Supersymmetric brick wall diagrams and the dynamical fishnet

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Integrability, Q-systems and Cluster Algebras
Varna
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RTG 2575: Rethinking Quantum Field Theory

Overview

Plan of the talk:

- Derivation of the superspace action of double-scaled β-deformation of $\mathcal{N} = 4$ SYM
- Super-Feynman-rules and superspace integral relations
- Free energy in the thermodynamic limit and critical coupling of QFTs
- Critical coupling of double-scaled β-deformation of $\mathcal{N} = 4$ SYM

Superspace formulation of double-scaled β-deformation of $\mathcal{N}=4$ SYM

$\mathcal{N}=4$ $\mathrm{SU(N)}$ SYM in $\mathcal{N}=1$ superspace formulation
[Penati,Santambrogio:0107071]

$$S=\int \mathrm{d}^4x\,\mathrm{d}^2\theta\mathrm{d}^2\bar{\theta}\,\sum_{i=1}^{3}\mathrm{tr}\left[\mathrm{e}^{-gV}\Phi_i^\dagger\mathrm{e}^{gV}\Phi_i\right]+\frac{1}{2g^2}\int \mathrm{d}^4x\,\mathrm{d}^2\theta\,\mathrm{tr}\left[W^\alpha W_\alpha\right]$$
$$+\,\mathrm{i}g\int \mathrm{d}^4x\,\mathrm{d}^2\theta\,\mathrm{tr}\left[\Phi_1[\Phi_2,\Phi_3]\right]+\mathrm{i}g\int \mathrm{d}^4x\,\mathrm{d}^2\bar{\theta}\,\mathrm{tr}\left[\Phi_1^\dagger[\Phi_2^\dagger,\Phi_3^\dagger]\right]$$

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β-deformation:
$\Phi_i\cdot\Phi_j\to\Phi_i\star\Phi_j:=\mathrm{e}^{\mathrm{i}\,\det\left(\boldsymbol{\gamma}|\mathbf{q}_i|\mathbf{q}_j\right)}\Phi_i\cdot\Phi_j$
with $\boldsymbol{\gamma}=(\beta,\beta,\beta)$,$\mathbf{q}_1=\left(\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right)$, $\mathbf{q}_2=\left(-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right)$ and
$\mathbf{q}_3=\left(-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right)$ [Lunin,Maldacena:0502086]

Superspace formulation of double-scaled β-deformation of $\mathcal{N} = 4$ SYM

β-deformed $\mathcal{N} = 4$ SYM in $\mathcal{N} = 1$ superspace formulation
[Jin,Roiban:1201.5012]

$$S = \int \mathrm{d}^4x \, \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta} \, \sum_{i=1}^{3} \mathrm{tr}\left[\mathrm{e}^{-gV}\Phi_i^\dagger \mathrm{e}^{gV}\Phi_i\right] + \frac{1}{2g^2}\int \mathrm{d}^4x \, \mathrm{d}^2\theta \, \mathrm{tr}\left[W^\alpha W_\alpha\right]$$
$$+ \mathrm{i}g \int \mathrm{d}^4x \, \mathrm{d}^2\theta \, \mathrm{tr}\left[q \, \Phi_1\Phi_2\Phi_3 - q^{-1}\Phi_1\Phi_3\Phi_2\right] + \mathrm{h.c.}$$

with $q = \mathrm{e}^{\mathrm{i}\beta}$

Superspace formulation of double-scaled β-deformation of $\mathcal{N}=4$ SYM

β-deformed $\mathcal{N}=4$ SYM in $\mathcal{N}=1$ superspace formulation
[Jin,Roiban:1201.5012]

