Uncovering new integrable structures via the ODE/IQFT correspondence

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ODE/IQFT (ODE/IM, . . .) correspondence

[Voros'92; Dorey Tateo'98; BLZ'98,03; LZ'10]

classical/quantum correspondence that holds for any $\hbar$

Quantum side: spectral problem of commuting family of operators in $1+1$ dimensional integrable QFT

Classical side: analysis of classically integrable $1+1$ dimensional partial differential equation

Powerful applications, e.g.,

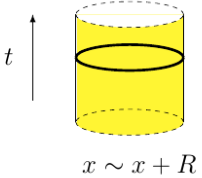
- first principles quantization of integrable QFT [with Bazhanov, Lacroix, Lukyanov, Teschner]
- scaling limit of integrable, critical spin chains [with Bazhanov, Frahm, Gehrmann, Koval, Lukyanov, Shabetnik]

ODE/IQFT for sine-Gordon [Lukyanov, Zamolodchikov'10]

Quantum side: sine-Gordon integrable QFT

$$\mathcal{L} = \frac{1}{16\pi}\Big((\partial_t\varphi)^2 - (\partial_x\varphi)^2\Big) + 2\mu\,\cos(\beta\varphi)$$

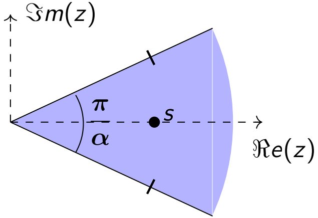
$$\partial_\mu\varphi(t, x+R) = \partial_\mu\varphi(t,x)$$



Classical side: modified sinh-Gordon integrable PDE

$$\partial_z\partial_{\bar z}\eta = \mathrm{e}^{2\eta} - |z^{2\alpha} - s^{2\alpha}|^2\,\mathrm{e}^{-2\eta}$$

$$\eta(z\,\mathrm{e}^{+\frac{\mathrm{i}\pi}{\alpha}}, \bar z\,\mathrm{e}^{-\frac{\mathrm{i}\pi}{\alpha}}) = \eta(z,\bar z)$$



Modified sinh-Gordon equation

$$\partial_z \partial_{\bar{z}} \eta = \mathrm{e}^{2\eta} - |z^{2\alpha} - s^{2\alpha}|^2 \, \mathrm{e}^{-2\eta} \qquad\qquad (\alpha, s > 0)$$

Integrability $\implies$ infinitely many conservation laws:

$$\partial_{\bar{z}} F_{2n} = \partial_z G_{2n-2}\,, \qquad\qquad \partial_z \bar{F}_{2n} = \partial_{\bar{z}} \bar{G}_{2n-2} \qquad\qquad (n = 1, 2, \ldots)$$

e.g.,

$$F_2 = \frac{1}{4\sqrt{p}} \left((\partial_z \eta)^2 - \partial_z^2 \eta + \frac{p''}{4p} - \frac{5\,(p')^2}{16p^2} \right), \qquad\qquad G_0 = 2\sqrt{\bar{p}}\,\big(|p|\,\mathrm{e}^{-2\eta} - 1\big)$$

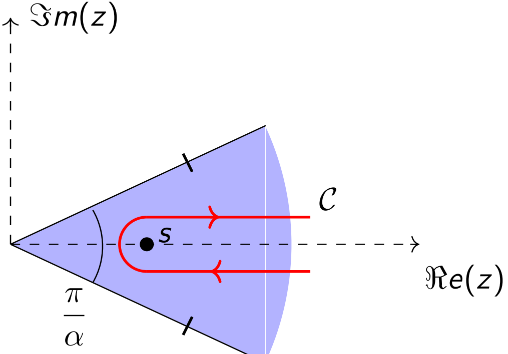
with $p = z^{2\alpha} - s^{2\alpha}\,,\ \bar{p} = \bar{z}^{2\alpha} - s^{2\alpha}$

F_{2n}, G_{2n-2} in terms of Gel'fand-Dikii polys [Lukyanov, Zamolodchikov'10]

Conserved quantities

$$\mathfrak{I}_{2n-1}[\eta] = \oint_{\mathcal{C}} F_{2n}\, \mathrm{d}z + G_{2n-2}\, \mathrm{d}\bar{z}\,, \qquad \bar{\mathfrak{I}}_{2n-1}[\eta] = \oint_{\mathcal{C}} \bar{F}_{2n}\, \mathrm{d}\bar{z} + \bar{G}_{2n-2}\, \mathrm{d}z$$

remain unchanged under continuous deformations of $\mathcal{C}$



$$\partial_z \partial_{\bar{z}} \eta = \mathrm{e}^{2\eta} - \left|z^{2\alpha} - s^{2\alpha}\right|^2 \mathrm{e}^{-2\eta}$$

sine-Gordon model

$$\mathcal{L} = \frac{1}{16\pi}\Big((\partial_t\varphi)^2 - (\partial_x\varphi)^2\Big) + 2\mu\,\cos(\beta\varphi)$$

Infinite number of conservation laws

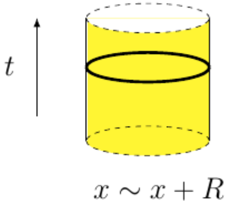
$$\partial_+ T_{2n} = \partial_- \Theta_{2n-2}\,, \qquad \partial_+ T_{-2n} = \partial_- \Theta_{2-2n}\,, \qquad \big(\partial_\pm = \tfrac{1}{2}\,(\partial_t \pm \partial_x)\big)$$

with $n = 1, 2, \ldots$.

