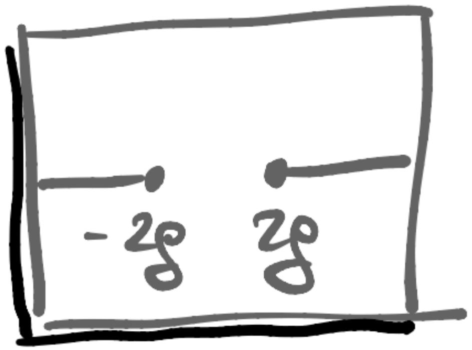
$$\tilde{P}_a = \mu_{ab} P^b$$



long-cut periodicity

ABA from QSC

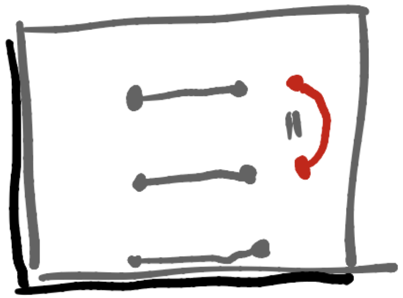
Qω - system



Q_i

monodromy

$$\tilde{Q}_i = w_{ij} Q^j$$



$$w_{ij}(u+i) = w_{ij}(u)$$

short-cut periodicity

ABA from QSC

$P\mu \longleftrightarrow Q\omega$

$$P_a = Q_{a i} Q^i$$

rotation $P \longleftrightarrow Q$

$$\mu_{ab} = Q_{a i} Q_{b j} \omega^{ij}$$

ABA (local operators) limit : $\omega^{12} \sim g^{-2L-4} \to \infty$

ABBA (BFKL limit) : $\omega^{13} \sim g^{-L-2} \to \infty$

- key simplification

not periodic

$$F = \frac{\mu_{ab}(u+i/2)}{\mu_{ab}(u-i/2)} = \frac{Q^+_{a i}\, Q^+_{b j}\, \omega^{ij}(u+i/2)}{Q^-_{a i}\, Q^-_{b j}\, \omega^{ij}(u-i/2)}$$

periodic

- $\Rightarrow F = \dfrac{\mu_{ab}(u+i/2)}{\mu_{ab}(u-i/2)} = \dfrac{Q^+ Q^+}{Q^- Q^-}$ ← analytic UHP

- $\overline{\mu_{ab}(u+i/2)} = \pm \mu_{ab}(u-i/2) \Rightarrow \bar{F} = \dfrac{1}{F}$

- $\mu_{ab}(u+i/2) = \tilde{\mu}_{ab}(u-i/2) \Rightarrow \tilde{F} = \dfrac{1}{F}$

- Conclusion: whenever one $w_{ij} \to \infty$

$F, \tilde{F}$ both are functions with one cut

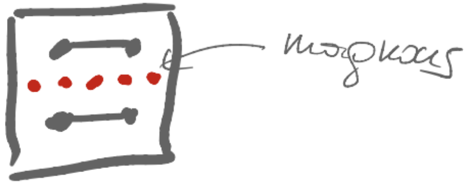
$\Rightarrow F =$ rational function of x

$x + \dfrac{1}{x} = \dfrac{u}{g}$ ⇒ BAE

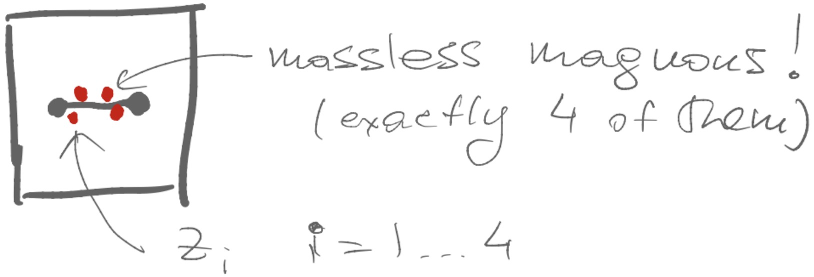
ABA vs ABBA

- $\omega_{12} \longleftrightarrow \omega_{13}$ same as $\Delta \longleftrightarrow S$
- μ_{12} was $\sim u^{\Delta}$, in ABBA $\mu_{12} \sim u^{S} e^{2\pi u}$

change of gluing



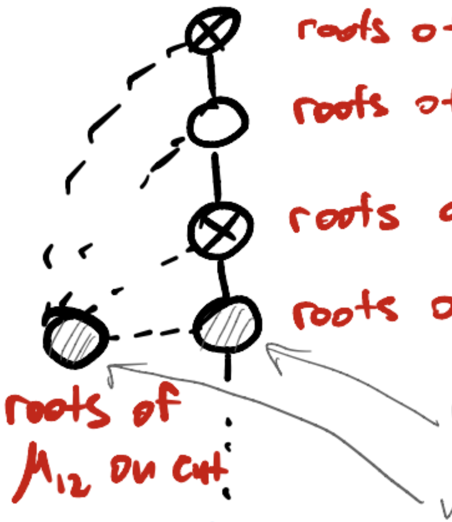
- $\mu_{12}(u+i/2)$ roots become magnons
- we found new type of rooms


massless magnons!
(exactly 4 of them)

