Wrapping the (twisted) pair of pants

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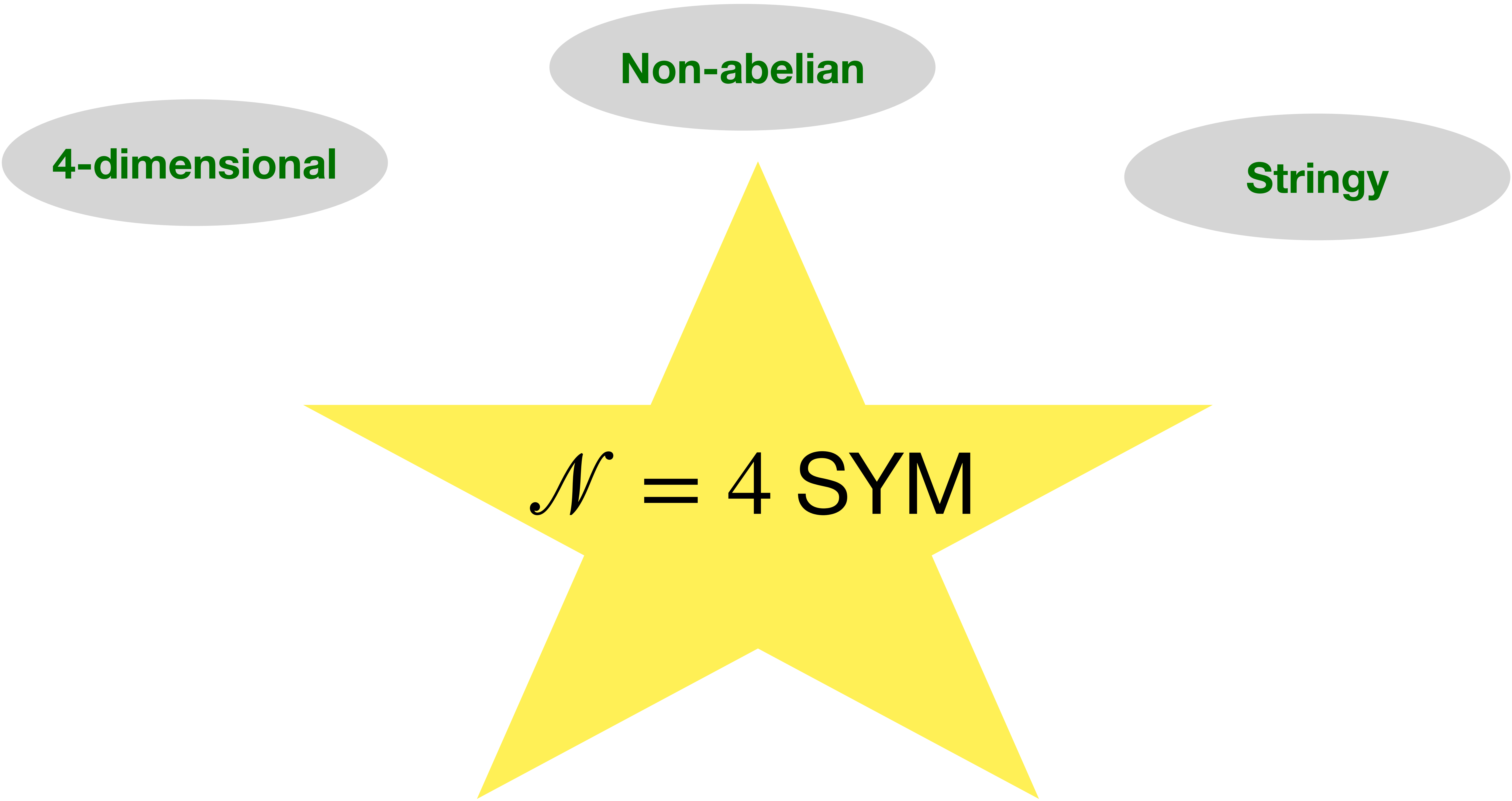
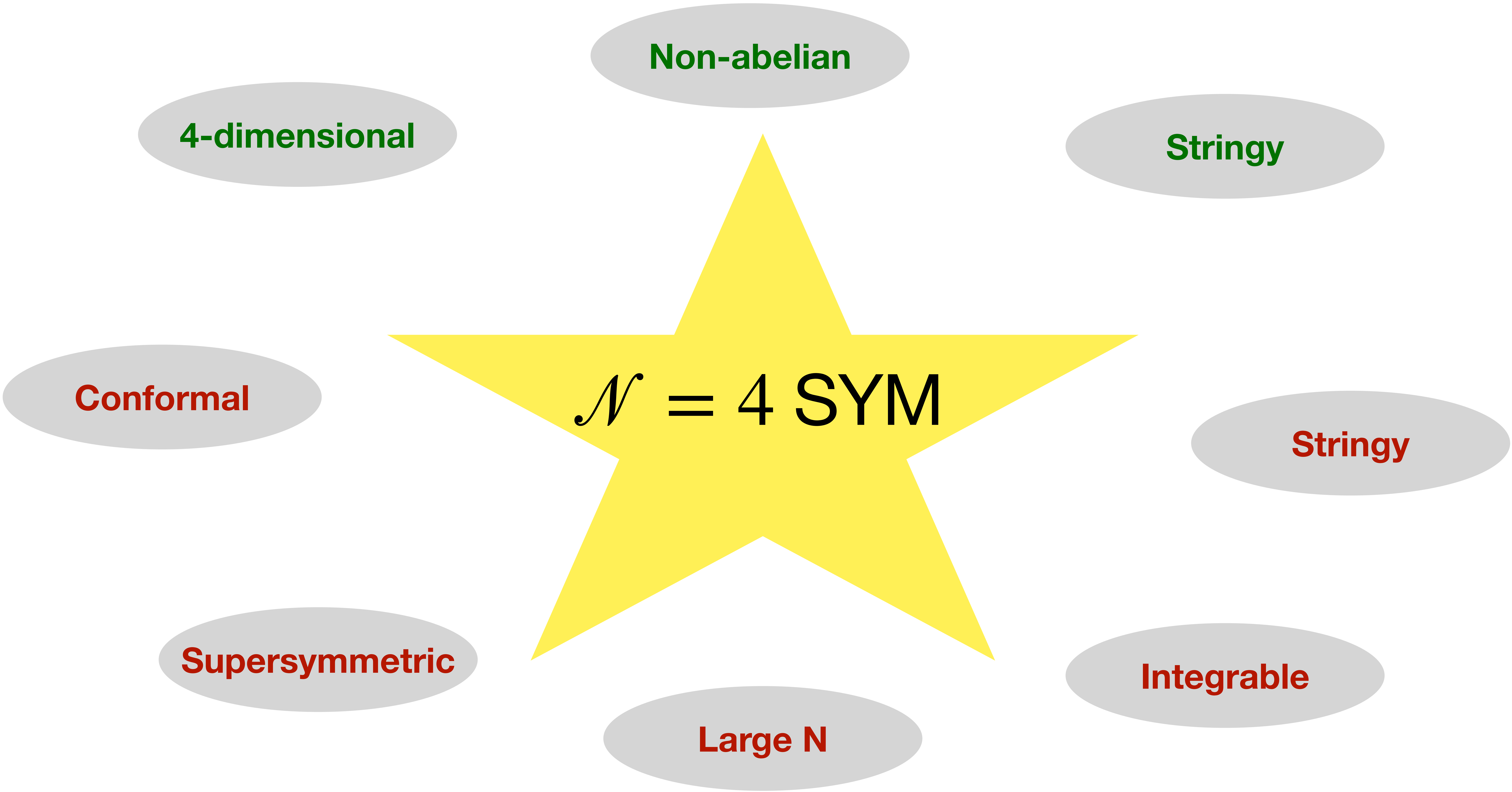
ICTP SAIFR