Local Integrals of Motion (IM) in finite volume:

$$\mathbb{J}_{2n-1} = \int_0^R \mathrm{d}x\,\big(T_{2n} - \Theta_{2n-2}\big)$$

$$\bar{\mathbb{J}}_{2n-1} = \int_0^R \mathrm{d}x\,\big(T_{-2n} - \Theta_{2-2n}\big)$$



Integrability assumes

$$[\mathbb{J}_{2n-1}, \mathbb{J}_{2m-1}] = [\bar{\mathbb{J}}_{2n-1}, \mathbb{J}_{2m-1}] = [\bar{\mathbb{J}}_{2n-1}, \bar{\mathbb{J}}_{2m-1}] = 0$$

Quantum/classical correspondence

- **Classical side:** modified sinh-Gordon equation integrable PDE

$$\partial_z \partial_{\bar{z}} \eta = \mathrm{e}^{2\eta} - |z^{2\alpha} - s^{2\alpha}|^2 \, \mathrm{e}^{-2\eta} = 0$$

with infinite number of 'conserved charges'

$$\mathfrak{I}_{2n-1}[\eta] = \oint_{\mathcal{C}} F_{2n} \, \mathrm{d}z + G_{2n-2} \, \mathrm{d}\bar{z} \,, \qquad \bar{\mathfrak{I}}_{2n-1}[\eta] = (\ldots)$$

Fix solution

$$\underbrace{\eta^{(\mathrm{vac},\ell)}}_{\mathrm{real}} = \begin{cases} 2\ell \log|z| + O(1) & \text{as} \quad |z| \to 0 \\ \alpha \log|z| + o(1) & \text{as} \quad |z| \to \infty \end{cases}$$

with $-\frac{1}{2} < \ell < \frac{1}{2}$ (no other singularities allowed)

Quantum/classical correspondence

- **Quantum side:** sine-Gordon model

$$\mathcal{L} = \frac{1}{16\pi}\Big((\partial_t\varphi)^2 - (\partial_x\varphi)^2\Big) + 2\mu\,\cos(\beta\varphi)$$

with infinite set of commuting IM

$$[\mathbb{J}_{2n-1}, \mathbb{J}_{2m-1}] = [\bar{\mathbb{J}}_{2n-1}, \mathbb{J}_{2m-1}] = [\bar{\mathbb{J}}_{2n-1}, \bar{\mathbb{J}}_{2m-1}] = 0$$

Fix state

Vacuum in sector with quasi-momentum k:

$$\varphi \mapsto \varphi + 2\pi/\beta\,, \qquad |\Psi_k^{(\mathrm{vac})}\rangle \mapsto \mathrm{e}^{2\pi\mathrm{i}k}|\Psi_k^{(\mathrm{vac})}\rangle \qquad (-\tfrac{1}{2} < k < \tfrac{1}{2})$$

Consider eigenvalues of IM computed on state

$$\mathbb{J}_{2n-1}|\Psi_k^{(\mathrm{vac})}\rangle = J_{2n-1}^{(\mathrm{vac},k)}|\Psi_k^{(\mathrm{vac})}\rangle\,, \qquad \bar{\mathbb{J}}_{2n-1}\big|\Psi_k^{(\mathrm{vac})}\big\rangle = \bar{J}_{2n-1}^{(\mathrm{vac},k)}\big|\Psi_k^{(\mathrm{vac})}\big\rangle$$

Quantum/classical correspondence

- **Classical problem:**

$$\mathfrak{I}_{2n-1}[\eta^{(\mathrm{vac},\ell)}] = \oint_{\mathcal{C}} F_{2n}\,\mathrm{d}z + G_{2n-2}\,\mathrm{d}\bar{z}\,, \qquad \bar{\mathfrak{I}}_{2n-1}[\eta^{(\mathrm{vac},\ell)}] = (\ldots)$$

- **Quantum problem:**

$$\mathbb{J}_{2n-1}|\Psi_k^{(\mathrm{vac})}\rangle = J_{2n-1}^{(\mathrm{vac},k)}|\Psi_k^{(\mathrm{vac})}\rangle\,, \qquad \bar{\mathbb{J}}_{2n-1}\big|\Psi_k^{(\mathrm{vac})}\big\rangle = \bar{J}_{2n-1}^{(\mathrm{vac},k)}\big|\Psi_k^{(\mathrm{vac})}\big\rangle$$

- **Classical/quantum correspondence:** [Lukyanov, Zamolodchikov'10]

$$J_{2n-1}^{(\mathrm{vac},k)} = C_n\,\mathfrak{I}_{2n-1}\big[\eta^{(\mathrm{vac},\ell)}\big] \qquad \bar{J}_{2n-1}^{(\mathrm{vac},k)} = C_n\,\bar{\mathfrak{I}}_{2n-1}\big[\eta^{(\mathrm{vac},\ell)}\big]$$

with

$$\beta^2 = \tfrac{1}{\alpha+1}\,, \qquad \underbrace{R\mu^{\frac{1}{2-2\beta^2}}}_{\text{dimensionless}} \propto s^{\alpha+1}\,, \qquad 2|k| = \ell + \tfrac{1}{2}$$

How to explore the ODE/IQFT correspondence?