$$S=\int \mathrm{d}^4x\, \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta} \sum_{i=1}^{3} \mathrm{tr}\left[\mathrm{e}^{-gV}\Phi_i^{\dagger}\mathrm{e}^{gV}\Phi_i\right] + \frac{1}{2g^2}\int \mathrm{d}^4x\, \mathrm{d}^2\theta\, \mathrm{tr}\left[W^{\alpha}W_{\alpha}\right]$$
$$+\mathrm{i}g\int \mathrm{d}^4x\, \mathrm{d}^2\theta\ \mathrm{tr}\left[q\ \Phi_1\Phi_2\Phi_3 - q^{-1}\Phi_1\Phi_3\Phi_2\right] + \mathrm{h.c.}$$

with $q=\mathrm{e}^{\mathrm{i}\beta}$
Then

- 't Hooft limit: Rescale the fields for genus expansion and $g\to 0$ and $\mathrm{N}\to\infty$, while $\lambda = g^2\mathrm{N}$ fixed
- Double-scaling limit: $\lambda\to 0$ and $\beta\to -\mathrm{i}\infty \Rightarrow q\to\infty$, while $\xi := \lambda\cdot q$ fixed [Gürdogan,Kazakov:1512.06704]

Superspace formulation of double-scaled β-deformation of $\mathcal{N} = 4$ SYM

Double-scaled β-deformed $\mathcal{N} = 4$ SYM in $\mathcal{N} = 1$ superspace formulation [MK,Staudacher:2408.05805]

$$S = \mathrm{N} \int \mathrm{d}^4 x \, \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta} \left\{ \sum_{i=1}^{3} \mathrm{tr} \left[\Phi_i^\dagger \Phi_i \right] \right.$$
$$\left. + \mathrm{i}\xi \cdot \bar{\theta}^2 \, \mathrm{tr} \left[\Phi_1 \Phi_2 \Phi_3 \right] + \mathrm{i}\xi \cdot \theta^2 \, \mathrm{tr} \left[\Phi_1^\dagger \Phi_2^\dagger \Phi_3^\dagger \right] \right\}$$

Superspace formulation of double-scaled β-deformation of $\mathcal{N}=4$ SYM

Double-scaled β-deformed $\mathcal{N}=4$ SYM in $\mathcal{N}=1$ superspace formulation [MK,Staudacher:2408.05805]

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In components: χ-CFT dynamical fishnet
[Gürdogan,Kazakov:1512.06704]

$$S = \mathrm{N}\int \mathrm{d}^4x\,\mathrm{tr}\left\{\sum_{i=1}^{3}\left[\phi_i^\dagger\Box\phi_i-\mathrm{i}\bar{\psi}_i\bar{\sigma}^\mu\partial_\mu\psi_i\right]\right.$$
$$+\xi^2\left[\phi_1\phi_2\phi_1^\dagger\phi_2^\dagger+\phi_3\phi_1\phi_3^\dagger\phi_1^\dagger+\phi_2\phi_3\phi_2^\dagger\phi_3^\dagger\right]$$
$$-\mathrm{i}\xi\left[\phi_1\psi_2\psi_3+\phi_2\psi_3\psi_1+\phi_3\psi_1\psi_2\right]$$
$$\left.-\mathrm{i}\xi\left[\phi_1^\dagger\bar{\psi}_2\bar{\psi}_3+\phi_2^\dagger\bar{\psi}_3\bar{\psi}_1+\phi_3^\dagger\bar{\psi}_1\bar{\psi}_2\right]+\mathcal{L}_{\mathrm{dt}}\right\}$$

Super-Feynman rules

Chiral superfield at point $z = (x, \theta, \bar{\theta})$ superspace

$$\Phi_i(z) = \mathrm{e}^{\mathrm{i}\theta\sigma^\mu\bar{\theta}\partial_\mu} \left[\phi_i(x) + \sqrt{2}\,\theta\psi_i(x) + \theta^2 F_i(x)\right]$$

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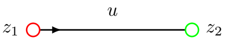
Generalized Superfield propagator