$z_i \quad i=1\ldots4$

BES	$s\ell_2$
roots of P_1	no roots
roots of $Q_{1\vert 1}$	no roots
roots of $Q_{1\vert 12}$	no roots
roots of $Q_{12\vert 12}$	S roots

$$\Delta = L + S + 2ig^2 \sum_k \left(\frac{1}{x_k^+} - \frac{1}{x_k^-}\right)$$



ABBA	$s\ell_2$
roots of P_1	no roots
roots of $Q_{1\vert 1}$	no roots
roots of $Q_{1\vert 13}$	no roots
roots of $Q_{12\vert 13}$	no roots

massive magnon

massles magnon

roots of A_{12} on cut

4-roots

$$S = -L + 1 + ig \sum_{k=1}^{4} \left(z_k - \frac{1}{z_k}\right)$$

$$+ 2ig^2 \sum_k \left(\frac{1}{x_k^+} - \frac{1}{x_k^-}\right)$$

sl_2 Sector

- P_a : $P_1(u) = \sigma_0(u)\, x^{-\frac{L}{2}-1}$

$\nwarrow$ massless analog of BES phase

$$\ln \sigma_0(z) = \oint \frac{dx}{2\pi i} \oint \frac{dy}{2\pi i} \frac{1}{z-x} \left(\frac{1}{y-z_i} - \frac{1}{y-1/z_i} \right) \ln \frac{\Gamma(1+iu_x-iu_y)}{\Gamma(1-iu_x+iu_y)}$$

- $Q_{a|i}$: $\left(\left(u+\frac{i}{2}\right)^2 - \theta_2^2\right) Q_{1|\beta}(u+i) + \left(\left(u+\frac{i}{2}\right)^2 - \theta_1^2\right) Q_{1|\beta}(u-i)$

$$= \left(2u^2 - \frac{1}{4}\left(\Delta^2 + 4\theta_1^2 + 4\theta_2^2 + 1\right)\right) Q_{1|\beta}(u)$$

$\hookrightarrow g\left(z_1 + \frac{1}{z_1}\right)$

$$\frac{(-1)^{-1/4}\, 2^{1-\Delta}\, \frac{e^{\frac{3i\pi\Delta}{4}}}{1+e^{i\pi\Delta}}\, \sqrt{\pi}\, \Gamma\left(1-\frac{\Delta}{2}\right) \Gamma\left(-iu+i\theta_1+\frac{1}{2}\right) \Gamma\left(-2i\theta_2\right)}{\Gamma\left(-iu-i\theta_2+\frac{1}{2}\right) \Gamma\left(-\frac{\Delta}{2}-i\theta_1-i\theta_2+\frac{1}{2}\right) \Gamma\left(-\frac{\Delta}{2}+i\theta_1-i\theta_2+\frac{1}{2}\right) \Gamma\left(i\theta_1+i\theta_2+1\right)}$$

$$_3F_2\left(iu+i\theta_2+\frac{1}{2}, -\frac{\Delta}{2}+i\theta_1+i\theta_2+\frac{1}{2}, \frac{\Delta}{2}+i\theta_1+i\theta_2+\frac{1}{2}; i\theta_1+i\theta_2+1, 2i\theta_2+1; 1\right)$$

For $\mathrm{tr}\, D_+^S z^L$

only 4 massless roots

$$(iz_k)^{2L+4} = \frac{1}{G_{kk}^{\frac{1+\Delta}{2}} G_{kk}^{\frac{1-\Delta}{2}}} \frac{\sigma_0^2(-z_k)}{\sigma_0^2(-1/z_k)} \prod_{\substack{n\neq 0 \\ j=1..4}} \frac{z_k^{[2n]} - z_j}{z_k^{[2n]} - 1/z_j} \prod_{e=1,2} \frac{G_{ke}^{\frac{1+\Delta}{2}} G_{ke}^{\frac{1-\Delta}{2}}}{G_{ke}^{1}}$$

$$G_{ke}^{\delta} = \frac{\Gamma(\delta + i\theta_k + i\theta_e)}{\Gamma(\delta - i\theta_k - i\theta_e)} \frac{\Gamma(\delta + i\theta_k - i\theta_e)}{\Gamma(\delta - i\theta_k + i\theta_e)}$$

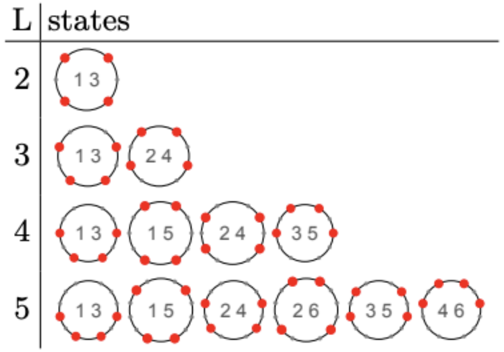
$$S = 1 - L + ig\sum_{k=1}^{4}\left(z_k - \frac{1}{z_k}\right) \leftarrow \text{„Energy"}$$