Important simplifying limit (UV limit):

$$\underbrace{R\mu^{\frac{1}{2-2\beta^2}}}_{\text{dimensionless}} \mapsto 0$$

- (non-)local IM in sine-Gordon model become integrable structure in CFT (KdV integrable structure [BLZ'94])
- modified sinh-Gordon PDE replaced by ODE (Schröidinger eq. for 3D anharmonic oscillator)

- **rigorously** prove the ODE/IQFT correspondence for vacua [BLZ'98]
- extend to all states in the theory [BLZ'03] (conjecture); [Conti, Masoero '20;'21] (proof)

UV limit for sine-Gordon model

- Lagrangian:

$$\mathcal{L} = \frac{1}{16\pi}\Big((\partial_t\varphi)^2 - (\partial_x\varphi)^2\Big) \;\cancel{+\,2\mu\cos(\beta\varphi)} \quad\mapsto\quad \text{free field theory}$$

- Quantization:

$$\varphi(t,x) = \phi(t+x) + \bar{\phi}(t-x)$$

ϕ, $\bar{\phi}$ generate two independent copies of Heisenberg algebra

- Space of states:

$$\mathcal{H} = \bigoplus_{p,\bar{p}\in\Sigma} \mathcal{F}_p \otimes \bar{\mathcal{F}}_{\bar{p}}$$

$\mathcal{F}_p$ ($\bar{\mathcal{F}}_{\bar{p}}$) Fock space for $\partial\phi$ ($\partial\bar{\phi}$) with zero-mode momentum p ($\bar{p}$)

UV limit for local IM [A. Zamolodchikov '89]

$$\lim_{R\to 0}\underbrace{\left[R^{2n-1}\,\mathbb{J}_{2n-1}\right]}_{\text{dimensionless}}=\mathbb{I}_{2n-1}\,,\qquad\qquad \lim_{R\to 0}\underbrace{\left[R^{2n-1}\,\bar{\mathbb{J}}_{2n-1}\right]}_{\text{dimensionless}}=\bar{\mathbb{I}}_{2n-1}$$

with

$$\mathbb{I}_{2n-1}\ :\ \mathcal{F}_p\mapsto\mathcal{F}_p\,,\qquad \left[\mathbb{I}_{2n-1},\mathbb{I}_{2m-1}\right]=0\qquad\qquad (\text{similarly for }\bar{\mathbb{I}}_{2n-1})$$

Example:

$$\mathbb{I}_1 \;=\; \int_0^{2\pi}\frac{\mathrm{d}x}{2\pi}\,(\partial\phi)^2$$

$$\mathbb{I}_3 \;=\; \int_0^{2\pi}\frac{\mathrm{d}x}{2\pi}\left(\big(\partial\phi\big)^4+\big(3-\beta^2-\beta^{-2}\big)\,\big(\partial^2\phi\big)^2\right)$$

$\{\mathbb{I}_{2n-1}\}_{n\geq 1}$ are local IM for **quantum KdV integrable structure**

Quantum KdV integrable structure [BLZ'94]

Family of commuting operators acting in Fock space:

$$\mathbb{O}\ :\quad \mathcal{F}_p \mapsto \mathcal{F}_p\,, \qquad\qquad [\mathbb{O},\mathbb{O}'] = 0$$

Consists of

- Local IM $\{\mathbb{I}_{2n-1}\}_{n\geq 1}$
- Reflection operator $\mathbb{R}$
- Non-local IM $\{\mathbb{H}_n\}_{n\geq 1}$
- Dual non-local IM $\{\widetilde{\mathbb{H}}_n\}_{n\geq 1}$

Focus on generating operator for all conserved charges:

$$\mathbb{A}_\pm(\lambda)\ :\ \mathcal{F}_p \mapsto \mathcal{F}_p\,, \qquad\qquad \big[\mathbb{A}_\pm(\lambda),\mathbb{A}_\pm(\lambda')\big] = \big[\mathbb{A}_\pm(\lambda),\mathbb{A}_\mp(\lambda')\big] = 0$$

similar to Q operator introduced in integrable lattice systems [Baxter'72]

Q operator for qKdV [BLZ'96]

Normalization factor so that $\mathbb{A}_\pm(0) = \mathbf{1}$

$$\mathbb{A}_+(\lambda) = Z^{-1}\,\mathrm{Tr}\left[q^{p\mathcal{H}}\,\overleftarrow{\mathcal{P}}\exp\left(\lambda\int_0^{2\pi}\mathrm{d}x\,\left(\mathrm{e}^{+2\mathrm{i}\beta\phi}\,q^{\frac{\mathcal{H}}{2}}\,\mathcal{E}_+ + \mathrm{e}^{-2\mathrm{i}\beta\phi}\,q^{-\frac{\mathcal{H}}{2}}\,\mathcal{E}_-\right)\right)\right]$$