$\mathcal{N} = 4$ SYM


Non-abelian
4-dimensional
Stringy
$\mathcal{N} = 4$ SYM


Non-abelian
4-dimensional
Stringy
$\mathcal{N} = 4$ SYM
Conformal
Stringy
Supersymmetric
Integrable
Large N

Outline

1. Ingredients
2. Making pair of pants
3. Wrapping pairs of pants
4. Twisting pairs of pants
5. Wrapping twisted pair of pants
6. Conclusions and future directions

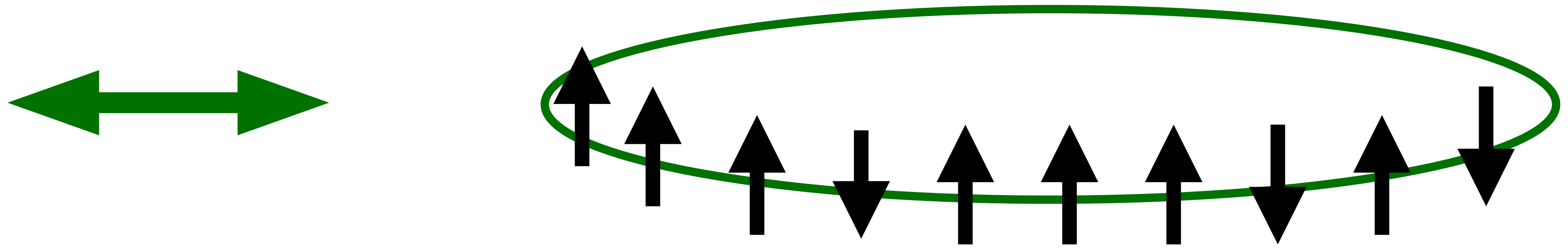
Ingredients

- The theory has 3 complex scalars (X, Y, Z), 4 complex fermions Ψ_a and the 4d gauge field A_μ. All in the adjoint rep. of SU(N)
- Bosonic symmetries: 4d conformal symmetry SO(2,4) ~ SU(2,2), R-symmetry SO(6) ~ SU(4)
- Together with SUSY generators: PSU(2,2|4)
- We focus on the single-trace operators, e.g. $\mathcal{O} = \mathrm{Tr}(ZZX\mathcal{D}_\mu\Psi_1 ZZ)$
- Ultimate goal: to find conformal data (Δ_j, C_{jkl}) at finite coupling in the planar limit
- Kolya and Simon's talks: Δ_j. Today: C_{jkl}