$g \to 0$ r.h.s. $\to 1$

$$(iz_1)^{L+2} = (iz_2)^{L+2} = \pm 1 \quad \Rightarrow \quad z_k = -i\, e^{\frac{\pi i n_k}{L+2}}$$

$n_k \to 1 \ldots L+1$

$\sim L^2/4$ states



perturbatively

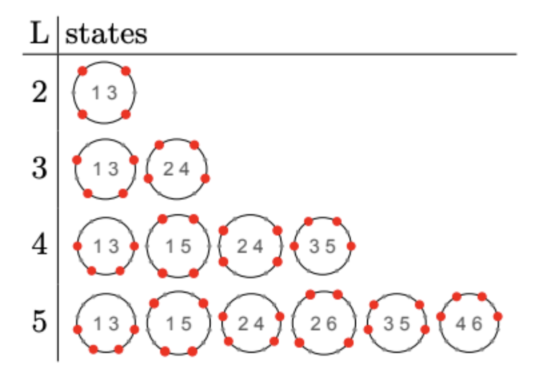
$$z_k = z_k^0 + \sum g^n a_{k,n}$$

Examples of the states

L=2:

$$-4\,g^2\,\chi(\Delta)+O(g^4)$$

$$\psi\left(\tfrac{1-\Delta}{2}\right)+\psi\left(\tfrac{1+\Delta}{2}\right)+2\gamma\equiv\chi\,,$$



L=3:

$$2\,g-4\,g^2\,\chi(\Delta)-\frac{2\,\pi^2\,g^3}{3}+g^4\left(24\,\chi''(\Delta)+\frac{4}{3}\,\pi^2\,\chi(\Delta)+28\,\zeta(3)\right)+O(g^5)$$

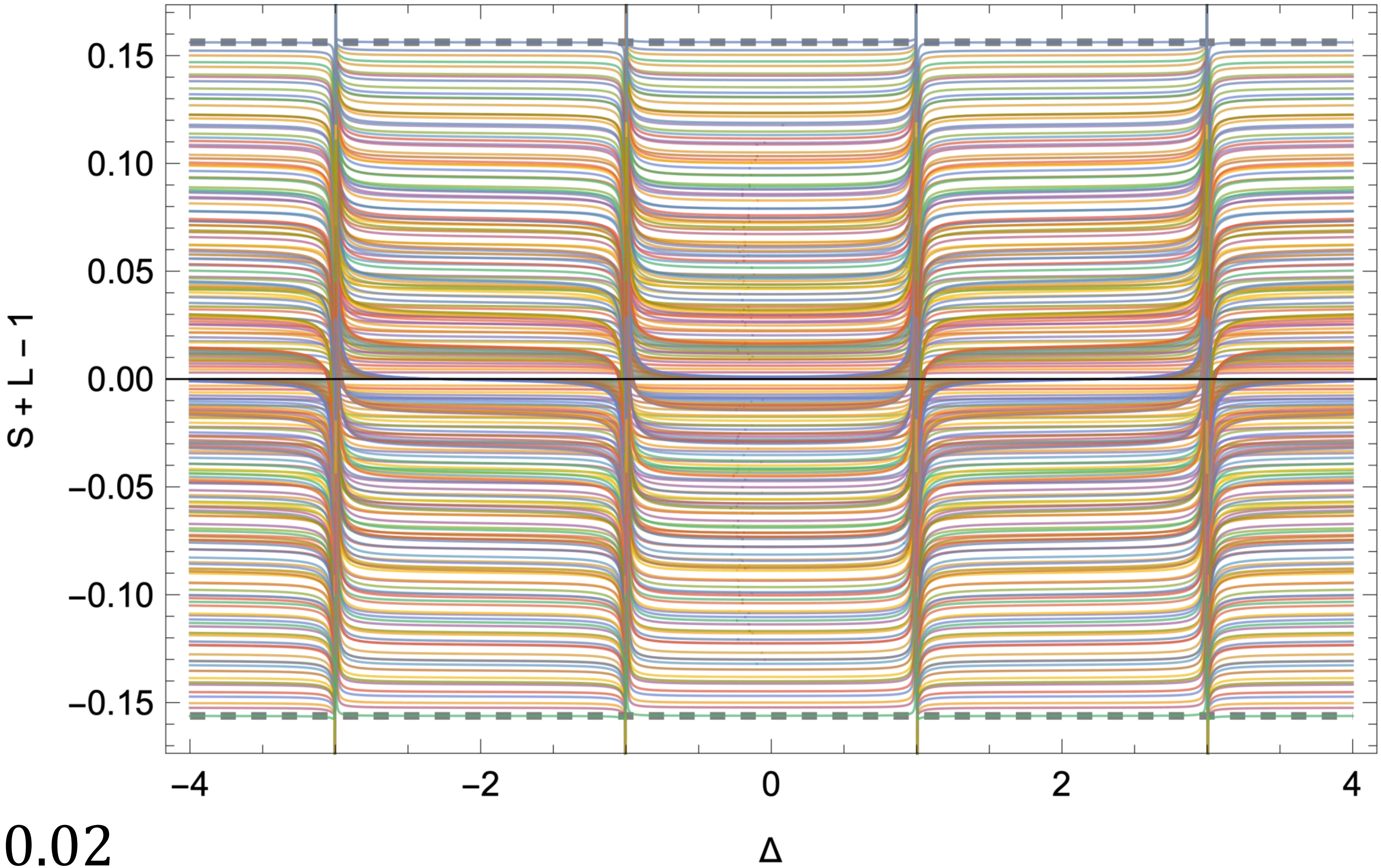
L=4:

$$2\sqrt{3}\,g-\frac{10}{3}\,g^2\,\chi(\Delta)-\frac{g^3\left(\chi(\Delta)^2+2\,\pi^2\right)}{\sqrt{3}}+\frac{4}{81}\,g^4\left(369\,\chi''(\Delta)+8\,\chi(\Delta)^3+18\,\pi^2\,\chi(\Delta)+531\,\zeta(3)\right)+$$
$$\frac{g^5\left(5\,\chi(\Delta)\left(9360\,\chi''(\Delta)-115\,\chi(\Delta)^3+324\,\pi^2\,\chi(\Delta)+3312\,\zeta(3)\right)+2376\,\pi^4\right)}{1620\,\sqrt{3}}+O(g^6)$$

L=5:

$$\begin{aligned}
&-2((-1)^{2/7}(1+(-1)^{4/7})(1+(-1)^{6/7}))g-\frac{4}{7}g^2((-1)^{3/7}(1-\sqrt[7]{-1}+(-1)^{2/7}-4(-1)^{4/7}+(-1)^{6/7})\text{chi}(\Delta))+\\
&\frac{2}{147}\sqrt[7]{-1}\,(1+\sqrt[7]{-1})g^3(7(-2+5\sqrt[7]{-1}-(-1)^{2/7}-3(-1)^{3/7}+2(-1)^{5/7})\pi^2-18\sqrt[7]{-1}\,((-1)^{2/7}-1)\text{chi}(\Delta)^2)-\\
&\frac{1}{1029}\times 4\,g^4\,((-1)^{6/7}\,(1+\sqrt[7]{-1})\,(98(-7+43\sqrt[7]{-1}-50(-1)^{2/7}+40(-1)^{3/7}-25(-1)^{4/7}+10(-1)^{5/7})\text{chi}''(\Delta)+\\
&\qquad 32(-1+\sqrt[7]{-1}-2(-1)^{2/7}+(-1)^{3/7}-(-1)^{4/7}+(-1)^{5/7})\text{chi}(\Delta)^3+7(-7+21\sqrt[7]{-1}-28(-1)^{2/7}+24(-1)^{3/7}-14(-1)^{4/7}+4(-1)^{5/7})\pi^2\,\text{chi}(\Delta)+\\
&\qquad 49\,(-11+101\sqrt[7]{-1}-112(-1)^{2/7}+80(-1)^{3/7}-56(-1)^{4/7}+32(-1)^{5/7})\,\zeta(3)))-\frac{1}{108045}\times 2\,g^5\\
&\ (\sqrt[7]{-1}\,(1+\sqrt[7]{-1})\,(5880\,((-1)^{2/7}-1)\,\text{chi}(\Delta)\,(5(5-11\sqrt[7]{-1}+5(-1)^{2/7})\text{chi}''(\Delta)+(1-34\sqrt[7]{-1}+(-1)^{2/7})\,\zeta(3))-150(-25-2\sqrt[7]{-1}+2(-1)^{3/7}+25(-1)^{4/7})\\
&\qquad\quad \text{chi}(\Delta)^4-210(-16-63\sqrt[7]{-1}+63(-1)^{3/7}+16(-1)^{4/7})\pi^2\,\text{chi}(\Delta)^2+49(-119+346\sqrt[7]{-1}-35(-1)^{2/7}-276(-1)^{3/7}+49(-1)^{4/7}+70(-1)^{5/7})\pi^4))-\\
&\frac{1}{756315}\times 4\,g^6\,((1+\sqrt[7]{-1})\,(4410\,\sqrt[7]{-1}\,((-1)^{2/7}-1)\,\text{chi}(\Delta)^2\,(8(17-15\sqrt[7]{-1}+17(-1)^{2/7})\text{chi}''(\Delta)+(113+26\sqrt[7]{-1}+113(-1)^{2/7})\,\zeta(3))+\\
&\qquad 343\,(42(370-495\sqrt[7]{-1}+389(-1)^{2/7}-185(-1)^{3/7}-19(-1)^{4/7}+125(-1)^{5/7})\text{chi}^{(4)}(\Delta)+10\sqrt[7]{-1}\,(-95+118\sqrt[7]{-1}-118(-1)^{3/7}+95(-1)^{4/7})\\
&\qquad\qquad \pi^2\,\text{chi}''(\Delta)+12\,285\,(-1)^{5/7}\,\zeta(5)+17\,136\,(-1)^{4/7}\,\zeta(5)-35\,595\,(-1)^{3/7}\,\zeta(5)+54\,054\,(-1)^{2/7}\,\zeta(5)-83\,475\,\sqrt[7]{-1}\,\zeta(5)+71\,190\,\zeta(5)+\\
&\qquad\quad 920\,(-1)^{5/7}\,\pi^2\,\zeta(3)-340\,(-1)^{4/7}\,\pi^2\,\zeta(3)-630\,(-1)^{3/7}\,\pi^2\,\zeta(3)+1600\,(-1)^{2/7}\,\pi^2\,\zeta(3)-2180\,\sqrt[7]{-1}\,\pi^2\,\zeta(3)+1260\,\pi^2\,\zeta(3))+\\
&\qquad 288\,\sqrt[7]{-1}\,(-5-27\sqrt[7]{-1}+27(-1)^{3/7}+5(-1)^{4/7})\text{chi}(\Delta)^5+70\sqrt[7]{-1}\,(-323+10\sqrt[7]{-1}-10(-1)^{3/7}+323(-1)^{4/7})\pi^2\,\text{chi}(\Delta)^3+\\
&\qquad 49(588-1409\sqrt[7]{-1}+1386(-1)^{2/7}-294(-1)^{3/7}-798(-1)^{4/7}+821(-1)^{5/7})\pi^4\,\text{chi}(\Delta)))+O(g^7)
\end{aligned}$$