Trace taken over infinite dimensional representation of q-oscillator algebra:

$$q\mathcal{E}_+\mathcal{E}_- - q^{-1}\mathcal{E}_-\mathcal{E}_+ = \frac{1}{q-q^{-1}}\,, \qquad [\mathcal{H},\mathcal{E}_\pm] = \pm 2\mathcal{E}_\pm\,, \qquad q = \mathrm{e}^{\mathrm{i}\pi\beta^2}$$

(similar formula for $\mathbb{A}_-$)

Construction based on rep. theory of $U_q\big(\widehat{\mathfrak{sl}}(2)\big)$

$$\Longrightarrow \qquad \big[\mathbb{A}_\pm(\lambda), \mathbb{A}_\pm(\lambda')\big] = \big[\mathbb{A}_\pm(\lambda), \mathbb{A}_\mp(\lambda')\big] = 0$$

and operator relations underlying integrability, e.g., quantum Wronskian relation

Generating function

Non-local IM appear in **small** λ expansion:

$$\log \mathbb{A}_+(\lambda) = -\sum_{n=1}^{\infty} \lambda^{2n}\, \mathbb{H}_n$$

Dual non-local/local IM appear in **large** λ expansion:

$$\log \mathbb{A}_+(\lambda) \asymp \log(\mathbb{R}) - \frac{p}{\beta^2} \log\big((-\lambda^2)/\mathrm{const}\big) + \sum_{n=0}^{\infty} B_n\, (-\lambda^2)^{\frac{1-2n}{2-2\beta^2}}\ \mathbb{I}_{2n-1}$$

$$+ \sum_{n=1}^{\infty} C_n\, (-\lambda^2)^{-\frac{n}{\beta^2}}\ \widetilde{\mathbb{H}}_n \qquad \text{as} \qquad \lambda^2 \to -\infty$$

ODE side: flat connection

Focus on flat connection

$$\partial_z\partial_{\bar z}\eta = \mathrm{e}^{2\eta} - |z^{2\alpha} - s^{2\alpha}|^2\,\mathrm{e}^{-2\eta} \qquad \Longleftrightarrow \qquad \big[\partial_z - \boldsymbol{A}_z(\lambda)\,,\,\partial_{\bar z} - \boldsymbol{A}_{\bar z}(\lambda)\big] = 0$$

which generates all IM, e.g., as $\lambda \to +\infty$

$$\mathrm{Tr}\,\overleftarrow{\mathcal{P}}\exp\bigg(\int_{\mathcal{C}} \boldsymbol{A}_z(\lambda)\,\mathrm{d}z + \boldsymbol{A}_{\bar z}(\lambda)\,\mathrm{d}\bar z\bigg) \asymp \exp\Big(C_0'\,\lambda + \sum_{n=1}^{\infty} C_n'\,\mathfrak{J}_{2n-1}\,\lambda^{-(2n-1)}\Big)$$

$$\partial_z - \boldsymbol{A}_z = \begin{pmatrix} \partial_z - \frac{1}{2}\,\partial_z\eta & \mathrm{e}^{\eta} \\ \lambda^{+2}\,(z^{2\alpha} - s^{2\alpha})\,\mathrm{e}^{-\eta} & \partial_z + \frac{1}{2}\,\partial_z\eta \end{pmatrix}$$

$$\partial_{\bar z} - \boldsymbol{A}_{\bar z} = \begin{pmatrix} \partial_{\bar z} + \frac{1}{2}\,\partial_{\bar z}\eta & \lambda^{-2}\,(\bar z^{2\alpha} - s^{2\alpha})\,\mathrm{e}^{-\eta} \\ \mathrm{e}^{\eta} & \partial_{\bar z} - \frac{1}{2}\,\partial_{\bar z}\eta \end{pmatrix}$$

UV limit of flat connection

- Rewrite as 2nd order ODE, e.g.,

$$(\partial_z - \boldsymbol{A}_z)\boldsymbol{\Psi} = 0 \qquad \Longleftrightarrow \qquad \Big[-\partial_z^2 + (\partial_z\eta)^2 - \partial_z^2\eta + \lambda^2\, z^{2\alpha}\Big]\psi = \lambda^2 s^{2\alpha}\psi$$

- **Multiple ways** to take limit. Send $s \propto R\mu^{\frac{1}{2-2\beta^2}} \to 0$ with $z, \lambda^{-1} \to 0$ keeping fixed

$$x = \lambda^{\frac{1}{1+\alpha}} z\,, \qquad E = \lambda^{\frac{2\alpha}{1+\alpha}}\, s^{2\alpha}$$

  (then $\partial_z - \boldsymbol{A}_z$ becomes differential operator encoding eigenvalues of unbarred IM, while $\partial_{\bar{z}} - \boldsymbol{A}_{\bar{z}}$ will turn out to be trivial)

- Substitute in

$$\eta \mapsto \eta^{(\mathrm{vac},\ell)} \sim \ell \log(z)$$

This yields

$$\Big[-\partial_x^2 + \frac{\ell(\ell+1)}{x^2} + x^{2\alpha}\Big]\psi = E\,\psi$$