Ingredients

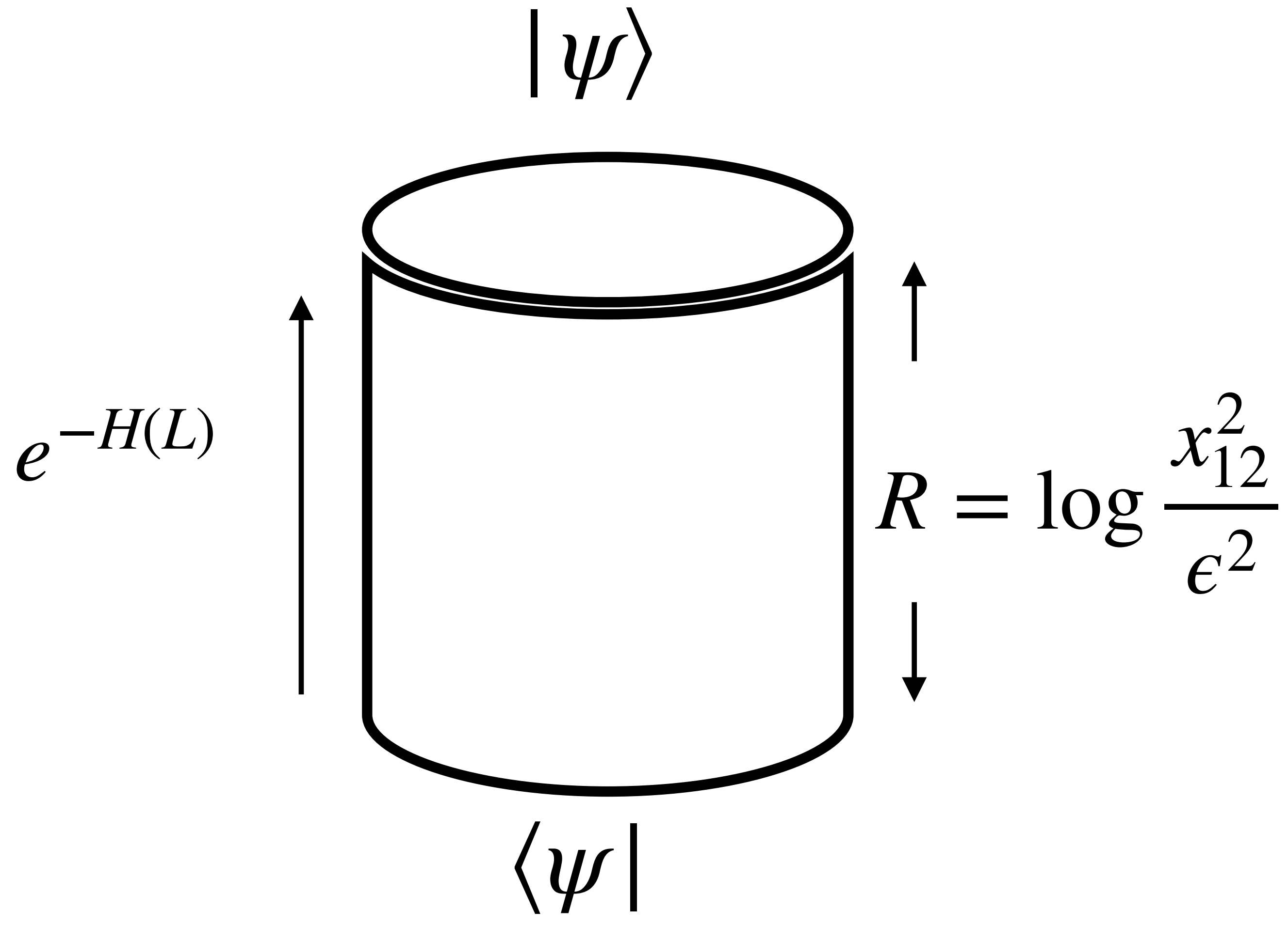
- At weak coupling, anomalous dimensions of single-trace operators are mapped into energies of spin-chain states

$$\mathcal{O} = \text{Tr}\ ZZZXZZZXZX$$

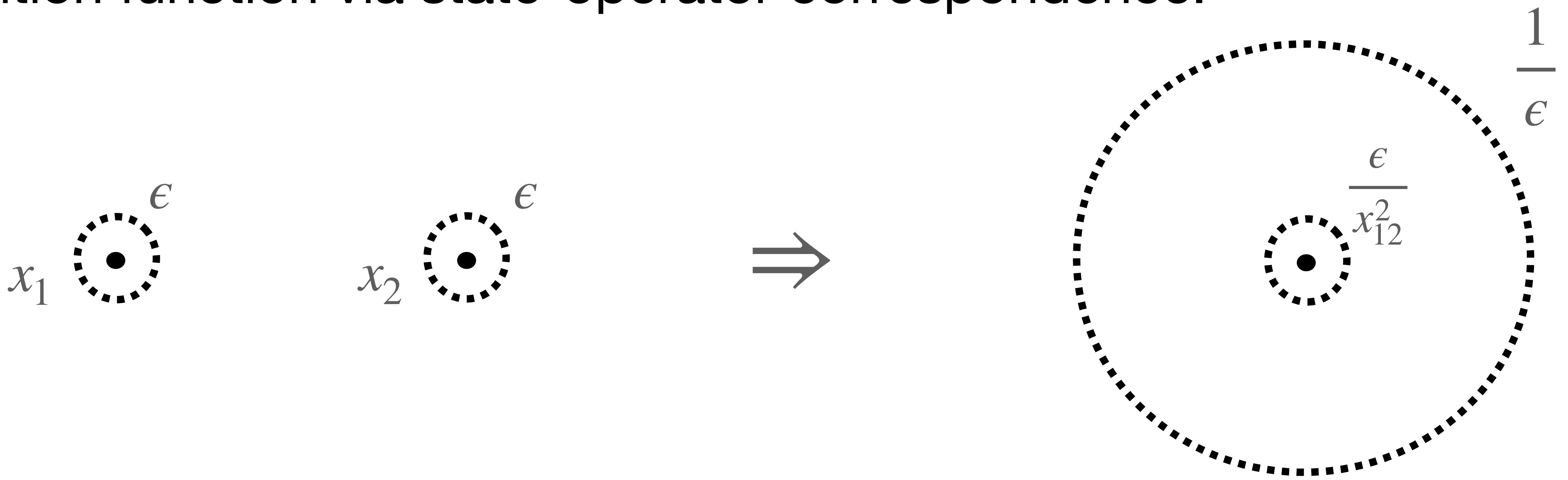


(D, Δ) $\qquad\qquad$ (H, E)

- The two-point function can be mapped to a cylinder partition function via state-operator correspondence:



"world-sheet of the string"



