# Lens Partition Functions and Integrability Properties

Integrability, Q-systems, and Cluster Algebras, Varna, 2024

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August 16, 2024

$$\mathcal{Z}_{\mathcal{S}_b^3/\mathbb{Z}_r} \boxed{\begin{array}{c} Gauge = SU(2) \\ Flavour\ Group : SU(6) \end{array}} = \mathcal{Z}_{\mathcal{S}_b^3/\mathbb{Z}_r} \boxed{\begin{array}{c} No\ Gauge\ Symmetry \\ 15\ Chiral\ Multiplets \end{array}}$$

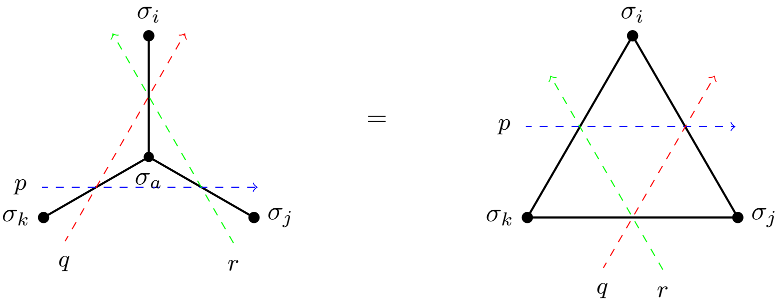
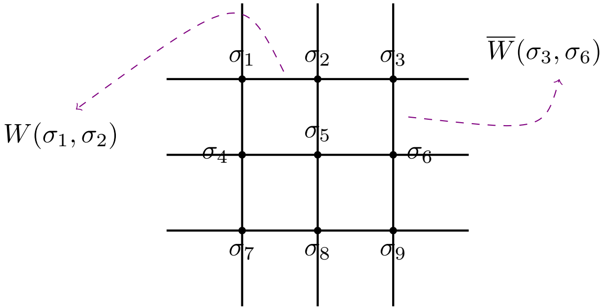



Figure: See [Yamazaki, 2018] and [Gahramanov and Shahriyar, 2017] for comprehensive review

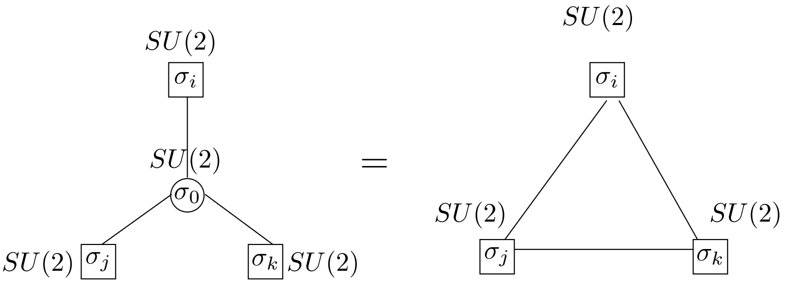
- The Ising model: 1920 (Lenz) - 1924 (Thesis) - 1925 (Paper)
- the Onsager's famous solution 1944
- the Fateev-Zamolodchikov model 1982
- the Kashiwara-Miwa model 1986
- the critical Potts model 1987

- the Faddeev-Volkov model 1992
- the Bazhanov-Sergeev model 2010
- the lens elliptic model (Yamazaki) 2013




$\sigma_j = (x_j, m_j)$ discrete and continuous spin packet

- $m_j$ for discrete spin values,
- $x_j$ for continuous spin values.