ODE/IQFT correspondence for quantum KdV

Schrödinger eq. for 3D anharmonic oscillator ⟷ Vacuum eigenvalues for quantum KdV IM

$$D_{\pm}(E) = A_{\pm}^{(\mathrm{vac})}(\lambda)$$

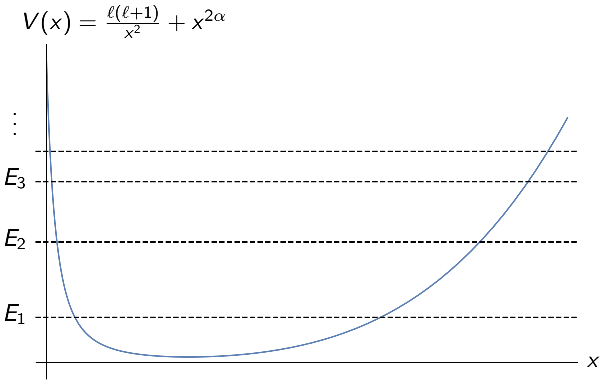
Spectral determinants of Schrödinger eq. → $D_{\pm}(E)$

Vacuum eigenvalue of $\mathbb{A}_{\pm}(\lambda)$ in $\mathcal{F}_p$ → $A_{\pm}^{(\mathrm{vac})}(\lambda)$

with $E \propto \lambda^2$, $\alpha = \beta^{-2} - 1$, $\ell = p - \frac{1}{2}$

- first discovered [Dorey, Tateo '98] (for $\ell = 0$)
- extended and proved [BLZ '98]
- generalized to excited states [BLZ '03] (proof in [Conti, Masoero '20;'21])

Spectral determinants



$$D_+(E) = \prod_{j=1}^{\infty}\left(1 - \frac{E}{E_j}\right)$$

(technical assumption $\alpha > 1$)

and $D_-(E|\ell) = D_+(E|-\ell-1)$

Comment on proof

Most important in quantum KdV is the **quantum Wronskian relation** [BLZ'96]:

$$(1) \qquad q^{p}\,\mathbb{A}_{+}(q^{\frac{1}{2}}\lambda)\,\mathbb{A}_{-}(q^{-\frac{1}{2}}\lambda) - q^{-p}\,\mathbb{A}_{+}(q^{-\frac{1}{2}}\lambda)\,\mathbb{A}_{-}(q^{+\frac{1}{2}}\lambda) = q^{p} - q^{-p}$$

On ODE side one can prove [BLZ'98]

$$(2) \qquad \mathrm{e}^{\frac{\mathrm{i}\pi}{\alpha+1}(\ell+\frac{1}{2})}\,D_{+}(\mathrm{e}^{\frac{2\mathrm{i}\pi}{\alpha+1}}E)\,D_{-}(E) - \mathrm{e}^{-\frac{\mathrm{i}\pi}{\alpha+1}(\ell+\frac{1}{2})}\,D_{+}(E)\,D_{-}(\mathrm{e}^{\frac{2\mathrm{i}\pi}{\alpha+1}}E) = \mathrm{e}^{\frac{\mathrm{i}\pi}{\alpha+1}(\ell+\frac{1}{2})} - \mathrm{e}^{-\frac{\mathrm{i}\pi}{\alpha+1}(\ell+\frac{1}{2})}$$

Relations (1) (specialized to common eigenvector) and (2) coincide if

$$q = \mathrm{e}^{\mathrm{i}\pi\beta^{2}} = \mathrm{e}^{\frac{\mathrm{i}\pi}{\alpha+1}}\,, \qquad p = \ell + \tfrac{1}{2}\,, \qquad E \propto \lambda^{2}$$

Additional **analyticity properties** required to show $D_{\pm}(E) = A_{\pm}^{(\mathrm{vac})}(\lambda)$

Other integrable structures: quantum AKNS

$$\left[-\partial_z^2+\frac{p^2-\frac{1}{4}}{z^2}+\frac{2\mathrm{i}s}{z}+1+\mu^{-2-n}\,z^n\right]\Psi=0$$

encodes vacuum eigenvalues of IM for quantum AKNS integrable structure [Fateev, Lukyanov '05]

$$\mathbb{O}\ :\quad \mathcal{F}_{p,s}\mapsto\mathcal{F}_{p,s}\,,\qquad\qquad [\mathbb{O},\mathbb{O}']=0$$

First few local IM:

$$\mathbb{I}_1 = \int_0^{2\pi}\frac{\mathrm{d}x}{2\pi}\,\Big((\partial\phi_1)^2+(\partial\phi_2)^2\Big)$$

$$\mathbb{I}_2 = \int_0^{2\pi}\frac{\mathrm{d}x}{2\pi}\,\left((\partial\phi_1)^3+\frac{3\,(n+2)}{3n+4}\,\partial\phi_1\,(\partial\phi_2)^2+\frac{3\mathrm{i}\,(n+1)\sqrt{n+2}}{3n+4}\,\partial\phi_1\,\partial^2\phi_2\right)$$

Massive ODE/IQFT correspondence has not been adequately explored, but is expected to involve complex sinh-Gordon I IQFT and its variants