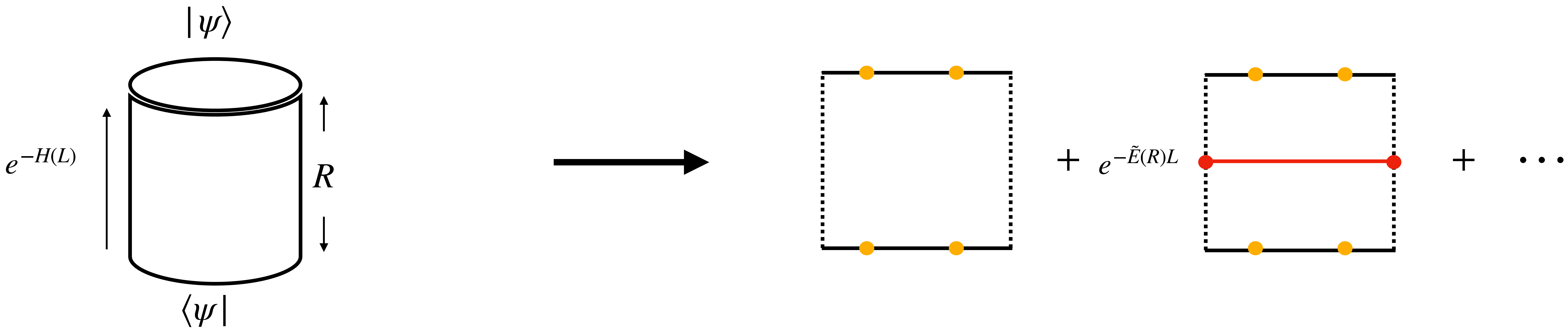
$$\langle\psi| e^{-RH(L)} |\psi\rangle = e^{-RE(L)} + \ldots = \frac{\epsilon^{2\Delta}}{x_{12}^{2\Delta}}$$

Ingredients

- To classify the excitations of the theory, look at symmetries of two-point function $\langle \text{Tr}(Z^L)(0)\ \text{Tr}(\bar{Z}^L)(\infty)\rangle$
  - Symmetry group: PSU(2|2) x PSU(2|2) + central extension (all dependence on the coupling is inside the central charges)
    - Bosonic part is clear: SO(1,3) in spacetime and SO(4) rotations in the scalars
  - Magnons: $\chi^A \dot{\chi}^{\dot{A}}$, where $\chi^A = (\phi^1, \phi^2, \psi^1, \psi^2)$
  - e.g. $X = \phi^1 \dot{\phi}^1$, $D^{\alpha\dot{\alpha}} = \psi^\alpha \dot{\psi}^{\dot{\alpha}}$
- Magnons scatter with PSU(2|2) x PSU(2|2) S-matrices, fixed by symmetry.
- The energy and momentum of each magnon are given by $e^{ip} = \dfrac{x^+(u)}{x^-(u)}, \quad E = \dfrac{1}{2}\dfrac{1 + x^+(u)x^-(u)}{1 - x^+(u)x^-(u)}$
- Where $x(u)$ is the Жуко́вский variable, given by $x(u) + \dfrac{1}{x(u)} = \dfrac{u}{g}$
- There's a crossing transformation that sends $x^\pm \to \dfrac{1}{x^\pm}$ and, consequently, $(p, E) \to (-p, -E)$

Ingredients

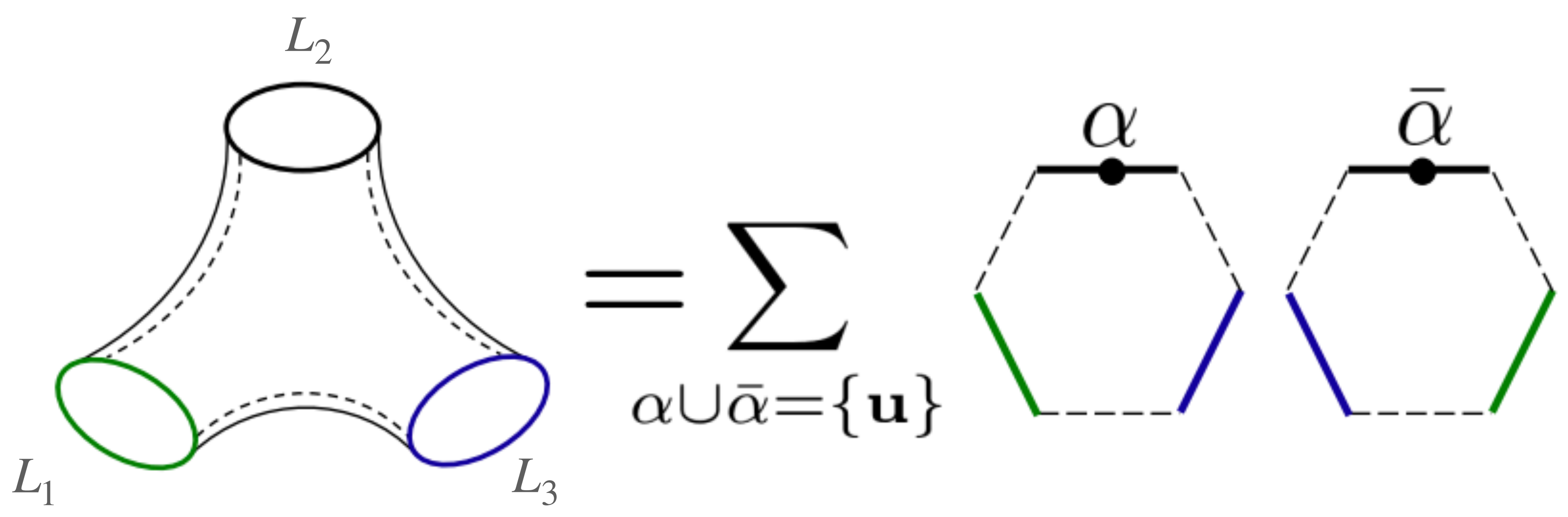
- Of course the S-matrix only makes sense at infinite volume, where one can have asymptotic states
- We need then to decompactify the cylinder by inserting a complete basis of mirror states along the dotted lines
- These virtual particles account for the finite size effects and are called wrapping effects
- The resummation of the finite size corrections is the TBA and can be upgraded to the QSC