$\left(x^\mu_{1\bar{2}} := x^\mu_1 - x^\mu_2 + \mathrm{i}\left[\theta_1\sigma^\mu\bar{\theta}_1 + \theta_2\sigma^\mu\bar{\theta}_2 - 2\theta_1\sigma^\mu\bar{\theta}_2\right]\right)$

$$\left\langle \Phi_i(z_1)\Phi^\dagger_j(z_2)\right\rangle_u = \mathrm{e}^{\mathrm{i}\left[\theta_1\sigma^\mu\bar{\theta}_1+\theta_2\sigma^\mu\bar{\theta}_2-2\theta_1\sigma^\mu\bar{\theta}_2\right]\partial_{1,\mu}}\frac{\delta_{ij}}{\left[x^2_{12}\right]^u} = \frac{\delta_{ij}}{\left[x^2_{1\bar{2}}\right]^u}$$

$=$ 

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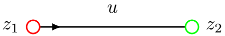
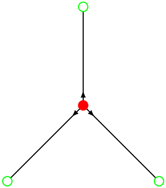
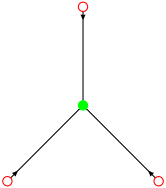
$=$ 

Super-vertices (chiral and anti-chiral)



$\sim -\xi\int \mathrm{d}^4x\ \mathrm{d}^2\theta\,\mathrm{d}^2\bar\theta\ \delta^{(2)}(\bar\theta)\,,$



$\sim -\xi\int \mathrm{d}^4x\ \mathrm{d}^2\theta\,\mathrm{d}^2\bar\theta\ \delta^{(2)}(\theta)\,.$

Useful superintegral relations

Osborn's formula [Osborn:9808041][Dolan,Osborn:0006098]

$$\mathrm{i}\int \mathrm{d}^4 x_0\ \mathrm{d}^2\theta_0\,\mathrm{d}^2\bar{\theta}_0\ \delta^{(2)}(\theta_0)\ \frac{1}{\left[x^2_{1\bar{0}}\right]^{u_1}}\frac{1}{\left[x^2_{2\bar{0}}\right]^{u_2}}\frac{1}{\left[x^2_{3\bar{0}}\right]^{u_3}}$$

$$\overset{u_1+u_2+u_3=3}{=} -4\,r(u_1,u_2,u_3)\ \frac{(\theta_{12}\theta_{13})\,x^2_{23,+}+(\theta_{23}\theta_{21})\,x^2_{31,+}+(\theta_{31}\theta_{32})\,x^2_{12,+}}{[x^2_{12,+}]^{2-u_3}[x^2_{23,+}]^{2-u_1}[x^2_{31,+}]^{2-u_2}}$$

with $x^\mu_{ij,+} := x^\mu_{i,+} - x^\mu_{j,+}$, $x^\mu_\pm = x^\mu \pm \mathrm{i}\theta\sigma^\mu\bar{\theta}$
and $r(u_1,u_2,u_3) := \pi^2 a(u_1)a(u_2)a(u_3)$, $a(u) := \frac{\Gamma(2-u)}{\Gamma(u)}$

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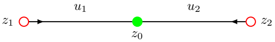
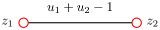
Chain relation:



$$=-4\,r(3-u_1-u_2,u_1,u_2)$$



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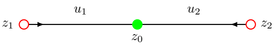
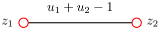
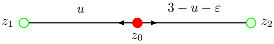
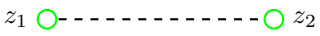


$= -4\,r(3-u_1-u_2, u_1, u_2)$


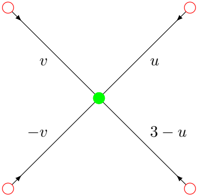
In particular

$\lim\limits_{\varepsilon\to 0}$


$= -4\,\pi^4\cdot a(u)\,a(3-u)$


Useful superintegral relations

Auxiliary relation: Super x-unity [MK,Staudacher:2408.05805]



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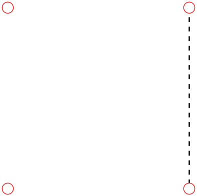