Pushing to large twist L=30



$+4g\left(\cos\frac{\pi}{L+2}+\cos\frac{3\pi}{L+2}\right)$

225 Trajec tories

$-4g\left(\cos\frac{\pi}{L+2}+\cos\frac{3\pi}{L+2}\right)$

$g = 0.02$

Can we penetrate from BFKL regime to DGLAP?

[Beccaria ’07] [Kotikov, Lipatov, Rej, Staudacher, Velizhanin ’07]

$$\frac{\gamma_2^{\rm ABA}(M)}{2} = 4\, S_1(\frac{M}{2})\,. \qquad \frac{\gamma_4^{\rm ABA}(M)}{4} = -2S_3 - 4S_1S_2\,,$$

$$\frac{\gamma_6^{\rm ABA}(M)}{8} = 2S_2S_3 + S_5 + 4S_{3,2} + 4S_{4,1} - 8S_{3,1,1} + S_1\left(4S_2^2 + 2S_4 + 8S_{3,1}\right)$$

$$\begin{aligned}\frac{\gamma_8^{\rm ABA}(M)}{16} = {} & S_1^3\left(\frac{40}{3}S_4 - \frac{32}{3}S_{3,1}\right) + S_1^2\left(20S_5 - 40S_{3,2} - 56S_{4,1} + 64S_{3,1,1}\right)\\ & +S_1\Big(7S_6 + 8S_{2,4} - 24S_{3,3} - 56S_{4,2} - 40S_{5,1} - 24S_{2,2,2} - 16S_{2,3,1}\\ & +88S_{3,1,2} + 88S_{3,2,1} + 120S_{4,1,1} - 192S_{3,1,1,1} - 8\zeta(3)S_3\Big) - \frac{56}{3}S_3S_4\\ & -\frac{107}{6}S_7 + 3S_{2,5} + \frac{41}{3}S_{3,4} + \frac{1}{3}S_{4,3} - 17S_{5,2} - \frac{20}{3}S_{6,1} - 4S_{2,2,3}\\ & -8S_{2,3,2} - 4S_{2,4,1} + \frac{104}{3}S_{3,1,3} + 52S_{3,2,2} + \frac{88}{3}S_{3,3,1} + 60S_{4,1,2}\\ & +60S_{4,2,1} + 40S_{5,1,1} + 8S_{2,3,1,1} - 120S_{3,1,1,2} - 120S_{3,1,2,1}\\ & -120S_{3,2,1,1} - 128S_{4,1,1,1} + 256S_{3,1,1,1,1}.\end{aligned}$$

(

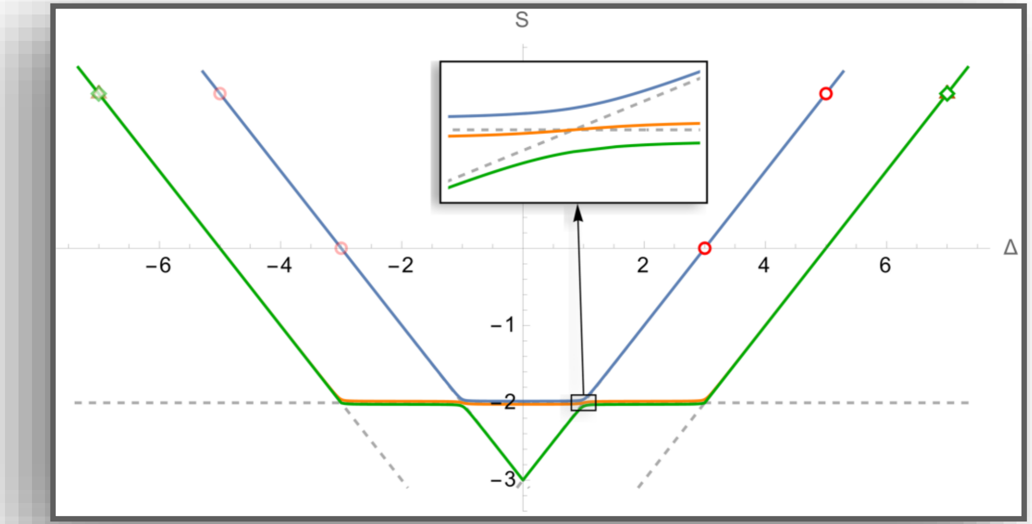
It is interesting to note that the double-logarithmic behavior of these states is different from the twist-two ones (3.10).