Making pair of pants

$$\ell_{ij} = \frac{L_i + L_j - L_k}{2}$$

- In the same spirit, we can represent the three-point function as a pair of pants
- Then, the idea is to construct the three-point function by sticking together two hexagons, summing over the ways of distributing the physical magnons
- This is analogous to the cutting of the cylinder in the two-point function



“closed string = (open string)^2”

Making pair of pants

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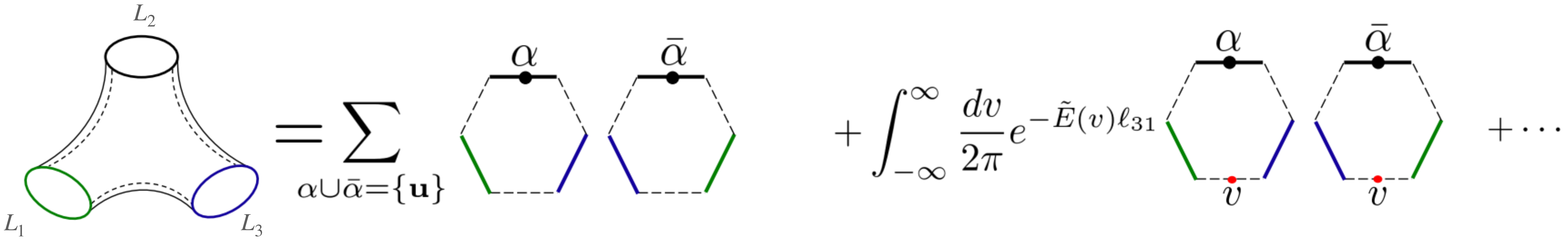
- When the bridge lengths are finite, we also need to sum over virtual magnons propagating from one operator to the other
- Concretely, this is realized by adding a virtual magnon and its antiparticle in each of the hexagons and identifying them by integrating over all rapidities and summing over all flavors

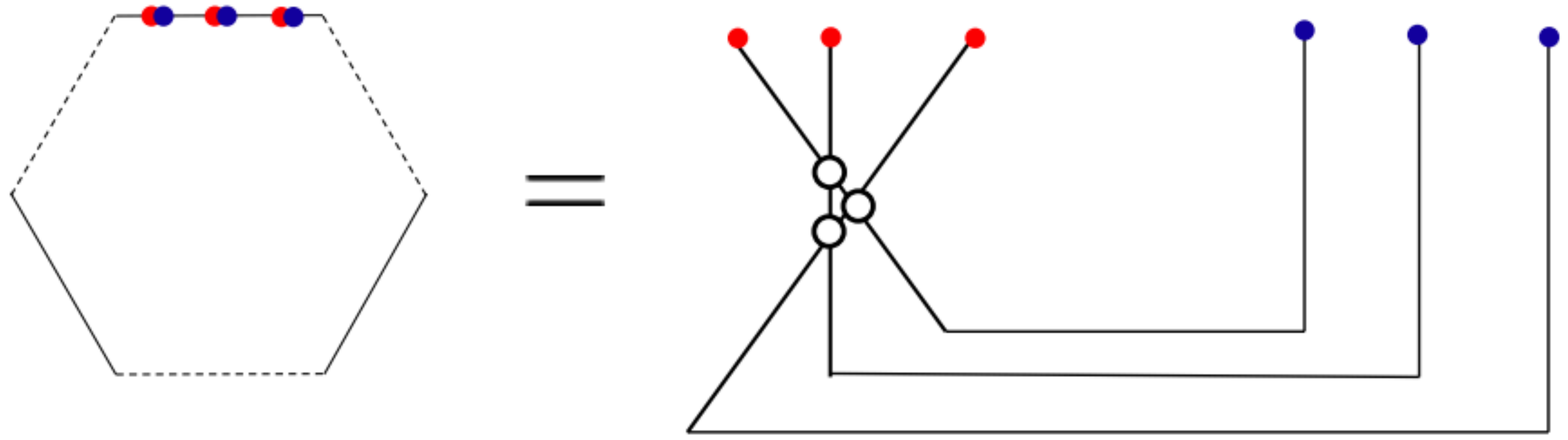


“closed string = (open string)^2”

[Basso, Komatsu, Vieira, Gonçalves, …]

Making pair of pants

- Each hexagon is pictorially represented as

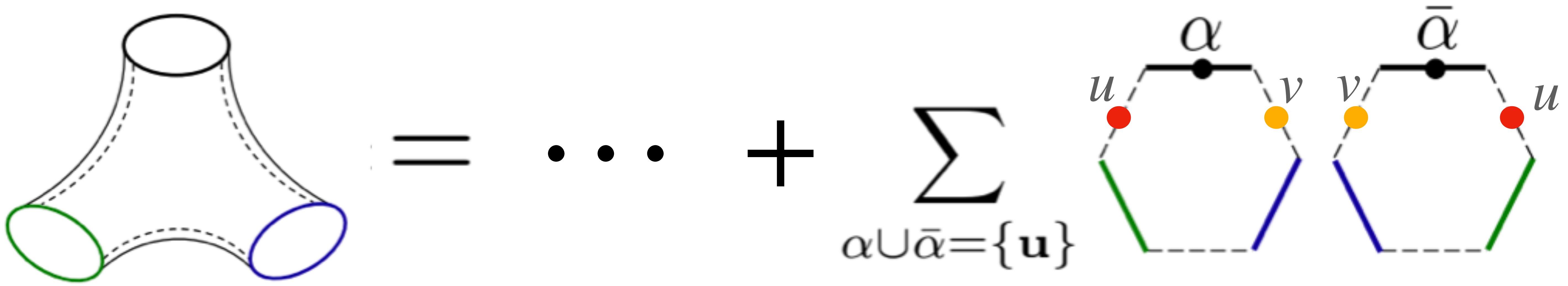


- Each white dot represents a factor $h(u,v) \times \mathcal{S}(u,v)$ where $\mathcal{S}$ is the PSU(2|2) S-matrix and h is fixed by symmetry.