Other integrable structures: Fateev

$$\left[-\partial_z^2 + \kappa^2 \mathcal{P}(z) + \sum_{j=1}^{3} \frac{p_j^2 - \frac{1}{4}}{(z-z_j)^2} - \frac{\gamma_j}{z-z_j} \right] \Psi(z) = 0$$

$$\mathcal{P}(z) = \prod_{j=1}^{3} (z-z_j)^{a_j - 2} \qquad \text{with} \qquad a_i > 0 : \ a_1 + a_2 + a_3 = 2$$

Encodes vacuum eigenvalues of Fateev integrable structure [Fateev '95]

$$\mathbb{O} \ : \quad \mathcal{F}_{p_1,p_2,p_3} \mapsto \mathcal{F}_{p_1,p_2,p_3}\,, \qquad [\mathbb{O}, \mathbb{O}'] = 0$$

Massive case studied in [Lukyanov '13; Bazhanov, Lukyanov '13]:

$$\partial_z \partial_{\bar{z}} \eta = e^{2\eta} - \rho^2 |\mathcal{P}(z)|^2 e^{-2\eta}$$

$$\mathcal{L} = \frac{1}{16\pi} \sum_{j=1}^{3} (\partial_\mu \varphi_j)^2 + 2\mu \left(e^{i\alpha_3\varphi_3} \cos(\alpha_1\varphi_1 + \alpha_2\varphi_2) + e^{-i\alpha_3\varphi_3} \cos(\alpha_1\varphi_1 - \alpha_2\varphi_2) \right) \qquad (a_i = 4\alpha_i^2)$$

So far: review of ODE/IQFT correspondence

Up next: some new results from
[Gehrmann, GK, Lukyanov'24]

Generalization of ODE

Re-write Schrödinger equation as:

$$\Big[-\partial_y^2 + p^2 + \mathrm{e}^{(n+r)y} - \tilde{E}^r\,\mathrm{e}^{ry}\Big]\Psi = 0$$

$\frac{\ell(\ell+1)}{x^2}$ → p^2, $x^{2\alpha}$ → $\mathrm{e}^{(n+r)y}$, E → $\tilde{E}^r\,\mathrm{e}^{ry}$

New parameters:

$$r = 1, 2, 3, \ldots\,, \qquad n > 0\,, \qquad p\,, \qquad \tilde{E}$$

Relation to old parameters

$$\alpha = \frac{n}{r}\,, \qquad p = \frac{r}{2}\left(\ell + \frac{1}{2}\right), \qquad \tilde{E} \propto E^{\frac{1}{r}}$$

and variables

$$y = \frac{2}{r}\log(x) + \frac{2}{n+r}\,\log\left(\frac{r}{2}\right), \qquad \Psi = x^{-\frac{1}{2}}\,\psi\,,$$

Generalization of ODE

$$\Big[-\partial_y^2 + p^2 + \mathrm{e}^{(n+r)y} - \tilde{E}^r \mathrm{e}^{ry} - \delta U(y) \Big] \Psi = 0$$

Extra term allowed to depend on $\tilde{E}$:

$$D_\pm(\tilde{E}^r) \mapsto D_\pm(\tilde{E})$$

$D_\pm(\tilde{E})$ can be introduced as connection coefficients of the ODE (to be discussed)

Key idea: choose δU such that quantum Wronskian relation is still satisfied:

$$\mathrm{e}^{+\frac{2\pi \mathrm{i}}{n+r}p} D_+(\tilde{q}^2 \tilde{E}) D_-(\tilde{E}) - \mathrm{e}^{-\frac{2\pi \mathrm{i}}{n+r}p} D_+(\tilde{E}) D_-(\tilde{q}^2 \tilde{E}) = \mathrm{e}^{+\frac{2\pi \mathrm{i}}{n+r}p} - \mathrm{e}^{-\frac{2\pi \mathrm{i}}{n+r}p}$$

with

$$\tilde{q}^{2r} = q^2 = \mathrm{e}^{\frac{2r\pi \mathrm{i}}{n+r}} \quad \Longrightarrow \quad \tilde{q} = \mathrm{e}^{\frac{\mathrm{i}\pi A}{r} + \frac{\mathrm{i}\pi}{n+r}} \qquad \text{where} \qquad A = 0, 1, 2, \ldots, r-1$$

Quantum Wronskian relation requires [BLZ '98]

- Invariance of ODE under symmetry transformation

$$\hat{\Omega}\ :\ y \mapsto y + \frac{2\pi \mathrm{i}}{n+r}\,, \qquad\qquad \tilde{E} \mapsto \tilde{q}^{-2}\,\tilde{E}$$

- Decay of potential at $y \to -\infty$

$$\lim_{y\to-\infty} \delta U(y) = 0$$

Definition of $D_{\pm}(\tilde{E})$ and proof of quantum Wronskian relation given in a moment ...