$\mathcal{Z}^A$ [*Gauge* $= SU(2)$; *Flavour Group* : $SU(6)$] $=$ $\mathcal{Z}^B$ [*No Gauge*; 15 *Chiral Multiplets*]

$$\sum_{m=-[r/2]}^{[r/2]} \int_{-\infty}^{\infty} \frac{\prod_{i=1}^{6} \gamma_h(a_i \pm x, u_i \pm m; \omega_1, \omega_2)}{\gamma_h(\pm 2ix, \pm 2m; \omega_1, \omega_2)} \frac{dx}{2r\sqrt{-\omega_1 \omega_2}} = \prod_{1 \leq i < j \leq 6} \gamma_h(a_i + a_j, u_i + u_j; \omega_1, \omega_2)$$

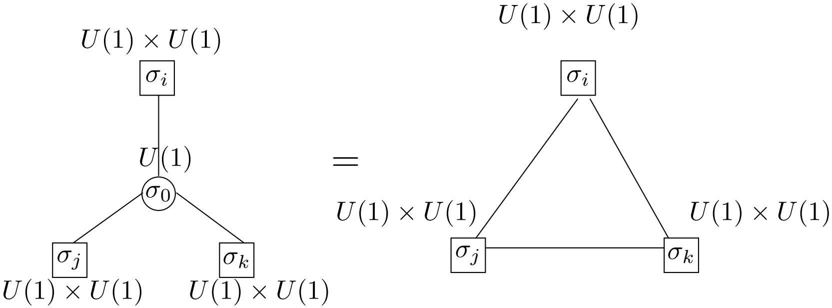
where balancing conditions are $\sum_{i=1}^{6} a_i = \omega_1 + \omega_2$ and $\sum_{i=1}^{6} u_i = 0$ and the Boltzmann weights are

$$W_{pq}(\sigma_i, \sigma_j) = \gamma_h(q - p \pm x_i \pm x_j, \pm m_i \pm m_j; \omega_1, \omega_2) \tag{1}$$

The Boltzmann weight satisfies the following star-triangle relation

$$\sum_{\sigma_0} \int dx_0 \overline{W}_{qr}(\sigma_i, \sigma_0) W_{pr}(\sigma_j, \sigma_0) \overline{W}_{pq}(\sigma_k, \sigma_0)$$
$$= R(p, q, r) W_{pq}(\sigma_j, \sigma_k) \overline{W}_{pr}(\sigma_i, \sigma_k) W_{qr}(\sigma_j, \sigma_i) \, . \tag{2}$$

# Star-triangle quiver diagram for U(1) gauge



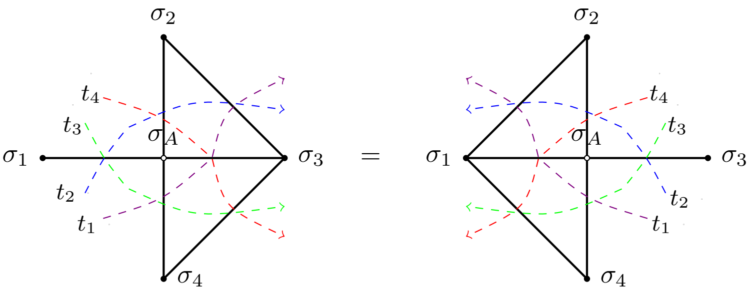

Obtained by gauge symmetry breaking [Spiridonov, 2010]

$\mathcal{Z}$ [ *Gauge* $= U(1)$; 6 *Chiral Multiplets*; *Global Symmetry* : $SU(3) \times SU(3) \times U(1)$ ] $=$ $\mathcal{Z}$ [ *No Gauge*; 9 *Free Mesons*; *Global Symmetry* : $SU(3) \times SU(3) \times U(1)$ ]

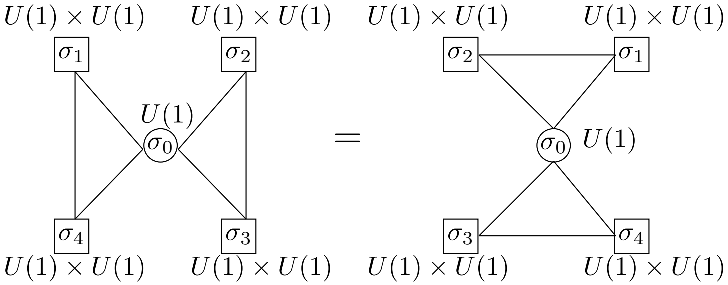
$$\sum_{m=-[r/2]}^{[r/2]} \int_{-\infty}^{\infty} \prod_{i=1}^{3} \gamma_h(a_i - x, u_i - m; \omega_1, \omega_2) \gamma_h(b_i + x, v_i + m; \omega_1, \omega_2) \frac{dx}{2r\sqrt{-\omega_1 \omega_2}} = \prod_{i,j=1}^{3} \gamma_h(a_i + b_j, u_i + u_j; \omega_1, \omega_2) \tag{3}$$