- Putting everything together we can compute three-point functions at any coupling **except** by *wrapping corrections.*

Wrapping pair of pants

- Consider the term in the hexagon expansion given by



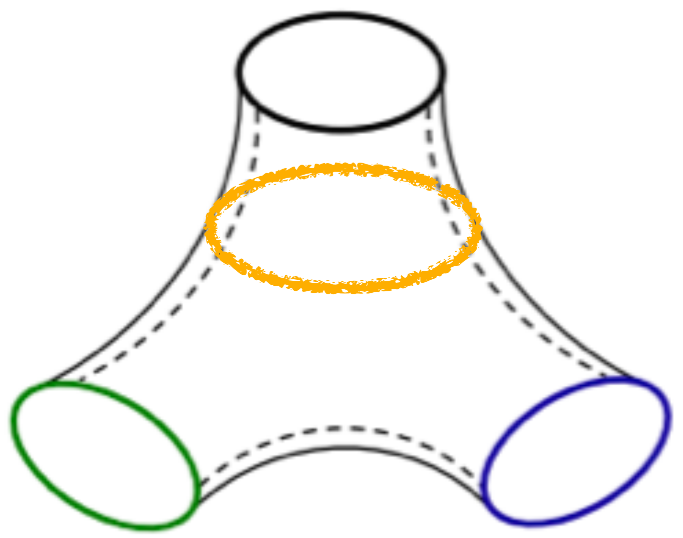
- We immediately encounter a problem, since there'll be a term of the form $h(u^{\gamma}, v^{5\gamma})h(v^{\gamma}, u^{5\gamma}) \sim \dfrac{1}{(u-v)^2}$ inside the double integral in u and v

- It's necessary to regularize the prescription. Idea: shift rapidities in the left/right hexagon by +/- $i\epsilon$

- $\dfrac{1}{(u-v)^2}(\ldots) \to \dfrac{1}{(u-v+i\epsilon)^2}(\ldots) + \dfrac{1}{\epsilon}2\pi\delta(u-v)(\ldots)\Big|_{\epsilon\to 0} + \pi\delta(u-v)i\partial_{\epsilon}(\ldots)\Big|_{\epsilon\to 0}$

Wrapping pair of pants

- This contribution corresponds to a propagating bound state of virtual particles wrapping around one of our operators:



- Some of these corrections are present in the two-point function and are taken into account by TBA (or the more powerful QSC) and some of them are new. How to re-sum them systematically?

- Some results for the full re-summation are available **[Basso, Georgoudis, Klemenchuk-Sueiro]** but we'd like to understand what's a regularization scheme that works for any number of wrapping magnons.

- In what follows, we are going to move to a twisted theory where the three-point functions of vacuum states is non-trivial and it contains wrapping corrections in order to understand better what's happening and to check our understanding against known results.

Twisting pair of pants

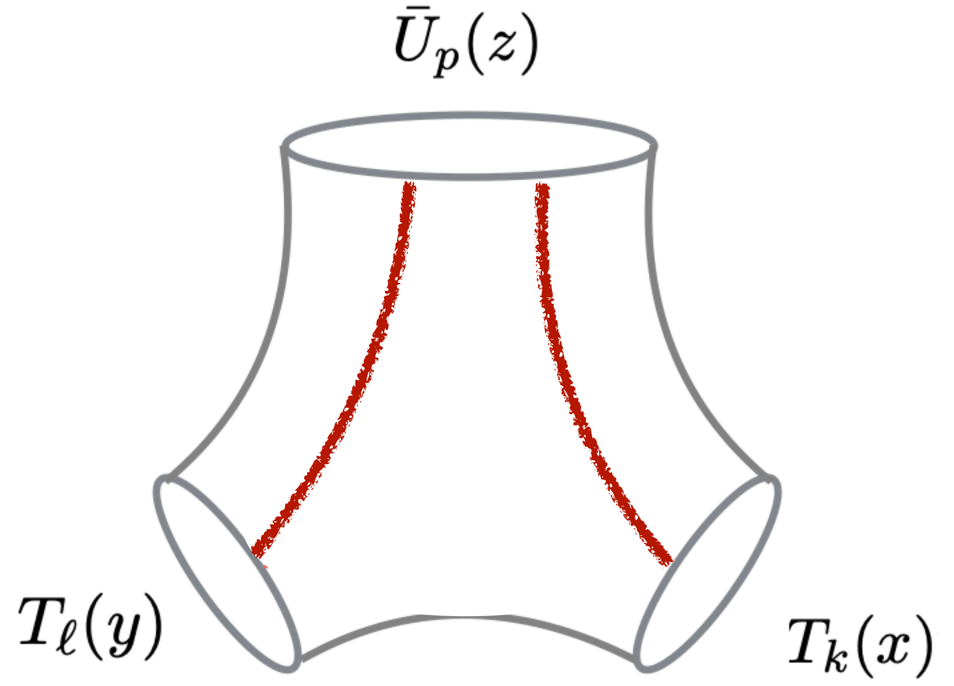
- We consider now a Z_2 orbifold of $\mathcal{N} = 4$ SYM. In this theory we have now a gauge group $SU\left(\frac{N}{2}\right) \times SU\left(\frac{N}{2}\right)$ instead of $SU(N)$.
- In particular, the scalar Z is broken into two pieces $Z = \begin{pmatrix} Z_0 & 0 \\ 0 & Z_1 \end{pmatrix}$ and we have a twist $\tau = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.
  - This twist acts on the right PSU(2|2) of the fundamental magnons as follows: $(\dot{\phi}^1, \dot{\phi}^2) \to -(\dot{\phi}^1, \dot{\phi}^2)$
  - As a consequence, $\tau(X, \bar{X}, Y, \bar{Y})\tau = -(X, \bar{X}, Y, \bar{Y})$
- Now we have two different vacua: $U_k = \frac{1}{\sqrt{2}}\text{Tr}(Z^k)$ and $T_k = \frac{1}{\sqrt{2}}\text{Tr}(\tau Z^k)$.
- This model is still integrable! Many observables computed using supersymmetric localization.

