

Quantum Affine Algebras and Cluster Algebras



David Hernandez and Bernard Leclerc

To Vyjayanthi Chari on her birthday

Abstract This article is an extended version of the minicourse given by the second author at the summer school of the conference *Interactions of quantum affine algebras with cluster algebras, current algebras and categorification*, held in June 2018 in Washington. The aim of the minicourse, consisting of three lectures, was to present a number of results and conjectures on certain monoidal categories of finite-dimensional representations of quantum affine algebras, obtained by exploiting the fact that their Grothendieck rings have the natural structure of a cluster algebra.

1 A Forerunner: Chari and Pressley's Paper on $U_q(\widehat{\mathfrak{sl}}_2)$

In [3], Chari and Pressley launched a systematic study of tensor categories of finite-dimensional representations of quantum affine algebras by investigating in detail the case of $U_q(\widehat{\mathfrak{sl}}_2)$. They gave a classification of simple objects, as well as a concrete description of them as tensor products of evaluation modules. They also gave a necessary and sufficient condition for such tensor products to be irreducible, and they described the composition factors of a reducible tensor product of two evaluation representations.

D. Hernandez

Université de Paris, Univ Paris Diderot, CNRS Institut de Mathématiques de Jussieu-Paris Rive Gauche UMR 7586, Paris, France

Institut Universitaire de France, Paris, France

e-mail: david.hernandez@imj-prg.fr

B. Leclerc (✉)

Normandie Univ, UNICAEN, CNRS, LMNO, Caen, France

e-mail: bernard.leclerc@unicaen.fr

© Springer Nature Switzerland AG 2021

J. Greenstein et al. (eds.), *Interactions of Quantum Affine Algebras with Cluster Algebras, Current Algebras and Categorification*, Progress in Mathematics 337, https://doi.org/10.1007/978-3-030-63849-8_2

In retrospect, these results may be seen as providing a cluster algebra structure on the Grothendieck ring of this category, predating by 10 years the invention of cluster algebras by Fomin and Zelevinsky [14]. We will therefore start our lectures by reviewing these results.

1.1 The Hopf Algebra $U_q(\widehat{\mathfrak{sl}}_2)$

Throughout the paper, we fix $q \in \mathbb{C}^*$ not a root of unity. The algebra $U_q(\widehat{\mathfrak{sl}}_2)$ is generated over \mathbb{C} by

$$E_0, F_0, K_0, K_0^{-1}, E_1, F_1, K_1, K_1^{-1},$$

subject to the following relations:

$$K_i K_i^{-1} = 1, \quad (1.1)$$

$$K_i K_j = K_j K_i, \quad (1.2)$$

$$K_i E_i K_i^{-1} = q^2 E_i, \quad (1.3)$$

$$K_i E_j K_i^{-1} = q^{-2} E_j, \quad (1.4)$$

$$K_i F_i K_i^{-1} = q^{-2} F_i, \quad (1.5)$$

$$K_i F_j K_i^{-1} = q^2 F_j, \quad (1.6)$$

$$E_i F_i - F_i E_i = \frac{K_i - K_i^{-1}}{q - q^{-1}}, \quad (1.7)$$

$$E_i F_j - F_j E_i = 0, \quad (1.8)$$

$$E_i^3 E_j - (q^2 + 1 + q^{-2}) E_i^2 E_j E_i + (q^2 + 1 + q^{-2}) E_i E_j E_i^2 - E_j E_i^3 = 0, \quad (1.9)$$

$$F_i^3 F_j - (q^2 + 1 + q^{-2}) F_i^2 F_j F_i + (q^2 + 1 + q^{-2}) F_i F_j F_i^2 - F_j F_i^3 = 0, \quad (1.10)$$

where $i \neq j$ are indices in $\{0, 1\}$. Moreover $U_q(\widehat{\mathfrak{sl}}_2)$ is a Hopf algebra, with comultiplication Δ given by

$$\Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i,$$

$$\Delta(F_i) = F_i \otimes 1 + K_i^{-1} \otimes F_i,$$

$$\Delta(K_i) = K_i \otimes K_i.$$

It follows that a tensor product of finite-dimensional $U_q(\widehat{\mathfrak{sl}}_2)$ -modules is again a $U_q(\widehat{\mathfrak{sl}}_2)$ -module.

1.2 Simple Finite-Dimensional $U_q(\widehat{\mathfrak{sl}}_2)$ -Modules

Let E, F, K, K^{-1} denote the generators of $U_q(\mathfrak{sl}_2)$. (They are subject to the same relations as (1.1), (1.3), (1.5) and (1.7).)

For every $a \in \mathbb{C}^*$, we have a surjective algebra homomorphism $ev_a : U_q(\widehat{\mathfrak{sl}}_2) \rightarrow U_q(\mathfrak{sl}_2)$ such that

$$ev_a(E_1) = E, \quad ev_a(F_1) = F, \quad ev_a(E_0) = q^{-1}aF, \quad ev_a(F_0) = qa^{-1}E.$$

Hence, every simple finite-dimensional $U_q(\mathfrak{sl}_2)$ -module M becomes a finite-dimensional $U_q(\widehat{\mathfrak{sl}}_2)$ -module $M(a)$ by pull-back through ev_a .

It is well-known that the simple finite-dimensional $U_q(\mathfrak{sl}_2)$ -modules¹ are classified by their dimension: for every $n \in \mathbb{Z}_{\geq 0}$ there is a unique (up to isomorphism) simple module V_n with dimension $n + 1$. Therefore, pulling back by the evaluation morphisms ev_a , we get for all $n \in \mathbb{Z}_{\geq 0}$ a one-parameter family of simple $U_q(\widehat{\mathfrak{sl}}_2)$ -modules $V_n(a)$ ($a \in \mathbb{C}^*$) with dimension $n + 1$. The representations $V_0(a)$ are all equal to the trivial representation. Otherwise, for $n \geq 1$, the simple modules $V_n(a)$ and $V_n(b)$ are non-isomorphic if $a \neq b$. The modules $V_n(a)$ are called *evaluation modules*.

Theorem 1.1 ([3]) *Every non-trivial simple finite-dimensional $U_q(\widehat{\mathfrak{sl}}_2)$ -module M is isomorphic to a tensor product of evaluation modules, that is,*

$$M \simeq V_{n_1}(a_1) \otimes \cdots \otimes V_{n_k}(a_k)$$

for some $k \in \mathbb{Z}_{>0}$, $n_1, \dots, n_k \in \mathbb{Z}_{>0}$, and $a_1, \dots, a_k \in \mathbb{C}^*$.

Note that tensor products of evaluation modules are *not* always irreducible. The next task is therefore to find some necessary and sufficient condition of irreducibility. In order to formulate this condition we introduce the notion of a *string*. This is a subset of \mathbb{C}^* of the form:

$$\Sigma(n, a) := \{aq^{-n+1}, aq^{-n+3}, \dots, aq^{n-1}\}, \quad (n \in \mathbb{Z}_{\geq 0}, a \in \mathbb{C}^*).$$

(In fact, $\Sigma(n, a)$ is nothing else than the set of roots of the Drinfeld polynomial of $V_n(a)$.) We say that two strings Σ_1 and Σ_2 are *in general position* if and only if

- (i) $\Sigma_1 \cup \Sigma_2$ is not a string, or
- (ii) $\Sigma_1 \subseteq \Sigma_2$ or $\Sigma_2 \subseteq \Sigma_1$.

¹In these lectures, we will only consider type I representations of quantum enveloping algebras. All representations can be obtained from the type I representations by twisting with some signs, see e.g. [4, §10.1].

Theorem 1.2 ([3]) *The tensor product $V_{n_1}(a_1) \otimes \cdots \otimes V_{n_k}(a_k)$ is irreducible if and only if for every $(i, j) \in \{1, \dots, k\}^2$ the strings $\Sigma(n_i, a_i)$ and $\Sigma(n_j, a_j)$ are in general position.*

Two strings which are not in general position are called *in special position*. What can we say about the tensor product $V_{n_1}(a_1) \otimes V_{n_2}(a_2)$ when the strings $\Sigma_1 := \Sigma(n_1, a_1)$ and $\Sigma_2 := \Sigma(n_2, a_2)$ are in special position? It turns out that in this case the tensor product always has two non-isomorphic composition factors. These two irreducible modules are, by Theorems 1.1 and 1.2, parametrized by two collections of strings in general position. Here is how to obtain them from Σ_1 and Σ_2 .

Because of Theorem 1.2 (i), $\Sigma_3 := \Sigma_1 \cup \Sigma_2$ is a string. Clearly, $\Sigma_4 := \Sigma_1 \cap \Sigma_2$ is also a string, contained in Σ_3 . Removing from Σ_3 the points of Σ_4 together with its two nearest neighbours, we are left with the union of two strings Σ_5 and Σ_6 . It is easy to see that the two pairs of strings (Σ_3, Σ_4) and (Σ_5, Σ_6) are in general position. For instance, if

$$\Sigma_1 = \{1, q^2, q^4, q^6, q^8\}, \quad \Sigma_2 = \{q^6, q^8, q^{10}, q^{12}, q^{14}, q^{16}\},$$

then

$$\Sigma_3 = \{1, q^2, q^4, q^6, q^8, q^{10}, q^{12}, q^{14}, q^{16}\}, \quad \Sigma_4 = \{q^6, q^8\}$$

and

$$\Sigma_5 = \{1, q^2\}, \quad \Sigma_6 = \{q^{12}, q^{14}, q^{16}\}.$$

We can then state:

Proposition 1.3 ([3, Proposition 4.9]) *Let Σ_1 and Σ_2 be two strings in special position. With the above notation, in the Grothendieck ring the following relation holds:*

$$[V(\Sigma_1) \otimes V(\Sigma_2)] = [V(\Sigma_3) \otimes V(\Sigma_4)] + [V(\Sigma_5) \otimes V(\Sigma_6)]. \quad (1.11)$$

Here, $V(\Sigma_i)$ denotes the evaluation module whose associated string is Σ_i .