where balancing conditions are $\sum_{i=1}^{3} a_i + b_i = \omega_1 + \omega_2$ and $\sum_{i=1}^{3} u_i + v_i = 0$ and the Boltzmann weights are

$$W_{pq}(\sigma_i, \sigma_j) = \gamma_h(q - p + x_i - x_j, m_i - m_j; \omega_1, \omega_2) \times \gamma_h(q - p - x_i + x_j, m_i + m_j; \omega_1, \omega_2) \, . \tag{4}$$




$$R_{(t_{34}t_{21})}\begin{pmatrix} & \sigma_1 & \\ \sigma_2 & & \sigma_3 \\ & \sigma_4 & \end{pmatrix} = R_{(t_{21}t_{34})}\begin{pmatrix} & \sigma_1 & \\ \sigma_2 & & \sigma_3 \\ & \sigma_4 & \end{pmatrix} \tag{5}$$




$$\mathcal{Z}\left[\begin{array}{c} Gauge = U(1) \\ Flavour\ Symmetry: \\ SU(4)\times SU(4)\times U(1) \end{array}\right] = \mathcal{Z}\left[\begin{array}{c} Gauge = U(1) \\ Global\ Symmetry: \\ [SU(2)\times SU(2)]^2\times U(1) \end{array}\right]$$

$$
\begin{aligned}
&\sum_{m=-[r/2]}^{[r/2]} \int_{-\infty}^{\infty} \prod_{i=1}^{8} \gamma_h(a_i \pm x, u_i \pm m; \omega_1, \omega_2) \frac{dx}{2r\sqrt{-\omega_1\omega_2}} \\
= &\prod_{1\leq i<j\leq 4} \gamma_h(a_i + a_j, u_i + u_j; \omega_1, \omega_2)\gamma_h(a_{i+4} + a_{j+4}, u_{i+4} + u_{j+4}; \omega_1, \omega_2) \\
&\times \sum_{y=-[r/2]}^{[r/2]} \int_{-\infty}^{\infty} \prod_{i=1}^{8} \gamma_h(\tilde{a}_i \pm z, \tilde{u}_i \pm y; \omega_1, \omega_2) \frac{dz}{2r\sqrt{-\omega_1\omega_2}}
\end{aligned}
\tag{6}
$$

where balancing conditions are $\sum_{i=1}^{8} a_i = 2(\omega_1 + \omega_2)$ and $\sum_{i=1}^{8} u_i = 0$ and Boltzmann weights are

$$
W_{pq}(\sigma_i, \sigma_j) = \gamma_h(q - p \pm x_i \pm x_j, \pm m_i \pm m_j; \omega_1, \omega_2)\,. \tag{7}
$$

$$\sum_{m=-[r/2]}^{[r/2]} \int_{-\infty}^{\infty} \prod_{i=1}^{4} \gamma_h(a_i - x, u_i - m; \omega_1, \omega_2)\gamma_h(b_i + x, v_i + m; \omega_1, \omega_2) \frac{dx}{2r\sqrt{-\omega_1\omega_2}}$$

$$= \prod_{i,j=1}^{2} \gamma_h(a_i + b_j, u_i + u_j; \omega_1, \omega_2)$$

$$\sum_{y=-[r/2]}^{[r/2]} \int_{-\infty}^{\infty} \prod_{i=1}^{4} \gamma_h(\tilde{a}_i - z, \tilde{u}_i - y; \omega_1, \omega_2)\gamma_h(\tilde{b}_i + z, \tilde{v}_i + y; \omega_1, \omega_2) \frac{dz}{2r\sqrt{-\omega_1\omega_2}} \quad (8)$$

where balancing conditions are $\sum_{i=1}^{4} a_i + b_i = 2(\omega_1 + \omega_2)$ and $\sum_{i=1}^{4} u_i + v_i = 0$ and Boltzmann weights are

$$W_{pq}(\sigma_i, \sigma_j) = \gamma_h(q - p + x_i - x_j, m_i - m_j; \omega_1, \omega_2) \times\gamma_h(q - p - x_i + x_j, m_i + m_j; \omega_1, \omega_2)\,. \quad (9)$$