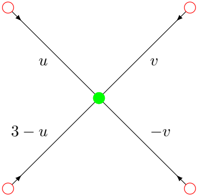

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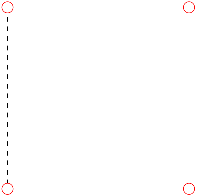
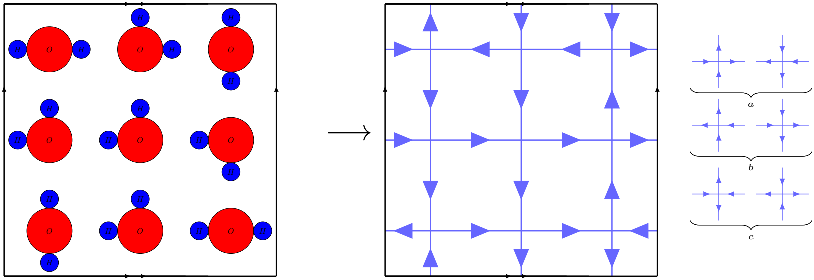


Free energy in the thermodynamic limit of water ice



Experimental residual entropy: 1.540 ± 0.001

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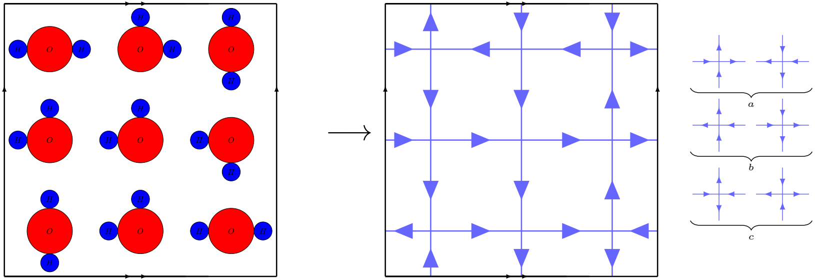


Experimental residual entropy: 1.540 ± 0.001

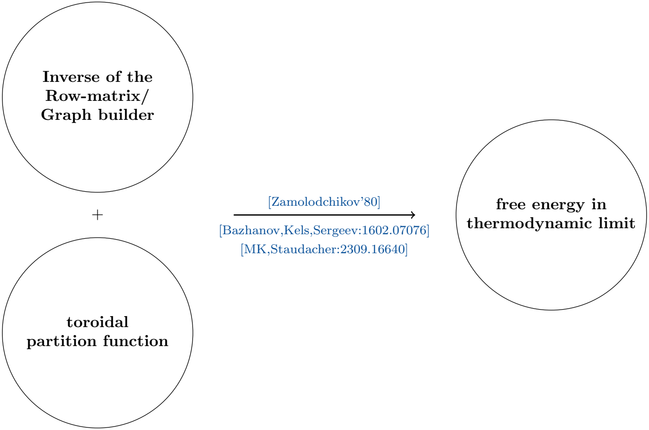
$$Z_{MN} = \sum_{\Omega \in \Lambda_{MN}} a^{n_1+n_2} b^{n_3+n_4} c^{n_5+n_6}$$

$$K(a,b,c) = \lim_{M,N\to\infty} (Z_{MN})^{1/MN}$$

$$K(1,1,1) = \left(\frac{4}{3}\right)^{3/2} \approx 1.5396$$ [Lieb '67]

Also obtained by **method of inversion relations** [Stroganov '79]

Method of inversion relations

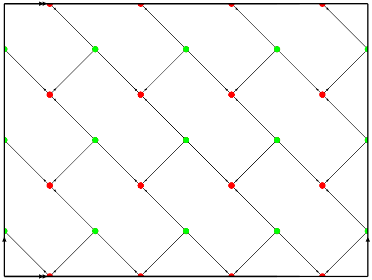


Applicable to **integrable QFTs** by interpreting them as integrable lattice models with **generalized propagators** as weights, results in **exact value for critical coupling**.