Twisting pair of pants

$p = k + l$

- Among the observables one can compute, there is the (extremal) three-point function:

$$C_{\bar{U}_p,T_k,T_l} = \frac{\det(1-K_{l+1})}{\sqrt{\det(1-K_l)\det(1-K_{l+2})}} \times \frac{\det(1-K_{k+1})}{\sqrt{\det(1-K_k)\det(1-K_{k+2})}}$$



- The result nicely factorizes and we can recognize the two numerators as the contributions coming from summing the bridge magnons on the two bridges with the twist

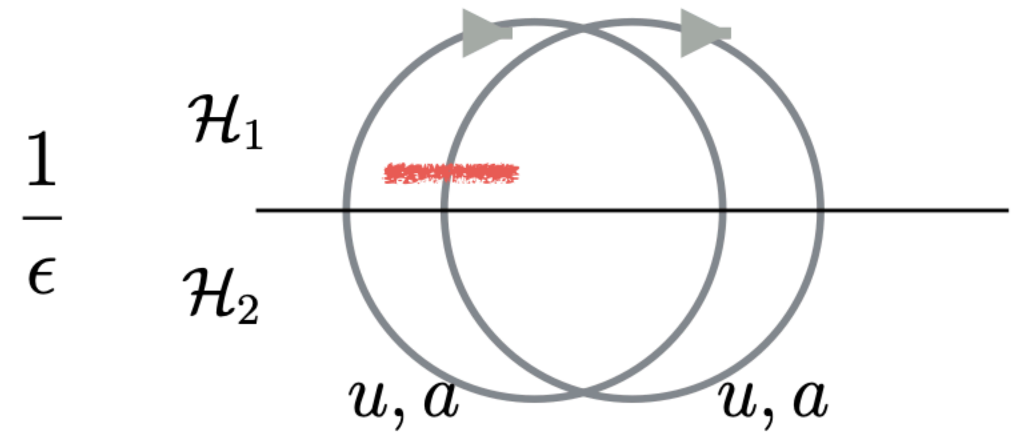
- All the other terms should come from wrapping and we should be able to reproduce them with hexagonalization

- In these formulas, K_l is the octagon kernel, given by the semi-infinite matrix:

$$(K_l)_{mn} = -2\sqrt{(l+2m)(l+2n)} \int_0^\infty \frac{dt}{t} \frac{e^t}{(1-e^t)^2} J_{2m+l}(2gt) J_{2n+l}(2gt)$$

Wrapping twisted pair of pants

- For concreteness, we'll take $k, p \gg 1$ and try to compute only one of the two wrapping terms



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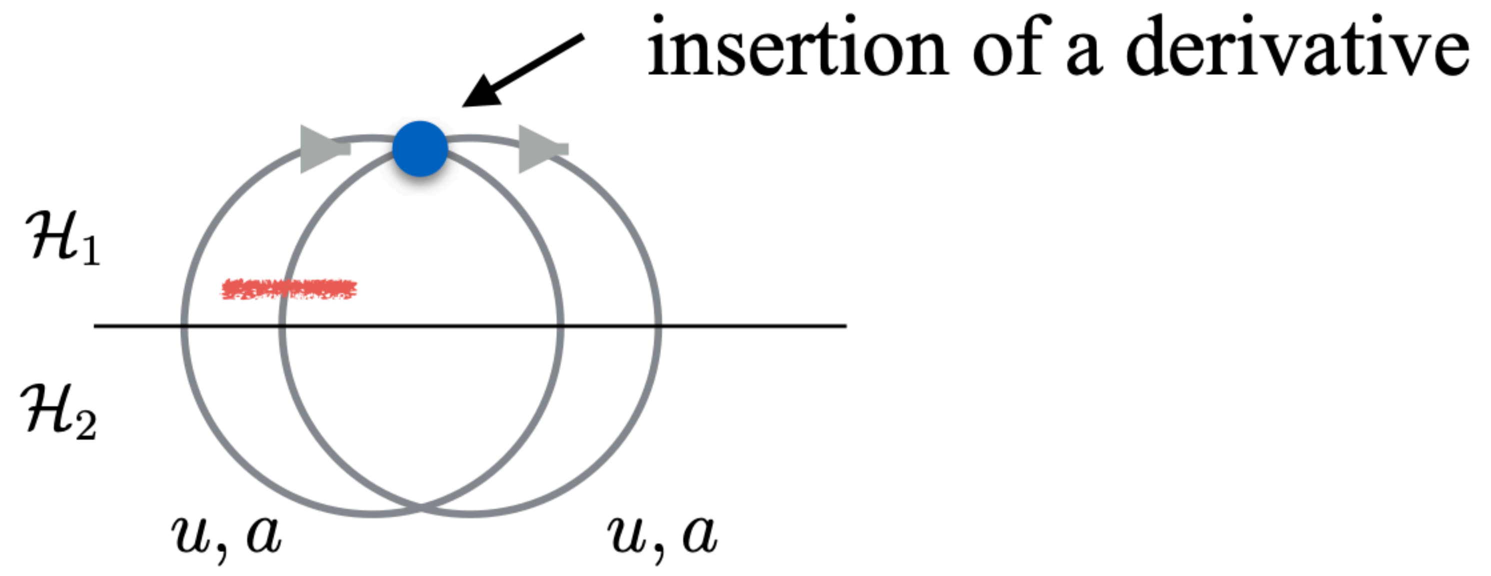
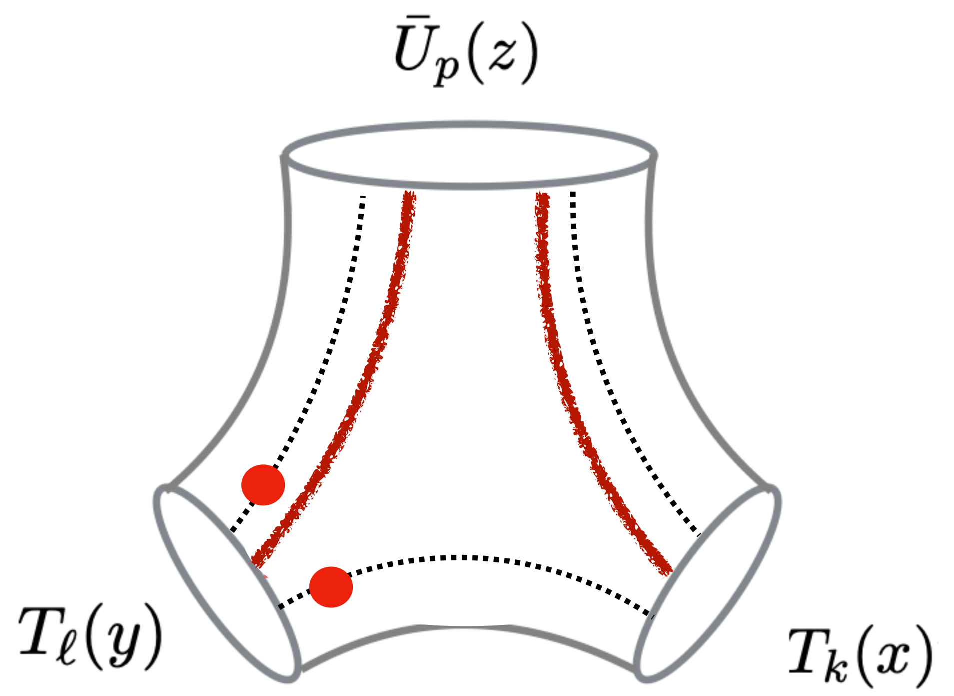