# What is the meaning of the gauge symmetry breaking from the statistical mechanics point of view?

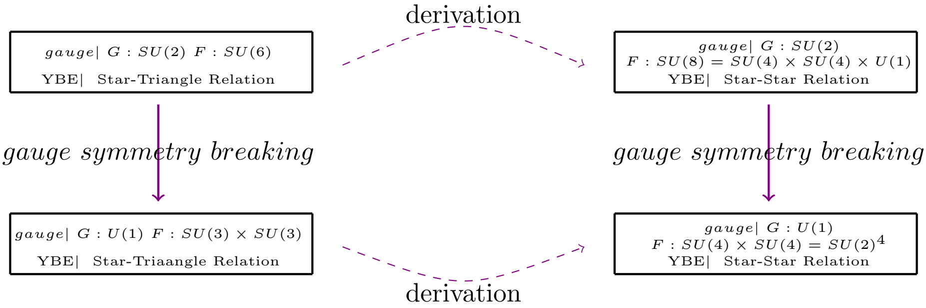
The Boltzmann weight for $SU(2)$ gauge symmetry

$$W_{pq}(\sigma_i, \sigma_j) = \gamma_h(q - p \pm x_i \pm x_j, \pm m_i \pm m_j; \omega_1, \omega_2) \tag{10}$$

The Boltzmann weight for $U(1)$ gauge symmetry

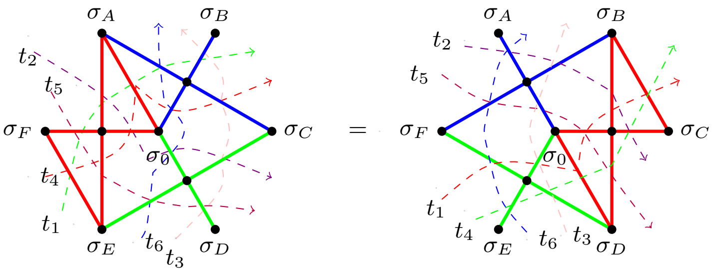
$$\begin{aligned} W_{pq}(\sigma_i, \sigma_j) = \gamma_h(q - p + x_i - x_j, m_i - m_j; \omega_1, \omega_2) \\ \times \gamma_h(q - p - x_i + x_j, m_i + m_j; \omega_1, \omega_2) \, . \end{aligned} \tag{11}$$

# The complete scheme [M. and Tas, 2021]]



Could we obtain IRF-type YBE

from the edge interacting models?




$$\sum_{m_0\in\mathbb{Z}}\int dx_0\ \ R_{t_{25}t_{41}}\begin{pmatrix} & \sigma_A & \\ \sigma_F & & \sigma_0 \\ & \sigma_E & \end{pmatrix} R_{t_{63}t_{25}}\begin{pmatrix} \sigma_0 & \sigma_C \\ \sigma_E & \sigma_D \end{pmatrix} R_{t_{41}t_{63}}\begin{pmatrix} \sigma_A & \sigma_B \\ \sigma_0 & \sigma_C \end{pmatrix}$$

$$= \sum_{m_0\in\mathbb{Z}}\int dx_0\ \ R_{t_{41}t_{63}}\begin{pmatrix} \sigma_F & \sigma_0 \\ \sigma_E & \sigma_D \end{pmatrix} R_{t_{63}t_{25}}\begin{pmatrix} \sigma_A & \sigma_B \\ \sigma_F & \sigma_0 \end{pmatrix} R_{t_{25}t_{41}}\begin{pmatrix} & \sigma_B & \\ \sigma_0 & & \sigma_C \\ & \sigma_D & \end{pmatrix}$$

If one has IRF-type models,

expects to acquire vertex-type models?