1.3 Relation with Cluster Algebras

A reader familiar with the definition of a cluster algebra will recognize in (1.11) an *exchange relation*. Let us make this more precise. First note that if two strings are in special position, all their points belong to the same class in $\mathbb{C}^*/q^{2\mathbb{Z}}$, that is, they all are of the form aq^k for some fixed $a \in \mathbb{C}^*$ and some $k \in 2\mathbb{Z}$. This motivates the following definition:

Definition 1.4 ([22]) Let $a \in \mathbb{C}^*$ and $\ell \in \mathbb{Z}_{>0}$. Let $\mathcal{C}_{a,\ell}$ be the full subcategory of the category of finite-dimensional $U_q(\widehat{\mathfrak{sl}}_2)$ -modules whose objects V satisfy:

Every composition factor of V is of the form $V_{n_1}(a_1) \otimes \cdots \otimes V_{n_k}(a_k)$ where all strings $\Sigma(n_i, a_i)$ are contained in $S := \{a, aq^{-2}, \dots, aq^{-2\ell}\}$.

The category $\mathcal{C}_{a,\ell}$ depends only on ℓ up to isomorphism. We can therefore restrict ourselves to the case $a = 1$, and write $\mathcal{C}_{1,\ell} = \mathcal{C}_\ell$. Then Theorems 1.1, 1.2 and Proposition 1.3 yield the following reformulation:

Theorem 1.5 ([22]) *The category \mathcal{C}_ℓ is a monoidal category, and its Grothendieck ring $K_0(\mathcal{C}_\ell)$ has the structure of a cluster algebra of finite type A_ℓ in the Fomin–Zelevinsky classification. More precisely, the cluster variables of $K_0(\mathcal{C}_\ell)$ are the classes of the evaluation modules contained in \mathcal{C}_ℓ , the class $[V_{\ell+1}(q^{-\ell})]$ being the only frozen variable. The cluster monomials are equal to the classes of the simple modules in \mathcal{C}_ℓ . Two cluster variables are compatible (i.e. belong to the same cluster) if and only if the corresponding strings are in general position. Otherwise they form an exchange pair with exchange relation given by (1.11).*

1.4 How Can We Generalize?

In an attempt to extend these results from $U_q(\widehat{\mathfrak{sl}}_2)$ to other quantum affine algebras, Chari and Pressley introduced in [7] the notion of a *prime* module: this is a simple finite-dimensional module that cannot be factored as a tensor product of modules of smaller dimension. It follows from Theorem 1.1 that the prime $U_q(\widehat{\mathfrak{sl}}_2)$ -modules are precisely the evaluation modules. This is no longer true for $U_q(\widehat{\mathfrak{sl}}_3)$, and Chari and Pressley have constructed an infinite class of prime $U_q(\widehat{\mathfrak{sl}}_3)$ -modules which are not evaluation modules, see [7]. In view of this, the following problems naturally arise.

Let \mathfrak{g} be a simple Lie algebra over \mathbb{C} , and let $U_q(\widehat{\mathfrak{g}})$ denote the corresponding untwisted quantum affine algebra.

- (P1) What are the prime $U_q(\widehat{\mathfrak{g}})$ -modules?
- (P2) Which tensor products of prime $U_q(\widehat{\mathfrak{g}})$ -modules are simple?

It is known that Kirillov–Reshetikhin modules (see below Definition 2.6) are prime. The *minimal affinization* modules introduced by Chari and Pressley [5] as replacements for evaluation modules which do not exist outside type A , are also prime. But this is not a complete list as we shall see below. Problem (P2) is also completely open. In [22] we proposed to use cluster algebras to shed new light on these questions.

2 Reminder on Finite-Dimensional $U_q(\widehat{\mathfrak{g}})$ -Modules

2.1 Cartan Matrix

Let $C = (c_{ij})_{i,j \in I}$ be the Cartan matrix of \mathfrak{g} . There is a diagonal matrix $D = \text{diag}(d_i \mid i \in I)$ with entries in $\mathbb{Z}_{>0}$ such that the product

$$B = DC = (b_{ij})_{i,j \in I}$$

is symmetric. We normalize D so that $\min\{d_i \mid i \in I\} = 1$, and we put $t := \max\{d_i \mid i \in I\}$. Thus

$$t = \begin{cases} 1 & \text{if } C \text{ is of type } A_n, D_n, E_6, E_7 \text{ or } E_8, \\ 2 & \text{if } C \text{ is of type } B_n, C_n \text{ or } F_4, \\ 3 & \text{if } C \text{ is of type } G_2. \end{cases}$$

Example 2.1 The Lie algebra $\mathfrak{g} = \mathfrak{so}_7$, of type B_3 in the Cartan–Killing classification, has Cartan matrix

$$C = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}$$

We have $D = \text{diag}(2, 2, 1)$ and the symmetric matrix B is given by

$$B = \begin{pmatrix} 4 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 2 \end{pmatrix}$$

We denote by α_i ($i \in I$) the simple roots of \mathfrak{g} , and by ϖ_i ($i \in I$) the fundamental weights. They are related by

$$\alpha_i = \sum_{j \in I} c_{ji} \varpi_j. \tag{2.1}$$

2.2 Classification

By Cartan–Killing theory, the simple finite-dimensional \mathfrak{g} -modules are in one-to-one correspondence with their highest weight, an element of the positive cone of integral dominant weights:

$$P_+ := \bigoplus_{i \in I} \mathbb{N}\varpi_i.$$

We denote by $L(\lambda)$ the simple \mathfrak{g} -module with highest weight $\lambda \in P_+$.

Chari and Pressley have obtained a similar classification of simple finite-dimensional $U_q(\widehat{\mathfrak{g}})$ -modules. To formulate it, we introduce the cone

$$\widehat{P}_+ := \bigoplus_{i \in I, a \in \mathbb{C}^*} \mathbb{N}(\varpi_i, a)$$

of dominant loop-weights.

Theorem 2.2 ([4]) *Up to isomorphism, the simple finite-dimensional $U_q(\widehat{\mathfrak{g}})$ -modules are in one-to-one correspondence with their highest loop-weight, an element of \widehat{P}_+ . \square*

We denote by $L(\widehat{\lambda})$ the simple $U_q(\widehat{\mathfrak{g}})$ -module with highest loop-weight $\widehat{\lambda} \in \widehat{P}_+$.

Example 2.3 Let $\mathfrak{g} = \mathfrak{so}_8$, of type D_4 . Thus $I = \{1, 2, 3, 4\}$, where we denote by 3 the trivalent node of the Dynkin diagram. Then the \mathfrak{g} -module $L(\varpi_3) = \bigwedge^2 \mathbb{C}^8$ is of dimension 28.

For $a \in \mathbb{C}^*$, the $U_q(\widehat{\mathfrak{g}})$ -module $L(\varpi_3, a)$ has dimension 29. This is a *minimal affinization* of $L(\varpi_3)$ in the sense of [5].

2.3 q -Characters

Finite-dimensional \mathfrak{g} -modules M are characterized by their character

$$\chi(M) := \sum_{\mu \in P} \dim(M_\mu) e^\mu,$$

where $P := \bigoplus_{i \in I} \mathbb{Z}\varpi_i$ is the weight lattice, $M := \bigoplus_{\mu \in P} M_\mu$ is the weight space decomposition of M , and e^μ is a formal exponential. So $\chi(M)$ is a Laurent polynomial in the variables $y_i := e^{\varpi_i}$, ($i \in I$).

Similarly, finite-dimensional $U_q(\widehat{\mathfrak{g}})$ -modules \widehat{M} have a loop-weight space decomposition

$$\widehat{M} := \bigoplus_{\widehat{\mu} \in \widehat{P}} \widehat{M}_{\widehat{\mu}},$$

where $\widehat{P} := \bigoplus_{i \in I, a \in \mathbb{C}^*} \mathbb{Z}(\varpi_i, a)$. Frenkel–Reshetikhin introduced the q -character

$$\chi_q(\widehat{M}) := \sum_{\widehat{\mu} \in \widehat{P}} \dim(\widehat{M}_{\widehat{\mu}}) e^{\widehat{\mu}},$$

a Laurent polynomial in the variables $Y_{i,a} := e^{(\varpi_i, a)}$, ($i \in I, a \in \mathbb{C}^*$).

Theorem 2.4 ([13]) Let \widehat{M} and \widehat{N} be two finite-dimensional $U_q(\widehat{\mathfrak{g}})$ -modules. The following are equivalent:

- (i) $\chi_q(\widehat{M}) = \chi_q(\widehat{N})$;
- (ii) $[\widehat{M}] = [\widehat{N}]$ in the Grothendieck ring $K_0(-\text{mod}(U_q(\widehat{\mathfrak{g}})))$;
- (iii) \widehat{M} and \widehat{N} have the same composition factors with the same multiplicities.

In particular, q -characters characterize simple $U_q(\widehat{\mathfrak{g}})$ -modules up to isomorphism.

Note also that, because of Theorem 2.4, χ_q descends to an injective ring homomorphism from $K_0(-\text{mod}(U_q(\widehat{\mathfrak{g}})))$ to the ring of Laurent polynomials $\mathbb{Z}[Y_{i,a}^{\pm 1} \mid i \in I, a \in \mathbb{C}^*]$.

Example 2.5 Let $\mathfrak{g} = \mathfrak{sl}_2$, of type A_1 . Then, as is well-known, we have

$$\begin{aligned}\chi(L(\varpi_1)) &= \chi(V_1) = y_1 + y_1^{-1} \\ \chi(L(2\varpi_1)) &= \chi(V_2) = y_1^2 + 1 + y_1^{-2} \\ &\dots \quad \dots\end{aligned}$$

On the other hand, for any $a \in \mathbb{C}^*$,

$$\begin{aligned}\chi_q(L(\varpi_1, a)) &= \chi_q(V_1(a)) = Y_{1,a} + Y_{1,aq^2}^{-1} \\ \chi_q(L(2(\varpi_1, a))) &= Y_{1,a}^2 + 2Y_{1,a}Y_{1,aq^2}^{-1} + Y_{1,aq^2}^{-2} \\ \chi_q(L((\varpi_1, a) + (\varpi_1, aq^2))) &= \chi_q(V_2(aq)) \\ &= Y_{1,a}Y_{1,aq^2} + Y_{1,a}Y_{1,aq^4}^{-1} + Y_{1,aq^2}^{-1}Y_{1,aq^4}^{-1}.\end{aligned}$$

This shows that $L((\varpi_1, a) + (\varpi_1, aq^2))$ is a minimal affinization of $L(2\varpi_1)$, but $L(2(\varpi_1, a))$ is not.