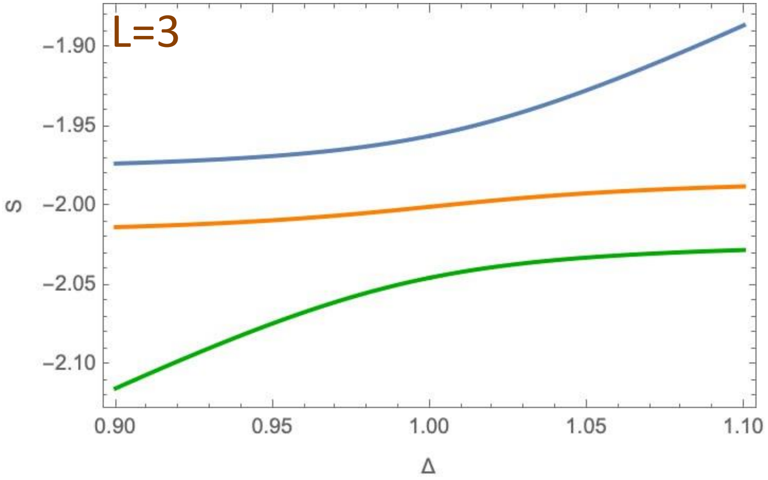
$$\gamma^{\rm ABA} = -8\frac{g^2}{\omega}\left(\frac{1}{1-t} - \zeta(2)\frac{1+3\,t^2}{(1-t)^2}\omega^2 + \ldots\right) + \ldots\,.$$

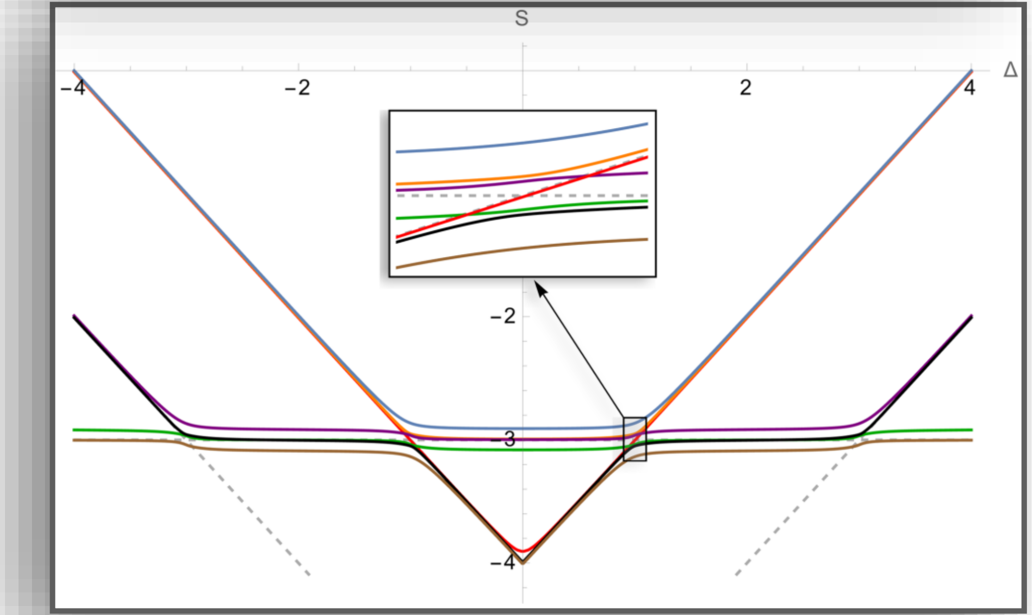
$$t = \frac{g^2}{\omega^2}.$$

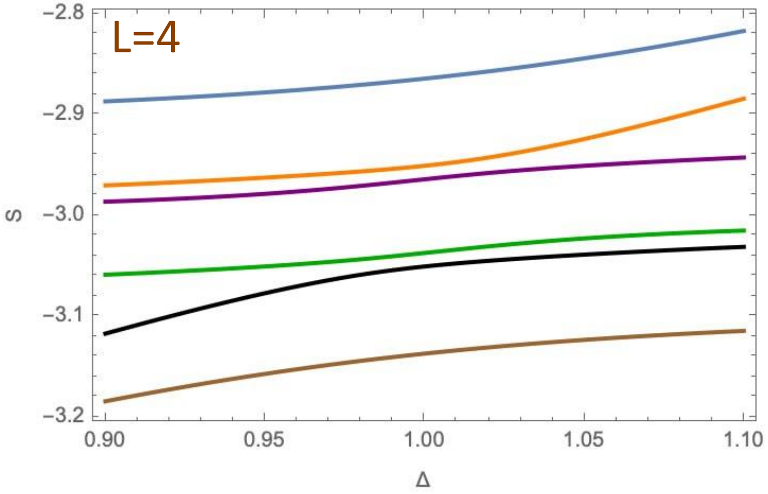
$$M = 2\,(\omega - 1)$$

Can we penetrate from BFKL regime to DGLAP?