Generalization of ODE [Gehrmann, GK, Lukyanov'24]

$$\Big[-\partial_y^2 + p^2 + \mathrm{e}^{(n+r)y} - (-1)^A \tilde{E}^r \mathrm{e}^{ry} - \delta U(y) \Big] \Psi = 0$$

with

$$\delta U(y) = \sum_{(\mu,j)\in\Xi_{r,A}} c_{\mu,j}\, \tilde{E}^{\mu}\, \mathrm{e}^{\big((A\mu - rj)\,\frac{n+r}{r} + \mu\big) y} \qquad (\mu, j \text{ integers})$$

- A integer such that $A = 0, 1, 2, \ldots, r-1$
- $c_{\mu,j}$ arbitrary continuous parameters
- $\Xi_{r,A}$ certain **non-negative integer** set fixed by requirements:

$$\lim_{y\to-\infty} \delta U(y) = 0\,, \qquad \lim_{y\to\infty} \mathrm{e}^{-(n+r)y}\delta U(y) = 0 \qquad (\forall n > 0)$$

(second condition imposed so that addition of $\delta U(y)$ in ODE is perturbation)

The set $\Xi_{r,A}$

In general,

$$\left[\tfrac{r+1}{4}\right] \le |\Xi_{r,A}| \le \left[\tfrac{r-1}{2}\right]$$

- For $A = 1, 2, \dots, r-2$:

$$\Xi_{r,A} \quad = \quad \left\{(\mu, j) \ : \quad \tfrac{rj}{A} < \mu < \tfrac{r}{A+1}\,(j+1) \ \ \& \ \ j \ge 0\right\}$$

- For $A = 0$:

$$\Xi_{r,0} \quad = \quad \left\{(\mu, j) \ : \quad \mu = 1, 2, \dots, r-1 \ \ \& \ \ j = 0\right\}$$

- For $A = r-1$:

$$\Xi_{r,r-1} = \left\{(\mu, j) \ : \quad \mu = j+1 \ \ \& \ \ j = 0, 1, \dots, r-2\right\}$$

Definition of $D_\pm(\tilde{E})$

$$\left[-\partial_y^2 + p^2 + \mathrm{e}^{(n+r)y} - (-1)^A \tilde{E}^r \mathrm{e}^{ry} - \delta U(y)\right]\Psi = 0 \qquad \text{with} \qquad \lim_{y\to-\infty} \delta U(y) = 0$$

- Introduce two solutions defined near $y = -\infty$:

$$\psi_+(y) \to \mathrm{e}^{py} \qquad \text{as} \qquad y \to -\infty\,, \qquad \psi_- = \psi_+(y|-p)$$

- Introduce decaying solution via WKB asymptotic:

$$\chi \asymp \exp\left(-\frac{n+r}{4}\,y - \frac{2}{n+r}\,\mathrm{e}^{\frac{n+r}{2}y} + \ldots\right) \qquad \text{as} \qquad y \to +\infty$$

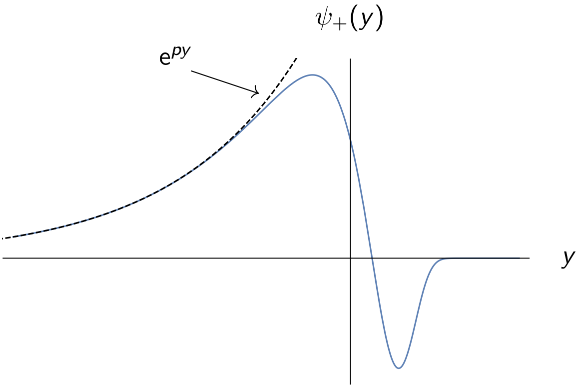
- Then

$$D_\pm(\tilde{E}) = \frac{\sqrt{\pi}}{\Gamma(1 \pm \frac{2p}{n+r})}\,(n+r)^{-\frac{1}{2}\mp\frac{2p}{n+r}}\,W[\chi, \psi_\pm]\,, \qquad D_\pm(0) = 1\,,$$

where $W[f,g] = f\partial_y g - g\partial_y f$ is the Wronskian

Connection to spectral determinant

$$D_+(\tilde{E}_*) = 0 \qquad \Longrightarrow \qquad W[\chi, \psi_+] = 0$$



$\Longrightarrow$ $\exists$ normalizable solution, i.e., E_* is an 'eigenvalue' of the linear ODE

Proof of quantum Wronskian relation [BLZ'98]

For ODE

$$\Big[-\partial_y^2 + p^2 + \mathrm{e}^{(n+r)y} - (-1)^A \tilde{E}^r \mathrm{e}^{ry} - \delta U(y)\Big]\Psi = 0$$

invariant under symmetry transformation

$$\hat{\Omega}\;:\; y \mapsto y + \frac{2\pi \mathrm{i}}{n+r}\,, \qquad\qquad \tilde{E} \mapsto \tilde{q}^{-2}\,\tilde{E}$$

quantum Wronskian relation is proved in three steps

(i) Expand χ into basis defined near $y \to -\infty$

$$\chi = B_+\psi_+ - B_-\psi_-$$

(ii) Apply $\hat{\Omega}$ to both sides:

$$\hat{\Omega}\circ\chi = B'_+\hat{\Omega}\circ\psi_+ + B'_-\hat{\Omega}\circ\psi_-$$