Definition 2.6 For $i \in I, k \in \mathbb{N}, a \in \mathbb{C}^*$, set

$$\widehat{\lambda}_{k,a}^{(i)} := \sum_{j=0}^{k-1} (\varpi_i, aq^{2di j}) \in \widehat{P}_+.$$

The simple $U_q(\widehat{\mathfrak{g}})$ -module $L(\widehat{\lambda}_{k,a}^{(i)})$ is called a *Kirillov–Reshetikhin module*. We often write for short $W_{k,a}^{(i)} = L(\widehat{\lambda}_{k,a}^{(i)})$. The modules $W_{1,a}^{(i)} = L((\varpi_i, a))$ are the *fundamental* $U_q(\widehat{\mathfrak{g}})$ -modules.

2.4 T -Systems

With the quantum affine algebra $U_q(\widehat{\mathfrak{g}})$ is associated a system of difference equations called a T -system [33]. Its unknowns are denoted by

$$T_{k,r}^{(i)}, \quad (i \in I, k \in \mathbb{N}, r \in \mathbb{Z}).$$

We fix the initial boundary condition

$$T_{0,r}^{(i)} = 1, \quad (i \in I, r \in \mathbb{Z}). \quad (2.2)$$

If \mathfrak{g} is of type A_n, D_n, E_n , the T -system equations are

$$T_{k,r+1}^{(i)} T_{k,r-1}^{(i)} = T_{k-1,r+1}^{(i)} T_{k+1,r-1}^{(i)} + \prod_{j: c_{ij}=-1} T_{k,r}^{(j)}, \quad (i \in I, k \geq 1, r \in \mathbb{Z}). \quad (2.3)$$

If \mathfrak{g} is not of simply-laced type, the T -system equations are more complicated. They can be written in the form

$$T_{k,r+d_i}^{(i)} T_{k,r-d_i}^{(i)} = T_{k-1,r+d_i}^{(i)} T_{k+1,r-d_i}^{(i)} + S_{k,r}^{(i)}, \quad (i \in I, k \geq 1, r \in \mathbb{Z}), \quad (2.4)$$

where $S_{k,r}^{(i)}$ is defined as follows. If $d_i \geq 2$, then

$$S_{k,r}^{(i)} = \prod_{j: c_{ji}=-1} T_{k,r}^{(j)} \prod_{j: c_{ji} \leq -2} T_{d_i k, r-d_i+1}^{(j)}. \quad (2.5)$$

If $d_i = 1$ and $t = 2$, then

$$S_{k,r}^{(i)} = \begin{cases} \prod_{j: c_{ij}=-1} T_{k,r}^{(j)} \prod_{j: c_{ij}=-2} T_{l,r}^{(j)} T_{l,r+2}^{(j)}, & \text{if } k = 2l, \\ \prod_{j: c_{ij}=-1} T_{k,r}^{(j)} \prod_{j: c_{ij}=-2} T_{l+1,r}^{(j)} T_{l,r+2}^{(j)} & \text{if } k = 2l + 1. \end{cases} \quad (2.6)$$

Finally, if $d_i = 1$ and $t = 3$, that is, if \mathfrak{g} is of type G_2 , denoting by j the other vertex we have $d_j = 3$ and

$$S_{k,r}^{(i)} = \begin{cases} T_{l,r}^{(j)} T_{l,r+2}^{(j)} T_{l,r+4}^{(j)} & \text{if } k = 3l, \\ T_{l+1,r}^{(j)} T_{l,r+2}^{(j)} T_{l,r+4}^{(j)} & \text{if } k = 3l + 1, \\ T_{l+1,r}^{(j)} T_{l+1,r+2}^{(j)} T_{l,r+4}^{(j)} & \text{if } k = 3l + 2. \end{cases} \quad (2.7)$$

Example 2.7 Let \mathfrak{g} be of type B_2 . The Cartan matrix is

$$C = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$$

and we have $d_1 = 2$ and $d_2 = 1$. The T -system reads

$$\begin{aligned} T_{k,r+2}^{(1)} T_{k,r-2}^{(1)} &= T_{k-1,r+2}^{(1)} T_{k+1,r-2}^{(1)} + T_{2k,r-1}^{(2)}, & (k \geq 1, r \in \mathbb{Z}), \\ T_{2l,r+1}^{(2)} T_{2l,r-1}^{(2)} &= T_{2l-1,r+1}^{(2)} T_{2l+1,r-1}^{(2)} + T_{l,r}^{(1)} T_{l,r+2}^{(1)}, & (l \geq 1, r \in \mathbb{Z}), \\ T_{2l+1,r+1}^{(2)} T_{2l+1,r-1}^{(2)} &= T_{2l,r+1}^{(2)} T_{2l+2,r-1}^{(2)} + T_{l+1,r}^{(1)} T_{l,r+2}^{(1)}, & (l \geq 0, r \in \mathbb{Z}). \end{aligned}$$

It was conjectured in [33], and proved in [39] (for \mathfrak{g} of type A, D, E) and [21] (general case), that the q -characters of the Kirillov–Reshetikhin modules of $U_q(\widehat{\mathfrak{g}})$ satisfy the corresponding T -system. More precisely, we have

Theorem 2.8 ([21, 38]) *For $i \in I$, $k \in \mathbb{N}$, $r \in \mathbb{Z}$,*

$$T_{k,r}^{(i)} = \chi_q \left(W_{k,q^r}^{(i)} \right),$$

is a solution of the T -system in the ring $\mathbb{Z} \left[Y_{i,q^r}^{\pm 1} \mid (i, r) \in I \times \mathbb{Z} \right]$. Equivalently, by Theorem 2.4,

$$T_{k,r}^{(i)} = \left[W_{k,q^r}^{(i)} \right],$$

is a solution of the T -system in the Grothendieck ring $K_0(-\text{mod}(U_q(\widehat{\mathfrak{g}})))$.

Remark 3.9

- (i) For $\mathfrak{g} = \mathfrak{sl}_2$, Theorem 2.8 is a particular case of Proposition 1.3.
- (ii) Theorem 2.8 allows to calculate inductively q -characters of Kirillov–Reshetikhin modules in terms of q -characters of fundamental modules. Note, however, that it is not straightforward to compute the q -characters of the fundamental modules in type E_8 or F_4 , say. An algorithm has been obtained by Frenkel and Mukhin [12]. Another one, based on cluster mutation, is described in [23].
- (iii) The T -system in the Grothendieck ring comes from a non-split short exact sequence

$$0 \rightarrow S_{k,r}^{(i)} \rightarrow W_{k,q^{r-d_i}}^{(i)} \otimes W_{k,q^{r+d_i}}^{(i)} \rightarrow W_{k-1,q^{r+d_i}}^{(i)} \otimes W_{k+1,q^{r+d_i}}^{(i)} \rightarrow 0,$$

where the module $S_{k,r}^{(i)}$ is defined as the tensor product of Kirillov–Reshetikhin modules associated with the factors of $S_{k,r}^{(i)}$ above (it does not depend on the order of the tensor product up to isomorphism). The representation $W_{k,q^{r-d_i}}^{(i)} \otimes$

$W_{k,q^{r+d_i}}^{(i)}$ is of length 2. By a general result of Chari on tensor products of Kirillov–Reshetikhin modules [2], it is cyclic generated by the tensor product of the highest weight vectors and so it is indecomposable.

3 Quivers, Subcategories, and Cluster Algebras

Following [25], we attach an infinite quiver to $U_q(\widehat{\mathfrak{g}})$, and we define some subcategories of the category of finite-dimensional $U_q(\widehat{\mathfrak{g}})$ -modules. We then introduce cluster algebras corresponding to finite segments of this infinite quiver.

3.1 Quivers

Put $\widetilde{V} = I \times \mathbb{Z}$. We introduce a quiver $\widetilde{\Gamma}$ with vertex set \widetilde{V} . Recall the symmetric matrix $B = (b_{ij})_{i,j \in I}$ of Sect. 2.1. The arrows of $\widetilde{\Gamma}$ are given by

$$((i, r) \rightarrow (j, s)) \iff (b_{ij} \neq 0 \text{ and } s = r + b_{ij}).$$

It is easy to check that the oriented graph $\widetilde{\Gamma}$ has two isomorphic connected components. We pick one of them and call it Γ . The vertex set of Γ is denoted by V . Examples in type A_3 and B_2 are shown in Fig. 1.

3.2 Subcategories

First, using the vertex set V , we introduce

$$\widehat{P}_{+, \mathbb{Z}} := \bigoplus_{(i,r) \in V} \mathbb{N}(\varpi_i, q^{r+d_i}),$$

a discrete subset of the positive cone \widehat{P}_+ of loop-weights.

Definition 3.1 Let $\mathcal{C}_{\mathbb{Z}}$ be the full subcategory of the category of finite-dimensional $U_q(\widehat{\mathfrak{g}})$ -modules whose objects M satisfy:

Every composition factor of M is of the form $L(\widehat{\lambda})$ with $\widehat{\lambda} \in \widehat{P}_{+, \mathbb{Z}}$.

By [23], $\mathcal{C}_{\mathbb{Z}}$ is a monoidal subcategory, i.e. it is stable under tensor products. Moreover, it is known that every simple finite-dimensional $U_q(\widehat{\mathfrak{g}})$ -module can be written as a tensor product of simple objects of $\mathcal{C}_{\mathbb{Z}}$ with spectral shifts. Therefore it is enough to study the simple objects of $\mathcal{C}_{\mathbb{Z}}$.