Can we penetrate from BFKL regime to DGLAP?

- Zoom near the edge: $\Omega \equiv \frac{S+2}{g}, D \equiv \frac{\Delta-1}{g}$
- The horizontal trajectories become: $\Omega_{1,2} \simeq \pm 2 - \frac{8}{D} + \frac{96}{D^3} + \mathcal{O}\left(\frac{1}{D^4}\right)$
- Riemann surface reduces to cubic poly (fixed by $\Omega_{1,2}$)

$$(\Omega-2)(\Omega+2)(\Omega-D)-16\Omega=0$$

- Solving for Δ

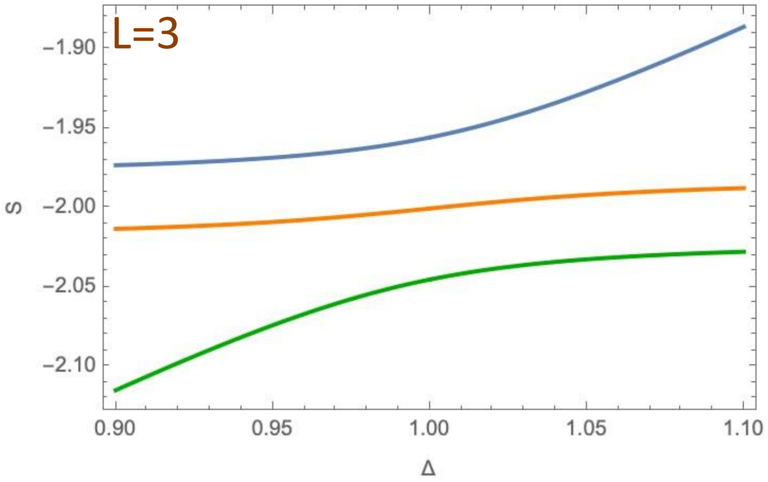
$$\Delta-1-\omega=-\frac{16\omega g^2}{\omega^2-4g^2} \simeq -\frac{16g^2}{\omega}-\frac{64g^4}{\omega^3}-\frac{256g^6}{\omega^5}-\frac{1024g^8}{\omega^7} \quad \omega=S+2$$

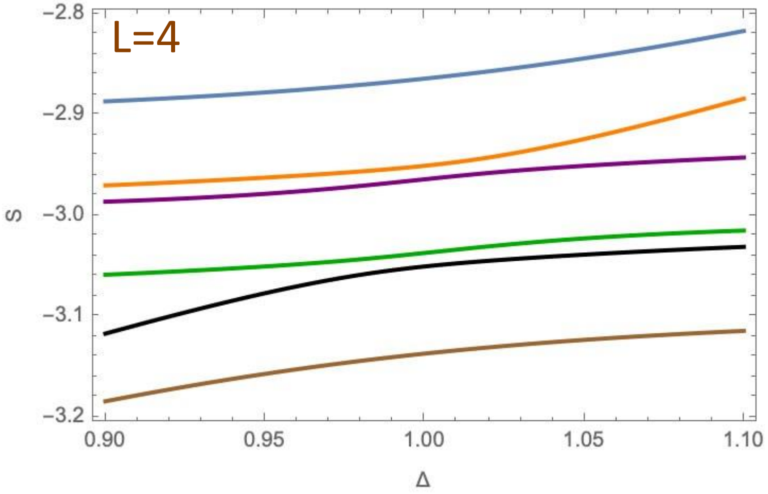
- Perfect agreement with explicit 4-loop for local operators!

[Beccaria '07] [Kotikov, Lipatov, Rej, Staudacher, Velizhanin '07]