Vacuum superdiagrams

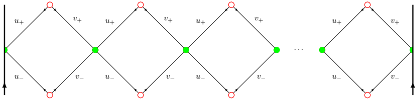
Row-matrices are stacked up and periodically identified to form a **toroidal vacuum superdiagrams** of double-scaled β-deformation of $\mathcal{N} = 4$ SYM:

$$Z_{3,4}\left(\begin{smallmatrix}0 & 1\\ 1 & 1\end{smallmatrix}\right) = \mathrm{Tr}\left[T_4\left(\begin{smallmatrix}0 & 1\\ 1 & 1\end{smallmatrix}\right)^3\right] =$$



with generalized row-matrix

$$T_N\left(\begin{smallmatrix}u_+ & v_+\\ u_- & v_-\end{smallmatrix}\right) =$$



Generalized vacuum diagrams

The generalized free energy

$$Z\left(\begin{smallmatrix} u_+ & v_+ \\ u_- & v_- \end{smallmatrix}\right) = \sum_{M,N=1}^{\infty} Z_{MN}\left(\begin{smallmatrix} u_+ & v_+ \\ u_- & v_- \end{smallmatrix}\right)(-\xi)^{2M\cdot N}$$

has the **radius of convergence/critical coupling** is related to

$$\begin{aligned} K\left(\begin{smallmatrix} u_+ & v_+ \\ u_- & v_- \end{smallmatrix}\right) &= \lim_{M,N\to\infty}\left|Z_{MN}\left(\begin{smallmatrix} u_+ & v_+ \\ u_- & v_- \end{smallmatrix}\right)\right|^{\frac{1}{MN}} \\ &= \lim_{N\to\infty}\left|\Lambda_{\mathrm{max},N}\left(\begin{smallmatrix} u_+ & v_+ \\ u_- & v_- \end{smallmatrix}\right)\right|^{\frac{1}{N}} \end{aligned}$$

Generalized vacuum diagrams

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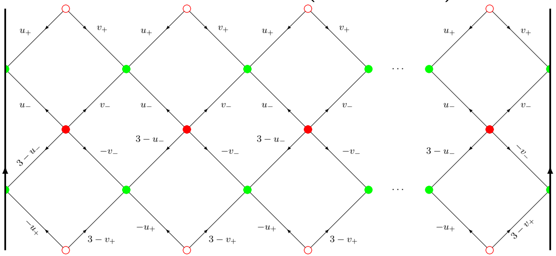
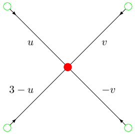
Goal: Find the inverse of the row-matrix, which should be itself at different spectral parameter point. Then let the product act on the eigenvector corresponding to the maximal eigenvalue $\Lambda_{\mathrm{max},N}$ to obtain **functional relations**.

There are **four representation of the inverse** of $T_N \left(\begin{smallmatrix} u_+ & v_+ \\ u_- & v_- \end{smallmatrix}\right)$.

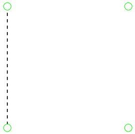
E.g.: $T_N \left(\begin{smallmatrix} u_+ & v_+ \\ u_- & v_- \end{smallmatrix}\right) \circ T_N \begin{pmatrix} 3-u_- & -v_- \\ -u_+ & 3-v_+ \end{pmatrix} \sim$



Auxiliary relation:



$= -4\,\pi^4 a(u)\,a(3-u)$



Inversion relations

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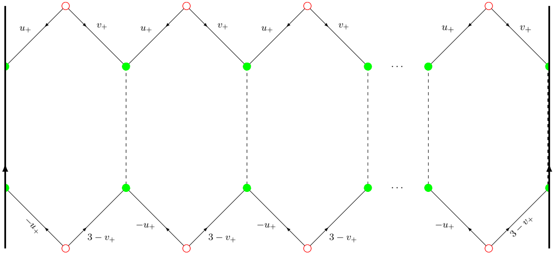
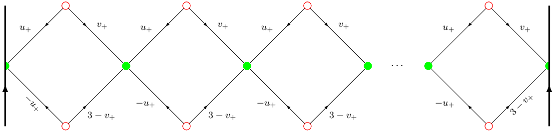
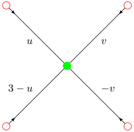


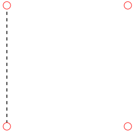
Inversion relations

There are **four representation of the inverse** of $T_N \begin{pmatrix} u_+ & v_+ \\ u_- & v_- \end{pmatrix}$.