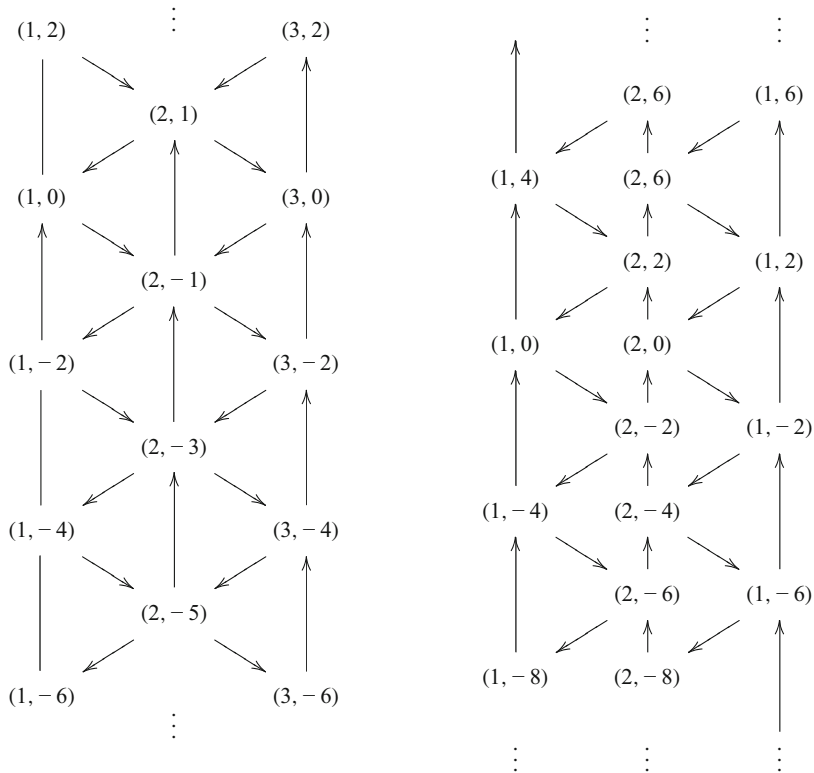


Fig. 1 The quivers Γ in type A_3 and B_2

In order to relate $\mathcal{C}_{\mathbb{Z}}$ with cluster algebras of finite rank, we need to “truncate” it, as we did in Sect. 1.3 for $\mathfrak{g} = \mathfrak{sl}_2$. Fix $\ell \in \mathbb{N}$, and put

$$\widehat{P}_{+, \ell} := \bigoplus_{(i,r) \in V, -2\ell-1 \leq r+d_i \leq 0} \mathbb{N}(\varpi_i, q^{r+d_i}).$$

Definition 3.2 Let \mathcal{C}_{ℓ} be the full subcategory of the category of finite-dimensional $U_q(\widehat{\mathfrak{g}})$ -modules whose objects M satisfy:

Every composition factor of M is of the form $L(\widehat{\lambda})$ with $\widehat{\lambda} \in \widehat{P}_{+, \ell}$.

Again by [23], \mathcal{C}_{ℓ} is a monoidal category and its Grothendieck ring is a polynomial ring in finitely many variables, namely, the classes of the fundamental modules contained in \mathcal{C}_{ℓ} :

$$K_0(\mathcal{C}_{\ell}) = \mathbb{Z} \left[[L((\varpi_i, q^{r+d_i})) \mid (i, r) \in V, -2\ell - 1 \leq r + d_i \leq 0] \right].$$

3.3 Cluster Algebras

We refer the reader to [16] and [19] for an introduction to cluster algebras, and for any undefined terminology.

Let Γ_ℓ denote the full subquiver of Γ with vertex set

$$V_\ell := \{(i, r) \in V \mid -2\ell - 1 \leq r + d_i \leq 0\}.$$

Let $\mathbf{z}_\ell := \{z_{(i,r)} \mid (i, r) \in V_\ell\}$ be a set of commuting indeterminates indexed by V_ℓ . The pair $(\mathbf{z}_\ell, \Gamma_\ell)$ can be regarded as a *seed*, in the sense of [16].

Definition 3.3 Let $\mathcal{A}_\ell \subset \mathbb{Q}(\mathbf{z}_\ell)$ be the cluster algebra with initial seed $(\mathbf{z}_\ell, \Gamma_\ell)$, where we consider the variables $z_{i,r}$ with $r - d_i < -2\ell - 1$ as *frozen variables*. \square

4 Main Conjecture

4.1 Statements and Examples

The category \mathcal{C}_ℓ and the cluster algebra \mathcal{A}_ℓ are related as follows:

Theorem 4.1 ([23]) For $(i, r) \in V_\ell$ put $m_{i,r} := \max\{k \mid r + (2k + 1)d_i \leq 0\} + 1$. The assignment $z_{i,r} \mapsto \left[W_{m_{i,r}, q^{r+d_i}}^{(i)} \right]$ extends to a ring isomorphism $\iota_\ell : \mathcal{A}_\ell \rightarrow K_0(\mathcal{C}_\ell)$.

We can now formulate our main conjecture from [22, 23].

Conjecture 4.2

- (i) The isomorphism ι_ℓ maps the subset of cluster monomials of \mathcal{A}_ℓ into the subset of classes of simple objects of \mathcal{C}_ℓ .
- (ii) The isomorphism ι_ℓ maps the subset of cluster variables of \mathcal{A}_ℓ into the subset of classes of *prime* simple objects of \mathcal{C}_ℓ .

In the situation of Theorem 4.1 and Conjecture 4.2, we say that \mathcal{C}_ℓ is a *monoidal categorification* of the cluster algebra \mathcal{A}_ℓ [22]. We illustrate the conjecture with simple examples.

Example 4.3 We take $\mathfrak{g} = \mathfrak{sl}_4$, of type A_3 , and we choose $\ell = 1$. The isomorphism ι_1 is displayed in Fig. 2, which shows the image of the initial seed of \mathcal{A}_1 . For instance, $\iota_1(z_{2,-1}) = \left[W_{1,q^0}^{(2)} \right]$. The 3 frozen variables are marked with a box.

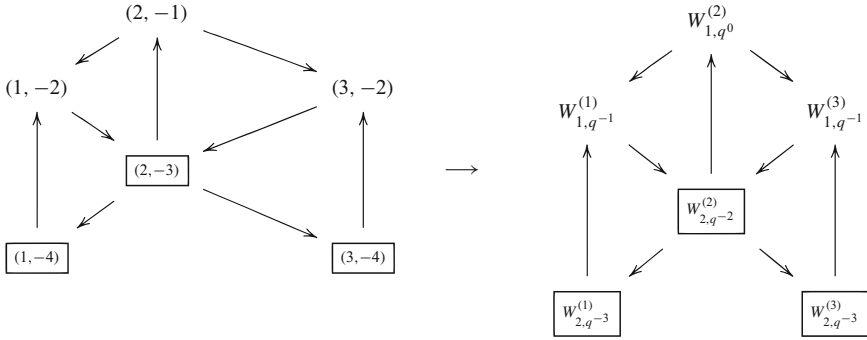


Fig. 2 The isomorphism ι_ℓ in type A_3 for $\ell = 1$

The cluster variable $z_{2,-1}^*$ obtained by mutating $z_{2,-1}$ in this initial seed is given by the exchange relation:

$$z_{2,-1} z_{2,-1}^* = z_{2,-3} + z_{1,-2} z_{3,-2},$$

which translates under ι_1 into the T -system equation:

$$\left[W_{1,q^0}^{(2)} \right] \left[W_{1,q^{-2}}^{(2)} \right] = \left[W_{2,q^{-2}}^{(2)} \right] + \left[W_{1,q^{-1}}^{(1)} \right] \left[W_{1,q^{-1}}^{(3)} \right].$$

Thus, $\iota_1(z_{2,-1}^*) = \left[W_{1,q^{-2}}^{(2)} \right]$.

In this case, \mathcal{A}_1 is a cluster algebra of finite type A_3 in the Fomin–Zelevinsky classification, and Conjecture 4.2 is proved [22]. Moreover, since \mathcal{A}_1 has finite type, the inclusions of Conjecture 4.2 are in fact bijections. The algebra \mathcal{A}_1 has 9 cluster variables plus 3 frozen variables. The prime simple modules of \mathcal{C}_1 corresponding to cluster variables are

$$\begin{aligned} W_{1,q^{-1}}^{(1)} &= L((\varpi_1, q^{-1})), & W_{1,q^{-2}}^{(2)} &= L((\varpi_2, q^{-2})), & W_{1,q^{-1}}^{(3)} &= L((\varpi_3, q^{-1})), \\ W_{1,q^{-3}}^{(1)} &= L((\varpi_1, q^{-3})), & W_{1,q^0}^{(2)} &= L((\varpi_2, q^0)), & W_{1,q^{-3}}^{(3)} &= L((\varpi_3, q^{-3})), \\ & & L((\varpi_1, q^{-3}) + (\varpi_2, q^0)), & & L((\varpi_1, q^{-3}) + (\varpi_2, q^0) + (\varpi_3, q^{-3})), \\ & & & & L((\varpi_2, q^0) + (\varpi_3, q^{-3})). \end{aligned}$$

There are 6 fundamental modules, and 2 minimal affinizations, but the 70-dimensional module $L((\varpi_1, q^{-3}) + (\varpi_2, q^0) + (\varpi_3, q^{-3}))$, which restricts to $L(\varpi_1 + \varpi_2 + \varpi_3) \oplus L(\varpi_2)$ as a $U_q(\mathfrak{sl}_4)$ -module, is *not* a minimal affinization.

By [15], there is a bijection between the set of cluster variables of \mathcal{A}_1 and the set of almost positive roots of a root system of type A_3 . A natural bijection was given in [22]:

$$\begin{aligned}
 L((\varpi_1, q^{-1})) &\leftrightarrow -\alpha_1 \\
 L((\varpi_2, q^{-2})) &\leftrightarrow -\alpha_2 \\
 L((\varpi_3, q^{-1})) &\leftrightarrow -\alpha_3 \\
 L((\varpi_1, q^{-3})) &\leftrightarrow \alpha_1 \\
 L((\varpi_2, q^0)) &\leftrightarrow \alpha_2 \\
 L((\varpi_3, q^{-3})) &\leftrightarrow \alpha_3 \\
 L((\varpi_1, q^{-3}) + (\varpi_2, q^0)) &\leftrightarrow \alpha_1 + \alpha_2 \\
 L((\varpi_2, q^0) + (\varpi_3, q^{-3})) &\leftrightarrow \alpha_2 + \alpha_3 \\
 L((\varpi_1, q^{-3}) + (\varpi_2, q^0) + (\varpi_3, q^{-3})) &\leftrightarrow \alpha_1 + \alpha_2 + \alpha_3
 \end{aligned}$$

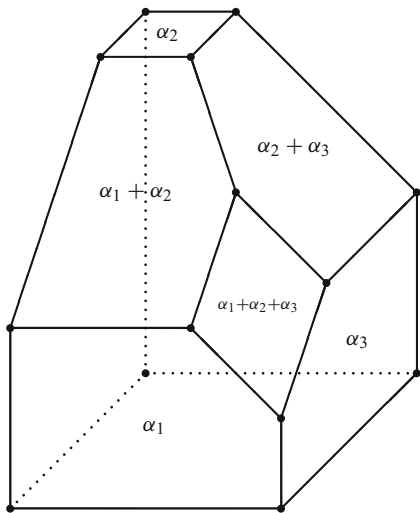
Following [15] and using this bijection one can read the 14 clusters of $\mathcal{A}_1 \cong K_0(\mathcal{C}_1)$ from the associahedron of Fig. 3. Every cluster variable corresponds to a face, indicated by the attached almost positive root (the negative simple root $-\alpha_i$ labels the unique rear face parallel to the face labelled by α_i). Every cluster corresponds to a vertex and consists of the 3 faces adjacent to it. For instance, there is a cluster

$$\{\alpha_1, \alpha_3, -\alpha_2\} \equiv \{L((\varpi_1, q^{-3})), L((\varpi_3, q^{-3})), L((\varpi_2, q^{-2}))\}.$$

The neat final result is that every simple module of \mathcal{C}_1 is a tensor product of prime simple modules belonging to a single cluster, and of frozen simple modules (corresponding to the frozen variables of \mathcal{A}_1).