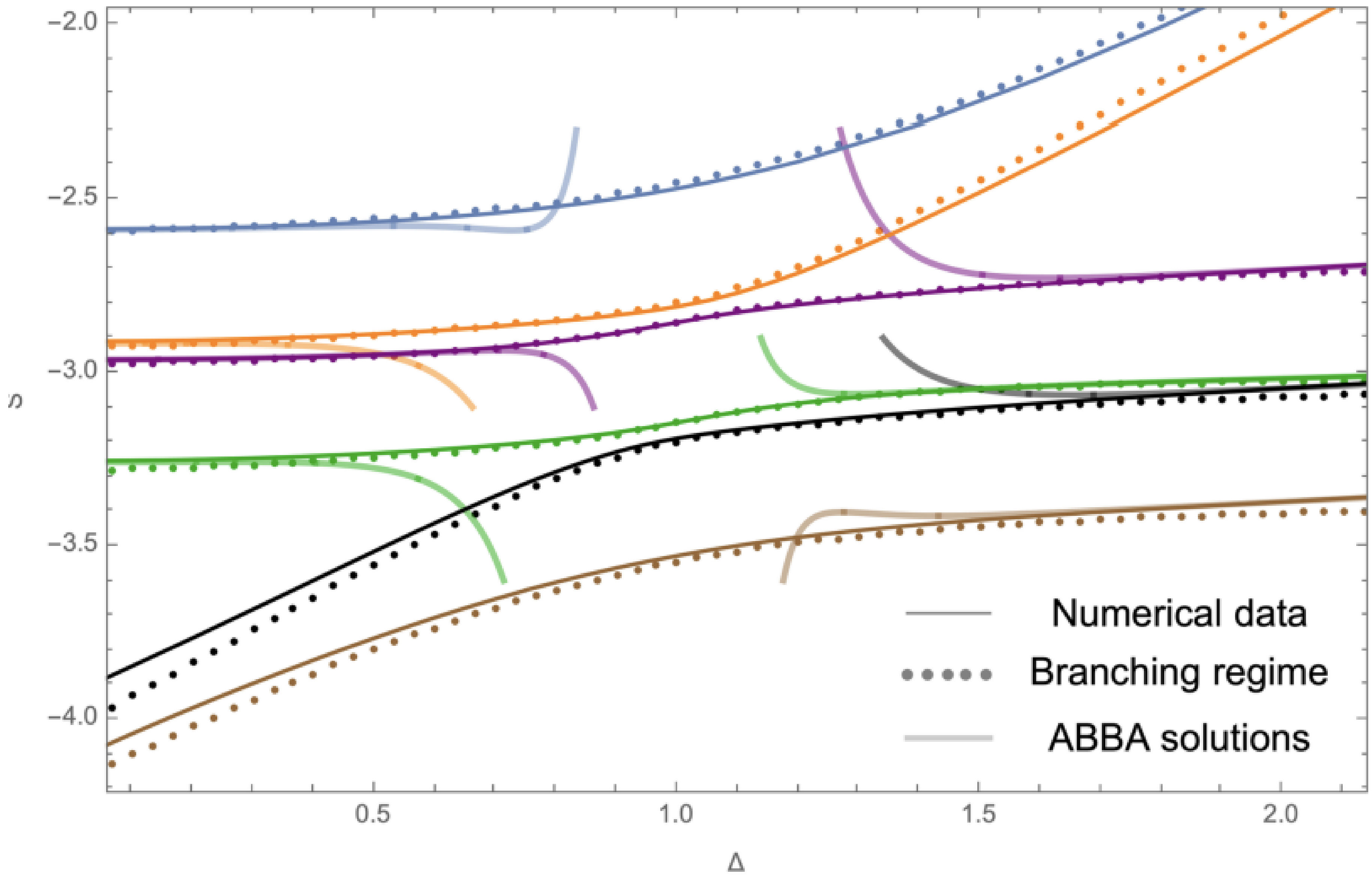
- Repeat the same procedure for $L=4$

$$(D^2-36)\Omega^4+(192-12D^2)\Omega^2-2D\Omega^5+48D\Omega^3-128D\Omega+\Omega^6-256=0$$





Can we penetrate from BFKL regime to DGLAP?



Spooky observation

- L=3 branching pole

$$\Delta - 1 - \omega = -\frac{16\omega g^2}{\omega^2 - 4g^2} \simeq \boxed{-\frac{16g^2}{\omega}} - \frac{64g^4}{\omega^3} - \frac{256g^6}{\omega^5} - \frac{1024g^8}{\omega^7}$$

- L=4 two trajectories turn (have pole)

$$\Delta_{\pm}^{L=4} - 1 \simeq \omega - 4g^2\frac{3 \pm \sqrt{5}}{\omega} - 16g^4\frac{25 \pm 11\sqrt{5}}{5\omega^3} + \ldots .$$

- In general:

$$\Delta_L \simeq \frac{B_n g^2}{\omega}$$

- # of trajectories with pole at $s = -L + 1$ match # of operators $\mathrm{tr}\, D_+^2 Z^L$
- The pole B_n is essentially the 1-loop anomalous dimension of this operator

$$B_n = 32 - 2\gamma_n^{(1)}$$ ← checked for $L = 2 \ldots 11$

- We don't know why, but this suggest a new ABA type of regime near the poles too!

Open questions

- Derivation from perturbation theory?

  Which mechanism gives odd powers of g? Dipole evolution, Light-Ray operators?

- Massless mode physics? AdS_3? (in progress with Simon and Bogdan)
- Evolution kernel at higher order from data?
- Is there some physics at $L \to \infty$ as Landau-Lifshitz for usual case?
- Simplification of QSC near the corner points where BFKL meets DGLAP? Counting?
- DGLAP at non-integer spin?
- Accessing lower Horizontal Trajectories $S \sim -L + 1 - 2n, n \in \mathbb{N}$?
- Generalise to all states in other sectors and remove parity restrictions?
- Hexagons? SoV?

Thank you!

IGST 2025
London 21-25 July