(iii) Compute Wronskian of χ and $\hat{\Omega}\circ\chi$

Proof of quantum Wronskian relation for $D_\pm(E)$

(i) Connection coefficient expressing χ in terms of $\psi_\pm$ are the Wronskians

$$\chi = \frac{1}{2p}\Big(W_+(\tilde{E})\,\psi_- - W_-(\tilde{E})\,\psi_+\Big) \qquad \text{with} \qquad W_\pm = \chi\,\partial_y\psi_\pm - \psi_\pm\,\partial_y\chi$$

(ii) Since $\psi_+ \sim e^{py}$ one has

$$\hat{\Omega}\circ\psi_+ = e^{+\frac{2\pi i}{n+r}p}\psi_+ \qquad \text{and similarly} \qquad \hat{\Omega}\circ\psi_- = e^{-\frac{2\pi i}{n+r}p}\,\psi_-$$

This way

$$\hat{\Omega}\circ\chi = \frac{1}{2p}\Big(e^{-\frac{2\pi i}{n+r}p}\,W_+(\tilde{q}^{-2}\tilde{E})\,\psi_- - e^{+\frac{2\pi i}{n+r}p}\,W_-(\tilde{q}^{-2}\tilde{E})\,\psi_+\Big)$$

Proof of quantum Wronskian relation for $D_\pm(E)$

(iii) Compute Wronskian of χ and $\hat{\Omega}\circ\chi$ in two ways:

- At large y via asymptotic formulae

$$\chi \asymp \exp\Big(-\tfrac{n+r}{4}\,y - \tfrac{2}{n+r}\,\mathrm{e}^{\frac{n+r}{2}y} + \ldots\Big)$$

$$\hat{\Omega}\circ\chi \asymp -\mathrm{i}\exp\Big(-\tfrac{n+r}{4}\,y + \tfrac{2}{n+r}\,\mathrm{e}^{\frac{n+r}{2}y} + \ldots\Big) \qquad \Longrightarrow \qquad W[\chi,\hat{\Omega}\circ\chi] = -2\mathrm{i}$$

- With χ and $\hat{\Omega}\circ\chi$ expressed in terms of $\psi_\pm$ $\Longrightarrow$

$$W[\chi,\hat{\Omega}\circ\chi] = -\frac{1}{2p}\Big(\mathrm{e}^{+\frac{2\pi\mathrm{i}}{n+r}p}\,W_+(\tilde{E})\,W_-(\tilde{q}^{-2}\tilde{E}) - \mathrm{e}^{-\frac{2\pi\mathrm{i}}{n+r}p}\,W_-(\tilde{E})\,W_+(\tilde{q}^{-2}\tilde{E})\Big)$$

$$\mathrm{e}^{+\frac{2\pi\mathrm{i}}{n+r}p}\,D_+(\tilde{q}^2\tilde{E})\,D_-(\tilde{E}) - \mathrm{e}^{-\frac{2\pi\mathrm{i}}{n+r}p}\,D_+(\tilde{E})\,D_-(\tilde{q}^2\tilde{E}) = \mathrm{e}^{+\frac{2\pi\mathrm{i}}{n+r}p} - \mathrm{e}^{-\frac{2\pi\mathrm{i}}{n+r}p}$$

$$D_\pm = C_\pm\,W_\pm \qquad \text{and} \qquad C_\pm = \sqrt{\pi}\,(n+r)^{-\frac{1}{2}\mp\frac{2p}{n+r}}\big/\Gamma\big(1\pm\tfrac{2p}{n+r}\big)$$

What about the IQFT?

ODE	✓
IQFT	?
Correspondence	?

For ODE with general $A = 0, 1, 2, \ldots r-1$ we have not discussed (if it even exists):

(i) Extended algebra of conformal symmetry

(ii) Construction of local/non-local IM

(iii) ODEs for excited states

Full answers are currently unknown except for ODE with $A = r-1$ studied in context of **generalized** $\mathfrak{sl}(2)$ **affine Gaudin model** [Kotousov, Lukyanov '21]

Lattice regularization

Main result of [Gehrmann, GK, Lukyanov'24] concerns integrable, critical 'inhomogeneous XXZ spin-$\frac{1}{2}$ chain' (to be discussed in Gehrmann's talk):

$$D_{\pm}(\tilde{E}) \quad = \quad \mathrm{slim}_{N\to\infty} A_{\pm}^{(N,\mathrm{vac})}$$

Connection coefficients (spectral determinants) of ODE

Scaling limit of ground state eigenvalues of Q operators for spin chain

- Spin chain Q operators can be thought of as "lattice regularization" of field theory Q operator (provides indirect definition)
- Allows for an alternative way of studying the ODE/IQFT correspondence

Summary

- ODE/IQFT correspondence for massive $1+1$D integrable QFT and classically integrable PDE

- UV limit $R/R_c \to 0$

  (i) IM in (massive) IQFT become integrable structure in CFT

  (ii) classical integrable PDE replaced by ODE

- ODE from [Gehrmann, GK, Lukyanov'24] that is expected to encode vacuum eigenvalues of new multiparametric integrable structure

- Connection to inhomogeneous XXZ spin-$\frac{1}{2}$ chain (to be discussed **in detail** in Gehrmann's talk)