Example 4.4 We take $\mathfrak{g} = \mathfrak{so}_5$, of type B_2 , and we choose $\ell = 2$. The isomorphism ι_2 is displayed in Fig. 4, which shows the image of the initial seed of \mathcal{A}_2 . For instance, $\iota_2(z_2, -2) = [W_{1, q^{-1}}^{(2)}]$. The 3 frozen variables are marked with a box.

Fig. 3 The associahedron in type A_3



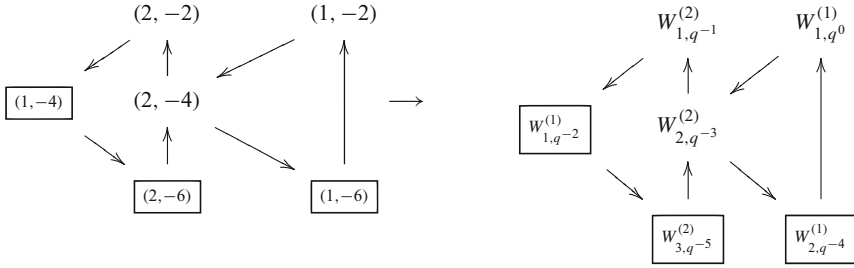


Fig. 4 The isomorphism ν_ℓ in type B_2 for $\ell = 2$

Again, \mathcal{A}_2 has finite cluster type A_3 . So \mathcal{C}_2 has 12 prime objects, namely, the 6 Kirillov–Reshetikhin modules of the initial seed, together with

$$W_{1,q^{-4}}^{(1)}, W_{1,q^{-3}}^{(2)}, W_{1,q^{-5}}^{(2)}, W_{2,q^{-5}}^{(2)}, L((\varpi_1, q^0) + (\varpi_2, q^{-5})),$$

$$L((\varpi_1, q^0) + (\varpi_2, q^{-3}) + (\varpi_2, q^{-5})).$$

A bijection between the highest loop-weights of the unfrozen primes and the almost positive roots of type A_3 , allowing to determine the clusters using the associahedron as in Example 4.3, is for instance:

(ϖ_1, q^{-4})	$\Leftrightarrow -\alpha_1$
$(\varpi_2, q^{-3}) + (\varpi_2, q^{-1})$	$\Leftrightarrow -\alpha_2$
(ϖ_2, q^{-1})	$\Leftrightarrow -\alpha_3$
(ϖ_1, q^0)	$\Leftrightarrow \alpha_1$
(ϖ_2, q^{-5})	$\Leftrightarrow \alpha_2$
(ϖ_2, q^{-3})	$\Leftrightarrow \alpha_3$
$(\varpi_1, q^0) + (\varpi_2, q^{-5})$	$\Leftrightarrow \alpha_1 + \alpha_2$
$(\varpi_2, q^{-3}) + (\varpi_2, q^{-5})$	$\Leftrightarrow \alpha_2 + \alpha_3$
$(\varpi_1, q^0) + (\varpi_2, q^{-3}) + (\varpi_2, q^{-5})$	$\Leftrightarrow \alpha_1 + \alpha_2 + \alpha_3.$

Example 4.5 We take \mathfrak{g} of type G_2 , and we choose $\ell = 3$. The isomorphism ν_3 is displayed in Fig. 5, which shows the image of the initial seed of \mathcal{A}_3 . There are 4 frozen variables. \mathcal{A}_3 has finite cluster type A_4 . So \mathcal{C}_3 has 18 prime objects, namely, the 8 Kirillov–Reshetikhin modules of the initial seed, together with

$$W_{1,q^{-6}}^{(1)}, W_{1,q^{-3}}^{(2)}, W_{1,q^{-5}}^{(2)}, W_{2,q^{-5}}^{(2)}, W_{2,q^{-7}}^{(2)}, W_{3,q^{-7}}^{(2)}, W_{1,q^{-7}}^{(2)},$$

$$L((\varpi_1, q^0) + (\varpi_2, q^{-7})), L((\varpi_1, q^0) + (\varpi_2, q^{-5}) + (\varpi_2, q^{-7})),$$

$$L((\varpi_1, q^0) + (\varpi_2, q^{-3}) + (\varpi_2, q^{-5}) + (\varpi_2, q^{-7})).$$

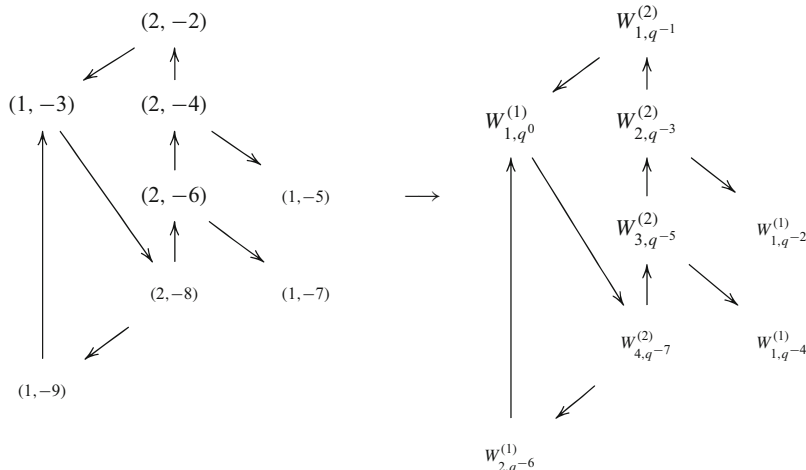


Fig. 5 The isomorphism ι_ℓ in type G_2 for $\ell = 3$

Remark 4.6

- (i) There is a similar Conjecture in which the finite quiver Γ_ℓ is replaced by the semi-infinite quiver Γ^- with vertex set

$$V^- := \{(i, r) \in V \mid r + d_i \leq 0\},$$

and the category \mathcal{C}_ℓ is replaced by the category \mathcal{C}^- of finite-dimensional $U_q(\widehat{\mathfrak{g}})$ -modules whose composition factors are of the form $L(\widehat{\lambda}), \widehat{\lambda} \in \bigoplus_{(i,r) \in V^-} \mathbb{N}(\varpi_i, q^{r+d_i})$, see [23].

- (ii) There is also a similar Conjecture in which the finite quiver Γ_ℓ is replaced by the doubly-infinite quiver Γ . In that case, the corresponding category is no longer a subcategory of the category of finite-dimensional $U_q(\widehat{\mathfrak{g}})$ -modules. We have to consider a certain subcategory of the category \mathcal{O} of (possibly infinite-dimensional) representations over a quantum Borel subalgebra $U_q(\widehat{\mathfrak{b}})$ of $U_q(\widehat{\mathfrak{g}})$, see [26]. The category $\mathcal{C}_\mathbb{Z}$ of Definition 3.1 can be regarded as a subcategory of this category of $U_q(\widehat{\mathfrak{b}})$ -modules. The initial seed consists of the classes of prefundamental representations, which are simple infinite-dimensional modules of $U_q(\widehat{\mathfrak{b}})$ which cannot be extended to $U_q(\widehat{\mathfrak{g}})$ -modules.

4.2 What Is Known?

4.2.1 Part (ii)

The difficult part of Conjecture 4.2 is (i). If (i) is known, then (ii) follows from a result of [17] which says that if a cluster algebra is a factorial ring, then every cluster variable is a prime element of this ring.

4.2.2 First Evidences

As explained in Sect. 1, when $\mathfrak{g} = \mathfrak{sl}_2$ Conjecture 4.2 follows from [3].

In [22], (i) was proved for type A_n and D_4 when $\ell = 1$. The proof was algebraic and combinatorial, and certain parts of the proof were more general.

In [38], Nakajima proved (i) for types A, D, E and $\ell = 1$, using the geometric approach to $U_q(\widehat{\mathfrak{g}})$ via quiver varieties. (An introduction to this proof is presented in [35].)

A variant of Conjecture 4.2 for type A_n and D_n when $\ell = 1$ was proved in [23]. This involves finite subquivers of Γ different from Γ_ℓ . Very recently Brito and Chari [1] generalized the results of [23] in type A using purely representation theoretic methods.

4.2.3 Proof in Simply-Laced Cases

In [40], Qin gave a proof of (i) for types A, D, E and arbitrary ℓ . The proof also relies on the geometric approach, and uses the t -deformation of $K_0(\mathcal{C}_\ell)$ introduced by Varagnolo–Vasserot and Nakajima in terms of quiver varieties.

4.2.4 Connection with Quiver Hecke Algebras

In type A, D, E , for $\ell = h/2 - 1$, where h is the Coxeter number supposed to be even, there is another approach as follows. In [24], we have shown that there is a ring isomorphism

$$i : \mathbb{C} \otimes_{\mathbb{Z}} K_0(\mathcal{C}_{h/2-1}) \longrightarrow \mathbb{C}[N],$$

where N is a maximal unipotent subgroup of a simple Lie group G with $\text{Lie}(G) = \mathfrak{g}$. The ring of polynomial functions $\mathbb{C}[N]$ has a well-known cluster algebra structure, and the isomorphism i transports the cluster structure of $\mathcal{A}_{h/2-1} \cong \mathbb{C} \otimes_{\mathbb{Z}} K_0(\mathcal{C}_{h/2-1})$ to the cluster structure of $\mathbb{C}[N]$. We have shown that the isomorphism i maps the basis of $\mathbb{C} \otimes_{\mathbb{Z}} K_0(\mathcal{C}_{h/2-1})$ consisting of classes of simple objects to the dual canonical basis (or upper global basis) of $\mathbb{C}[N]$. Therefore to prove (i) in this case amounts to prove:

(i') The cluster monomials of $\mathbb{C}[N]$ form a subset of the dual canonical basis of $\mathbb{C}[N]$.