E.g.: $T_N \begin{pmatrix} u_+ & v_+ \\ u_- & v_- \end{pmatrix} \circ T_N \begin{pmatrix} 3-u_- & -v_- \\ -u_+ & 3-v_+ \end{pmatrix} \sim$



Auxiliary relation: 

$= -4\,\pi^4 a(u)\, a(3-u)$ 

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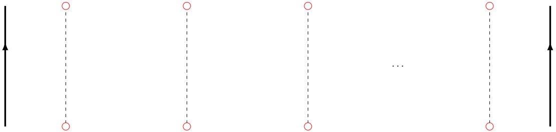

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Inversion relations

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E.g.:

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K \left(\begin{smallmatrix} u_+ & v_+ \\ u_- & v_- \end{smallmatrix}\right) \circ K \begin{pmatrix} -u_- & 3-v_- \\ 3-u_+ & -v_+ \end{pmatrix} &= 16\pi^8\, a(u_+)\, a(3-u_+)\, a(v_-)\, a(3-v_-) \\
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\end{aligned}
$$

Result for the critical coupling

Solve functional relations with ansatz

$$K\left(\begin{smallmatrix} u_+ & v_+ \\ u_- & v_- \end{smallmatrix}\right) = \kappa(u_+)\kappa(u_-)\kappa(v_+)\kappa(v_-)$$

satisfying

$$\kappa(u)\kappa(-u) \;=\; 1\,,$$

$$\kappa(u)\kappa(3-u) = 4\pi^4\; a(u)\, a(3-u) = 4\pi^4\,\frac{\Gamma(2-u)\Gamma(u-1)}{\Gamma(u)\Gamma(3-u)}$$

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The solution with the right analytic properties is

[Bazhanov,Kels,Sergeev'16;Zamolodchikov'77;Shankar,Witten'78;Bombardelli:1606.02949]

$$\kappa(u) = 12^{\frac{u}{3}}\pi^{\frac{4u}{3}}\frac{\Gamma\left(\frac{u+1}{3}\right)\Gamma(2-u)}{\Gamma\left(\frac{1}{3}\right)}\prod_{k=1}^{\infty}\frac{\Gamma(3k-u+2)\Gamma(3k+u)\Gamma(3k-2)}{\Gamma(3k+u-2)\Gamma(3k-u)\Gamma(3k+2)}$$

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The critical coupling of double-scaled β-deformed $\mathcal{N}=4$ is [MK,Staudacher:2408.05805]

$$\xi_{\mathrm{cr}} = \left[\mathbb{K}\left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)\right]^{-1/2} = \kappa\,(1)^{-3/2} = \frac{3}{2\pi^2\,\Gamma(\frac{1}{3})^{3/2}}$$

Outlook

- Commuting transfer matrices require non-compact superconformal $s\ell(4|1)$ R-matrix ($s\ell(2|1)$ by [Derkachov,Karakhanyan,Kirschner:0102024])
  Is Osborn's formula enough?
- Uplift many results from fishnet & family to double-scaled β-deformation (closer to $\mathcal{N}=4$ SYM)

  [Basso,Derkachov,Dixon,Ferrando,Gromov,Kazakov,Korchemsky,Kostov,Olivucci,...:15'-now]
- Construct the superfishnet, a 3D $\mathcal{N}=2$, as the double-scaled β-deformation of ABJM

Thanks for your attention!