$$\mathrm{STr}_a\, \mathbf{1}_a \times \mathrm{STr}_a\, \tau_a = 0$$

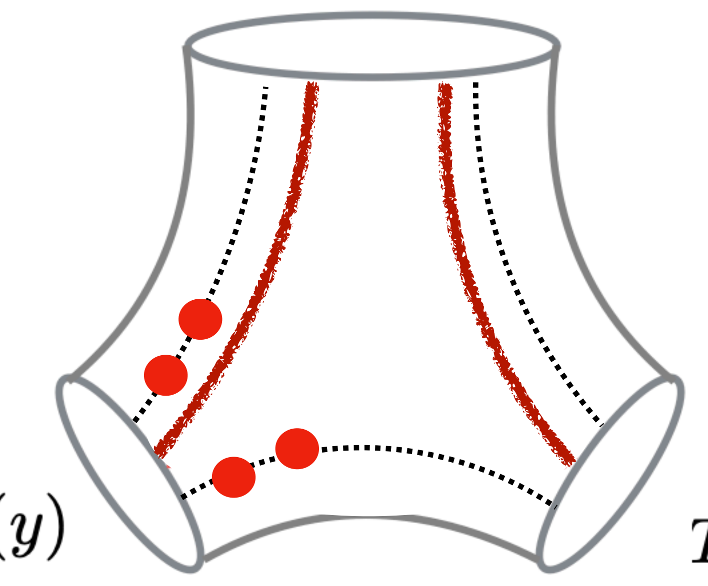
$$\widetilde{\mathcal{K}}_{aa}(u,u) = -i\, \mathrm{STr}_{a\otimes b}\{\mathcal{S}_{ba}(v^\gamma, u^\gamma)\,\tau_a\, \partial_u \mathcal{S}_{ab}(u^\gamma, v^\gamma)\}|_{v\to u;\ b\to a}$$

- As expected, wrapping does not lead to divergences since the energies of the vacua are protected

- Wrapping corrections take the form

$$W_1 = \frac{1}{2}\sum_{a\geq 1}(4a)\int_{-\infty}^{\infty}\frac{du}{2\pi}e^{-\tilde{E}_a l}\tilde{\mathcal{K}}_{aa}(u,u) = -\frac{1}{2}\sum_{a\geq 1}(4a)\int_{-\infty}^{\infty}\frac{du}{2\pi}e^{-\tilde{E}_a l}\left(e^{\tilde{E}_a} + e^{-\tilde{E}_a}\right) = \frac{1}{2}\mathrm{Tr}(K_l + K_{l+2})$$

Wrapping twisted pair of pants



- We can go on and compute the diagram with two magnons in each edge
- Simply shifting hexagons fails in this case
- Good way: start with a finite four-point function and go to OPE limit
- Hard to do in practice in $\mathcal{N} = 4$ SYM, but we can get some intuition from Fishnets
- With the intuition, we came up with a regularization prescription that reproduces the two-wrapping part of the result!

Future directions

1. How do the wrapping magnons nicely resum to $\frac{1}{\sqrt{\det(1-K_k)\det(1-K_{k+2})}}$?

2. How general is our regularization procedure? Can we test it in other setups?

3. Can we understand the results in **[Basso, Georgoudis, Klemenchuk-Sueiro]** from this regularization, without going to 4pt functions?

4. Can we write this very simple three-point function in an SoV-like form?

5. Or with Q-functions from QSC?

Thank you!

To be continued…