This was proved by Kang et al. [30]. They used the categorification of the dual canonical basis of $\mathbb{C}[N]$ by simple objects of a category \mathcal{H} of graded modules over quiver Hecke algebras. This raises the question of a relation between the two categories \mathcal{H} and $\mathcal{C}_{h/2-1}$. In [28], Kang, Kashiwara and Kim constructed a functor from \mathcal{H} to $\mathcal{C}_{h/2-1}$ inducing the isomorphism i^{-1} at the level of Grothendieck rings. In type A this can be regarded as a variant of the quantum affine Schur–Weyl

duality of Chari–Pressley and Ginzburg–Reshetikhin–Vasserot [6, 18]. Recently, Fujita [10, 11] proved that the KKK-functor is in fact an equivalence of categories.

4.2.5 Non-Simply-Laced Cases

For non-simply-laced types, let us start with the example of the category \mathcal{C}_2 in type B_2 discussed in Example 4.4. Comparing with the category \mathcal{C}_1 in type A_3 discussed in Example 4.3, we observe that not only the cluster algebras \mathcal{A}_1 in type A_3 and \mathcal{A}_2 in type B_2 have the same cluster type A_3 and the same numbers of frozen variables, but also the quivers in Figs. 3 and 4 are mutation-equivalent. This is illustrated in Fig. 6 with the mutation sequence at nodes $(3, -2)$, $(2, -1)$, $(1, -2)$. Arrows between frozen vertices may be omitted as this does not change the cluster algebra structure. Hence we get a distinguished isomorphism between Grothendieck rings

$$K_0(\mathcal{C}_1, A_3) \simeq K_0(\mathcal{C}_2, B_2)$$

which is compatible with the cluster algebra structures. This last point is the most important since we know already that the two rings are isomorphic to the polynomial ring in 6 variables.

This example is the first instance of a family of isomorphisms of cluster algebras

$$K_0(\mathcal{C}_{h/2-1, A_{2n-1}}) \simeq K_0(\mathcal{C}'_{B_n})$$

obtained by the first author in a joint work with Hironori Oya [27]. These are distinguished isomorphisms between the Grothendieck rings of the type A_{2n-1} categories $\mathcal{C}_{h/2-1, A_{2n-1}}$ already mentioned in Sect. 4.2.4 above, and remarkable subcategories \mathcal{C}'_{B_n} of finite-dimensional representations in type B_n . The proof is established at the level of quantum cluster algebras, in order to demonstrate a conjecture on quantum Grothendieck rings and related analogs of Kazhdan–Lusztig polynomials formulated in [20].

Using a completely different method based on functors from categories of representations of quiver Hecke algebras, Kashiwara–Kim–Oh and Kashiwara–Oh [31, 32] constructed isomorphisms of Grothendieck rings in types A_{2n-1}/B_n preserving the classes of simple modules. In fact, these match the distinguished isomorphisms obtained from cluster algebra structures in [27]. Hence the cluster algebra isomorphisms also preserve classes of simple modules. For instance, in the example above, the bijection between prime simples can be directly written from the bijection with almost positive roots in Examples 4.3 and 4.4. Consequently, combining these results, the analogue of Conjecture 4.2 for the subcategories \mathcal{C}'_{B_n} holds.

It is expected that this approach will be extended to larger categories and to more general types.

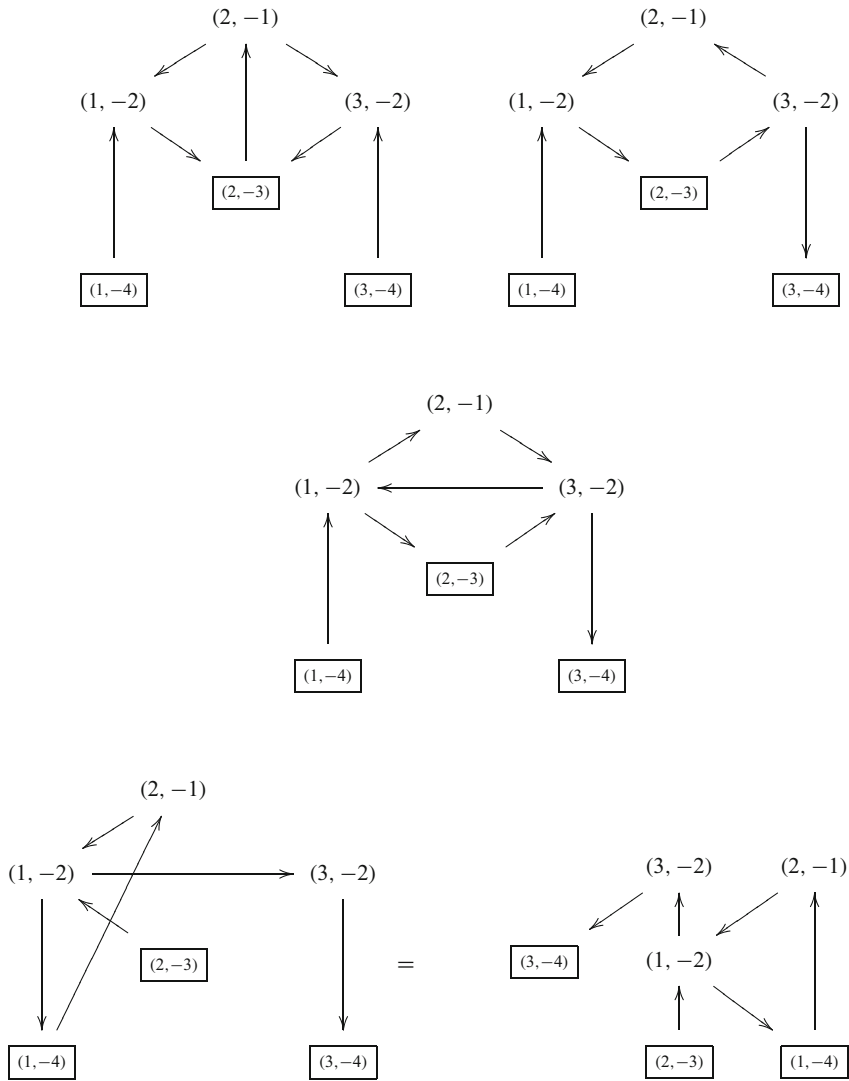


Fig. 6 Mutations : from \mathcal{A}_1 in type A_3 to \mathcal{A}_2 in type B_2

4.2.6 Real Modules

When the cluster algebra \mathcal{A}_ℓ is not of finite cluster type, cluster monomials do not span the vector space $\mathbb{C} \otimes \mathcal{A}_\ell$. This raises the question of describing the simple objects of \mathcal{C}_ℓ whose class in the Grothendieck ring is a cluster monomial.

Conjecture 4.7 ([22, 23]) The class of a simple object S of \mathcal{C}_ℓ in the Grothendieck ring is a cluster monomial if and only if S is a *real simple object*, that is, if and only if $S \otimes S$ is simple.

Let us assume that Conjecture 4.2 (i) holds. Then one direction of Conjecture 4.7 is obvious: the square of a cluster monomial is clearly a cluster monomial. The converse is wide open.

Real $U_q(\widehat{\mathfrak{g}})$ -modules have interesting properties. For instance Kang, Kashiwara, Kim and Oh proved the following theorem, which was conjectured in type A in [34].

Theorem 4.8 ([29]) *If S_1 and S_2 are two simple $U_q(\widehat{\mathfrak{g}})$ -modules, and one of them (at least) is real, then $S_1 \otimes S_2$ has a simple socle and a simple head. Moreover the socle and the head are isomorphic if and only if $S_1 \otimes S_2$ is simple.*

Classification of real simple modules (in terms of their highest loop-weight) is a difficult open problem. Recently Lapid and Minguez [36] classified in type A all real simples satisfying a certain regularity condition. Surprisingly, this classification is related to the classification of rationally smooth Schubert varieties in type A flag varieties.

5 Geometric Character Formulas

An important obstacle for proving Conjecture 4.2 in general is the absence of Nakajima’s geometric theory in the non-symmetric cases B_n, C_n, F_4, G_2 . It turns out that, applying the results of Derksen, Weyman and Zelevinsky [8, 9] to the cluster algebras \mathcal{A}_ℓ , one can define projective varieties whose Euler characteristics calculate the q -characters of the *standard* $U_q(\widehat{\mathfrak{g}})$ -modules in all types. These varieties can be seen as generalizations of the Nakajima graded varieties $\mathcal{L}^\bullet(V, W)$ for types A, D, E . We shall now review this theorem of [23].

5.1 Quiver Grassmannians and F -Polynomials

Let Q be a quiver with vertex set I . Let M be a representation of Q over the field \mathbb{C} of complex numbers. Let $\mathbf{e} = (e_i)_{i \in I} \in \mathbb{N}^I$ be a dimension vector, and write $e := \sum_{i \in I} e_i$. The variety $\text{Gr}(\mathbf{e}, M)$ is the closed subvariety of the Grassmannian $\text{Gr}(e, M)$ of e -dimensional subspaces of M whose points parametrize the subrepresentations of M with dimension vector \mathbf{e} . Thus, $\text{Gr}(\mathbf{e}, M)$ is a projective complex variety, called a *quiver Grassmannian*.

Definition 5.1 The F -polynomial of the representation M of Q is

$$F_M(\mathbf{v}) := \sum_{\mathbf{e} \in \mathbb{N}^I} \chi(\text{Gr}(\mathbf{e}, M)) \mathbf{v}^{\mathbf{e}},$$

where $\mathbf{v} := (v_i)_{i \in I}$ is a sequence of commutative variables, $\mathbf{v}^{\mathbf{e}} := \prod_i v_i^{e_i}$, and $\chi(V)$ denotes the Euler characteristic of a complex projective variety V .

5.2 The Algebra A

Recall that in Sect. 3.1 we have associated with $U_q(\widehat{\mathfrak{g}})$ an infinite quiver Γ . Recall also the notation of Sect. 2.1 for the Cartan matrix. For every negative entry $c_{ij} < 0$ of the Cartan matrix and every $(i, r) \in V$, the graph Γ contains an oriented cycle $\gamma_{i,j,r}$:

$$\begin{array}{ccc}
 (i, r) & & \\
 \uparrow & \searrow & \\
 (i, r - b_{ii}) & & (j, r + b_{ij}) \\
 \vdots & & \\
 (i, r + 2b_{ij} + b_{ii}) & & \\
 \uparrow & \swarrow & \\
 (i, r + 2b_{ij}) & &
 \end{array} \tag{5.1}$$

We define a *potential* S as the formal sum of all these oriented cycles $\gamma_{i,j,r}$ up to cyclic permutations, see [8, §3]. This is an infinite sum, but note that a given arrow of Γ can only occur in a finite number of summands. Hence all the cyclic derivatives of S , defined as in [8, Definition 3.1], are finite sums of paths in Γ . Let R be the list of all cyclic derivatives of S . Let J denote the two-sided ideal of the path algebra $\mathbb{C}\Gamma$ generated by R . Following [8], we now introduce

Definition 5.2 Let A be the infinite-dimensional \mathbb{C} -algebra $\mathbb{C}\Gamma/J$.

Example 5.3 Let $\mathfrak{g} = \mathfrak{sl}_3$, of type A_2 . The quiver Γ is displayed in Fig. 7. We slightly abuse notation using the same letter α for every arrow of the form $(2, r) \rightarrow (1, r - 1)$, and similarly for β, γ, δ . All cycles $\gamma_{i,j,r}$ are of length 3, and are either of the form (γ, β, α) or of the form (β, δ, α) . Therefore the potential is

$$S = \sum (\gamma, \beta, \alpha) + \sum (\beta, \delta, \alpha),$$

and the relations obtained by taking its cyclic derivatives are of the form:

$$\text{derivative with respect to } \alpha \rightsquigarrow \gamma\beta + \beta\delta = 0,$$

$$\text{derivative with respect to } \beta \rightsquigarrow \alpha\gamma + \delta\alpha = 0,$$

$$\text{derivative with respect to } \gamma \rightsquigarrow \beta\alpha = 0,$$

$$\text{derivative with respect to } \delta \rightsquigarrow \alpha\beta = 0.$$

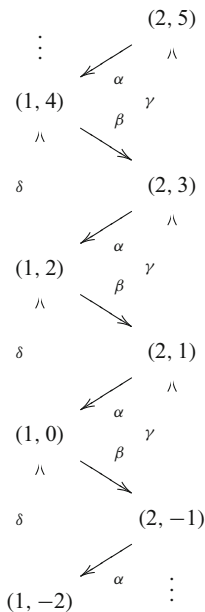


Fig. 7 The algebra A in type A_2

So A is the algebra defined by the quiver Γ of Fig. 7, subject to the above 4 families of relations.

5.3 Some A -Modules

The algebra A is infinite-dimensional. For every $(i, r) \in V$ there is a one-dimensional A -module $S_{(i,r)}$ supported on vertex (i, r) . Let $I_{(i,r)}$ denote the injective hull of $S_{(i,r)}$, an infinite-dimensional indecomposable A -module.

Example 5.4 We continue Example 5.3. The injective module $I_{(1,2)}$ is represented in Fig. 8. Each vertex occurring in the picture carries a one-dimensional vector space, and all occurring arrows are nonzero. The β arrows are all zero and therefore not represented. The infinite socle series of this module is

$$\begin{aligned}
 & \vdots \\
 & S_{(1,-2)} \oplus S_{(2,1)} \\
 & S_{(1,0)} \oplus S_{(2,3)} \\
 & S_{(1,2)}
 \end{aligned}$$

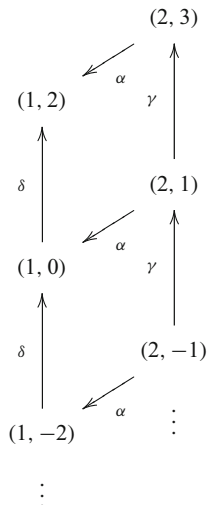
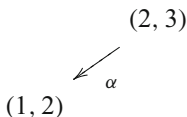


Fig. 8 The injective A -module $I_{(1,2)}$ in type A_2

Note that we have a short exact sequence of A -modules

$$0 \rightarrow K_{(1,2)} \rightarrow I_{(1,2)} \rightarrow I_{(1,0)} \rightarrow 0,$$

where $K_{(1,2)}$ is the two-dimensional module:



Proposition 5.5 *For every $(i, r) \in V$ there is a unique submodule $K_{(i,r)}$ of $I_{(i,r)}$ such that $I_{(i,r)}/K_{(i,r)}$ is isomorphic to $I_{(i,r-d_i)}$. The module $K_{(i,r)}$ is finite-dimensional.*

5.4 Geometric q -Character Formula

The next theorem gives a geometric description of the q -character of the fundamental $U_q(\widehat{\mathfrak{g}})$ -module $L((\varpi_i, q^{r-d_i}))$ in terms of the F -polynomial of the A -module $K_{(i,r)}$. To state it we need some more notation. To every $(i, r) \in V$ we attach a commutative variable $z_{(i,r)}$, and we set

$$\widehat{Y}_{(i,r)} := \prod_{(i,r) \rightarrow (j,s)} z_{(j,s)} \prod_{(k,l) \rightarrow (i,r)} z_{(k,l)}^{-1},$$

where the first product runs over all arrows of Γ starting at vertex (i, r) , and the second product over all arrows of Γ ending in vertex (i, r) .

Theorem 5.6 ([23]) *The F -polynomial $F_{K(i,r)}(\widehat{\mathbf{y}})$ of the A -module $K_{(i,r)}$ evaluated in the variables $\widehat{\mathbf{y}}_{(i,r)}$ can be expressed as a Laurent polynomial in the new variables*

$$Y_{j,q^{s-d_j}} := \frac{z_{(j,s-2d_j)}}{z_{(j,s)}}, \quad ((j, s) \in V).$$

Then we have

$$\chi_q(L((\varpi_i, q^{r-d_i}))) = Y_{i,q^{r-d_i}} F_{K(i,r)}(\widehat{\mathbf{y}}),$$

where in the right-hand side $F_{K(i,r)}(\widehat{\mathbf{y}})$ is expressed in terms of the variables $Y_{j,q^{s-d_j}}$.

Example 5.7 We continue Example 5.4. The A -module $K_{(1,2)}$ has exactly three submodules: $\{0\}$, $S_{(1,2)}$, and $K_{(1,2)}$. So there are three nonempty quiver Grassmannians, each reduced to a single point, hence each having Euler characteristic 1. It follows that

$$F_{K_{(1,2)}}(\mathbf{v}) = 1 + v_{(1,2)} + v_{(1,2)}v_{(2,3)}.$$

On the other hand

$$\widehat{y}_{(1,2)} = \frac{z_{(2,1)}z_{(1,4)}}{z_{(1,0)}z_{(2,3)}}, \quad \widehat{y}_{(2,3)} = \frac{z_{(1,2)}z_{(2,5)}}{z_{(1,4)}z_{(2,1)}}.$$

Hence

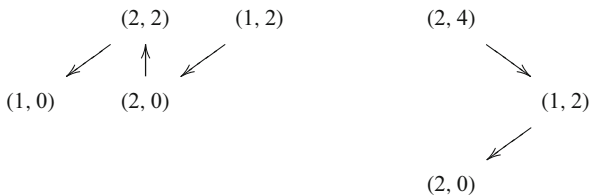
$$F_{K_{(1,2)}}(\mathbf{y}) = 1 + \frac{z_{(2,1)}z_{(1,4)}}{z_{(1,0)}z_{(2,3)}} + \frac{z_{(1,2)}z_{(2,5)}}{z_{(1,0)}z_{(2,3)}} = 1 + Y_{1,q}^{-1}Y_{1,q^3}^{-1}Y_{2,q^2} + Y_{1,q}^{-1}Y_{2,q^4}^{-1},$$

and

$$Y_{1,q} F_{K_{(1,2)}}(\mathbf{y}) = Y_{1,q} + Y_{1,q^3}^{-1}Y_{2,q^2} + Y_{2,q^4}^{-1} = \chi_q(L((\varpi_1, q))).$$

Example 5.8 We take $\mathfrak{g} = \mathfrak{so}_5$, of type B_2 . The A -modules $K_{(1,0)}$ and $K_{(2,0)}$ are displayed in Fig. 9. In this case too, the nonempty quiver Grassmannians are reduced

Fig. 9 The A -modules $K_{(1,0)}$ and $K_{(2,0)}$ in type B_2



to points, and we can calculate easily:

$$\begin{aligned} \chi_q(L((\varpi_1, q^{-2})) = Y_{1,q^{-2}} & \left(1 + \widehat{y}_{(1,0)} + \widehat{y}_{(1,0)}\widehat{y}_{(2,2)} + \widehat{y}_{(1,0)}\widehat{y}_{(2,2)}\widehat{y}_{(2,0)} \right. \\ & \left. + \widehat{y}_{(1,0)}\widehat{y}_{(2,2)}\widehat{y}_{(2,0)}\widehat{y}_{(1,2)} \right), \end{aligned}$$

$$\chi_q(L((\varpi_2, q^{-1})) = Y_{2,q^{-1}} \left(1 + \widehat{y}_{(2,0)} + \widehat{y}_{(2,0)}\widehat{y}_{(1,2)} + \widehat{y}_{(2,0)}\widehat{y}_{(1,2)}\widehat{y}_{(2,4)} \right).$$

To express these q -characters in terms of variables Y_{j,q^s} one uses the formulas:

$$\begin{aligned} \widehat{y}_{(1,r)} &= \frac{z_{(1,r+4)}z_{(2,r-2)}}{z_{(1,r-4)}z_{(2,r+2)}} = \frac{Y_{2,q^{r-1}}Y_{2,q^{r+1}}}{Y_{1,q^{r+2}}Y_{1,q^{r-2}}}, \\ \widehat{y}_{(2,r)} &= \frac{z_{(1,r-2)}z_{(2,r+2)}}{z_{(1,r+2)}z_{(2,r-2)}} = \frac{Y_{1,q^r}}{Y_{2,q^{r-1}}Y_{2,q^{r+1}}}. \end{aligned}$$

5.5 Comments on Theorem 5.6

5.5.1 Kirillov–Reshetikhin Modules

In [23] we give a similar q -character formula for every Kirillov–Reshetikhin module. One only needs to replace the A -modules $K_{(1,r)}$ by some larger finite-dimensional submodules of the injective modules $I_{(i,r)}$.

5.5.2 Standard Modules

Classical properties of Euler characteristics imply that, given two finite-dimensional A -modules M and N , we have

$$F_{M \oplus N}(\mathbf{v}) = F_M(\mathbf{v})F_N(\mathbf{v}).$$

On the other hand, q -characters are multiplicative on tensor products. So Theorem 5.6 readily extends to tensor products of fundamental $U_q(\widehat{\mathfrak{g}})$ -modules, that is, we have a similar geometric q -character formula for *standard* $U_q(\widehat{\mathfrak{g}})$ -modules (or local Weyl modules), in which one uses quiver Grassmannians of direct sums of A -modules $K_{(i,r)}$.

5.5.3 Relation to Nakajima’s Theory

If \mathfrak{g} is of type A, D, E it follows from results of Lusztig [37] and Savage–Tingley [41] that the quiver Grassmannians

$$G(\mathbf{e}, \mathbf{a}) := \text{Gr} \left(\mathbf{e}, \bigoplus_{(i,r) \in V} K^{\oplus a(i,r)} \right)$$

are homeomorphic to Nakajima varieties $\mathcal{L}^\bullet(V, W)$, where the graded dimension of V is encoded in the dimension vector \mathbf{e} , and the graded dimension of W is given by the multiplicity vector $\mathbf{a} = (a_{(i,r)})$. One can therefore regard the varieties $G(\mathbf{e}, \mathbf{a})$ in non-simply-laced types B, C, F, G as natural candidates for replacing the graded Nakajima varieties $\mathcal{L}^\bullet(V, W)$.

5.5.4 Beyond KR-Modules and Standard Modules

By the Derksen–Weyman–Zelevinsky theory, every cluster monomial of \mathcal{A}_ℓ has an expression of the form

$$m = \mathbf{z}^{\mathbf{g}} F_K(\widehat{\mathbf{y}})$$

for an appropriate A -module K . So, if Conjecture 4.2 is true, all the simple $U_q(\widehat{\mathfrak{g}})$ -modules corresponding to cluster monomials (all the real modules, if Conjecture 4.7 is true) have a similar geometric q -character formula in terms of quiver Grassmannians.

Acknowledgments D. Hernandez is supported in part by the European Research Council under the European Union’s Framework Programme H2020 with ERC Grant Agreement number 647353 QAffine.

References

1. M. BRITO, V. CHARI, *Tensor products and q -characters of HL-modules and monoidal categorifications*, J. Éc. polytech. math. 6 (2019), 581–619.
2. V. CHARI, *Braid group actions and tensor products*, Int. Math. Res. Not. (2002), no. 7, 357–382.
3. V. CHARI, A. PRESSLEY, *Quantum affine algebras*, Commun. Math. Phys. **142** (1991), 261–283.
4. V. CHARI, A. PRESSLEY, *A guide to quantum groups*. Cambridge University Press 1994.
5. V. CHARI, A. PRESSLEY, *Minimal affinizations of representations of quantum groups: the simply laced case*, J. Algebra **184** (1996), 1–30.
6. V. CHARI, A. PRESSLEY, *Quantum affine algebras and affine Hecke algebras*, Pacific J. Math. **174** (1996), 295–326.
7. V. CHARI, A. PRESSLEY, *Factorizations of representations of quantum affine algebras in Modular Interfaces*, (Riverside, Calif. 1995), AMS/IP Stud. Adv. Math. **4**, Amer. Math. Soc., Providence, 1997, 33–40.
8. H. DERKSEN, J. WEYMAN, A. ZELEVINSKY, *Quivers with potential and their representations I: Mutations*, Selecta Math., **14** (2008), 59–119.

9. H. DERKSEN, J. WEYMAN, A. ZELEVINSKY, *Quivers with potential and their representations II: Applications to cluster algebras*, J. Amer. Math. Soc. **23** (2010), 749–790.
10. R. FUJITA, *Affine highest weight categories and quantum affine Schur-Weyl duality of Dynkin quiver types*, arXiv:1710.11288
11. R. FUJITA, *Geometric realization of Dynkin quiver type quantum affine Schur-Weyl duality*, Int. Math. Res. Not. IMRN **22** (2020), 8353–8386.
12. E. FRENKEL, E. MUKHIN, *Combinatorics of q -characters of finite-dimensional representations of quantum affine algebras*, Comm. Math. Phys. **216** (2001), 23–57.
13. E. FRENKEL, N. RESHETIKHIN, *The q -characters of representations of quantum affine algebras*, Recent developments in quantum affine algebras and related topics, Contemp. Math. **248** (1999), 163–205.
14. S. FOMIN, A. ZELEVINSKY, *Cluster algebras I: Foundations*, J. Amer. Math. Soc. **15** (2002), 497–529.
15. S. FOMIN, A. ZELEVINSKY, *Cluster algebras II: Finite type classification*, Invent. Math. **154** (2003), 63–121.
16. S. FOMIN, A. ZELEVINSKY, *Cluster algebras: notes for the CDM-03 conference*, in Current developments in mathematics, 2003, 1–34, Int. Press, Somerville, MA, 2003.
17. C. GEISS, B. LECLERC, J. SCHRÖER, *Factorial cluster algebras*, Doc. Math. **18** (2013), 249–274.
18. V. GINZBURG, N. RESHETIKHIN, E. VASSEROT, *Quantum groups and flag varieties*, in Mathematical aspects of conformal and topological field theories and quantum groups (South Hadley, MA, 1992), 101–130, Contemp. Math., **175**, Amer. Math. Soc., Providence, RI, 1994.
19. M. GEKHTMAN, M. SHAPIRO, A. VAINSHTEIN, *Cluster algebras and Poisson geometry*, AMS Math. Survey and Monographs **167**, AMS 2010.
20. D. HERNANDEZ, *Algebraic approach to q,t -characters*, Adv. Math. **187** (2004), 1–52.
21. D. HERNANDEZ, *The Kirillov-Reshetikhin conjecture and solutions of T -systems*, J. Reine Angew. Math. **596** (2006), 63–87.
22. D. HERNANDEZ, B. LECLERC, *Cluster algebras and quantum affine algebras*, Duke Math. J. **154** (2010), 265–341.
23. D. HERNANDEZ, B. LECLERC, *Monoidal categorifications of cluster algebras of type A and D* , in Symmetries, integrable systems and representations, (K. Iohara, S. Morier-Genoud, B. Rémy, eds.), Springer proceedings in mathematics and statistics **40**, 2013, 175–193.
24. D. HERNANDEZ, B. LECLERC, *Quantum Grothendieck rings and derived Hall algebras*, J. Reine Angew. Math. **701** (2015), 77–126.
25. D. HERNANDEZ, B. LECLERC, *A cluster algebra approach to q -characters of Kirillov-Reshetikhin modules*, J. Eur. Math. Soc., **18** (2016), 1113–1159.
26. D. HERNANDEZ, B. LECLERC, *Cluster algebras and category \mathcal{O} for representations of Borel subalgebras of quantum affine algebras*, Algebra Number Theory **10** (2016), 2015–2052.
27. D. HERNANDEZ, H. OYA, *Quantum Grothendieck ring isomorphisms, cluster algebras and Kazhdan-Lusztig algorithm*, Adv. Math. **374** (2019), 192–272.
28. S.-J. KANG, M. KASHIWARA, M. KIM, *Symmetric quiver Hecke algebras and R -matrices of quantum affine algebras, II*, Duke Math. J. **164** (2015), 1549–1602.
29. S.-J. KANG, M. KASHIWARA, M. KIM, S.-J. OH, *Simplicity of heads and socles of tensor products*, Compos. Math. **151** (2015), 377–396.
30. S.-J. KANG, M. KASHIWARA, M. KIM, S.-J. OH, *Monoidal categorification of cluster algebras*, J. Amer. Math. Soc. **31** (2018), 349–426.
31. M. KASHIWARA, M. KIM, S.-J. OH, *Monoidal categories of modules over quantum affine algebras of type A and B* , Proc. London Math. Soc. **118** (2019) 43–77.
32. M. KASHIWARA, S.-J. OH, *Categorical relations between Langlands dual quantum affine algebras: doubly laced types*, to appear in J. Algebraic Combin. **49** (2019), 401–435. <https://doi.org/10.1007/s10801-018-0829-z>
33. A. KUNIBA, T. NAKANISHI, J. SUZUKI, *Functional relations in solvable lattice models: I. Functional relations and representation theory*, Int. J. Mod. Phys. **A9** (1994), 5215–5266.

34. B. LECLERC, *Imaginary vectors in the dual canonical basis of $U_q(n)$* , Transform. Groups **8** (2003), 95–104.
35. B. LECLERC, *Quantum loop algebras, quiver varieties, and cluster algebras*, in Representations of Algebras and Related Topics, (A. Skowroński and K. Yamagata, eds.), European Math. Soc. Series of Congress Reports, 2011, 117–152.
36. E. LAPID, A. MINGUEZ, *Geometric conditions for \square -irreducibility of certain representations of the general linear group over a non-Archimedean local field*, Adv. Math. **339** (2018), 113–190.
37. G. LUSZTIG, *On quiver varieties*, Adv. Math. **136** (1998), 141–182.
38. H. NAKAJIMA, *t -analogs of q -characters of Kirillov-Reshetikhin modules of quantum affine algebras*, Represent. Theory **7** (2003), 259–274.
39. H. NAKAJIMA, *Quiver varieties and cluster algebras*, Kyoto J. Math. **51** (2011), 71–126.
40. FAN QIN, *Triangular bases in quantum cluster algebras and monoidal categorification conjectures*, Duke Math. J. **166** (2017), 2337–2442.
41. A. SAVAGE, P. TINGLEY, *Quiver Grassmannians, quiver varieties and preprojective algebras*, Pacific J. Math. **251** (2011), 393–429.