**Definition**

Two functions $\alpha(x,m;t,p)$ and $\beta(x,m;t,p)$, where $x,t \in \mathbb{C}$ and $m,p \in \mathbb{Z}$, form an integral hyperbolic hypergeometric Bailey pair with respect to $t$ and $p$ if the functions satisfy

$$\beta(z,m;t,p) = M(t,p)_{z,m;x,j}\alpha(x,j;t,p) \ , \tag{12}$$

where $M(t,p)_{z,m;x,j}$ is an operator integrating over $x \in \mathbb{C}$ and summing over $j \in \mathbb{Z}$, which also called an integral-sum operator.

**Lemma (Bailey Lemma)**

*Suppose $\alpha(x,m;t,p)$ and $\beta(x,m;t,p)$ form an integral Bailey pair with respect to $t \in \mathbb{C}$ and $p \in \mathbb{Z}$. Then, the sequences of functions $\alpha'(x,k;t+s,p+q)$ and $\beta'(x,k;t+s,p+q)$, $k \in \mathbb{Z}$, defined by*

$$\alpha'(x,k;t+s,p+q) = D(s,q;y,l;x,k)\alpha(x,k;t,p)\,, \tag{13}$$

$$\beta'(x,k;t+s,p+q) = D(-t,-p;y,l;x,k)M(s,q)_{x,k;z,m}D(s+t,p+q;y,l;z,m)\beta(z,m;t,p)\,, \tag{14}$$

*form a Bailey pair with respect to the new parameters $t+s$ and $p+q$ where $s,y \in \mathbb{C}$, $q,l \in \mathbb{Z}$ are arbitrary and the operator $D(s,q;y,l;x,k)$ is described as above.*

$$\begin{aligned} M(s,q)_{w,k;z,m}D(s+t,q+p;y,l;z,m)M(t,p)_{z,m;x,j} \\ = D(t,p;y,l;w,k)M(s+t,q+p)_{w,k;x,j}D(s,q;y,l;x,j)\,. \end{aligned} \tag{15}$$

$$M(t,p)_{z,m;x,j} = \frac{1}{C(t,p)} \sum_{j=0}^{[r/2]} \int_{-\infty}^{\infty} \gamma_h(-t \pm z \pm x, \pm m - p \pm j; \omega_1, \omega_2) \frac{[\mathrm{d}_j x]}{2r\sqrt{-\omega_1\omega_2}} , \tag{16}$$

$$D(t,p;x,j;z,m) = \gamma_h(-t \pm z \pm x, \pm m - p \pm j; \omega_1, \omega_2)$$

The same operators also satisfy the star-star relation.

$$\begin{aligned} M(t,p)_{z,m;x,j} =& \frac{1}{C(t,p)} \sum_{j=0}^{[r/2]} \int_{-\infty}^{\infty} \gamma_h(-t + z + x, m - p + j; \omega_1, \omega_2) \\ & \times \gamma_h(-t - z - x, -m - p - j; \omega_1, \omega_2) \frac{[\mathrm{d}_j x]}{2r\sqrt{-\omega_1\omega_2}} , \end{aligned} \tag{17}$$

$$\begin{aligned} D(t,p;x,j;z,m) =& \gamma_h(-t + z + x, m - p + j; \omega_1, \omega_2) \\ & \times \gamma_h(-t - z - x, -m - p - j; \omega_1, \omega_2) \end{aligned}$$

What about higher-spin interacting lattice models

from the supersymmetric gauge theories?

How do we interpret the following equality of the partition functions?

$$\sum_{m=-[r/2]}^{[r/2]} \int_{-\infty}^{\infty} \prod_{i=1}^{8} \gamma_h(a_i \pm x, u_i \pm m; \omega_1, \omega_2) \frac{dx}{2r\sqrt{-\omega_1\omega_2}}$$
$$= \prod_{1\le i<j\le 8} \gamma_h(a_i + a_j, u_i + u_j; \omega_1, \omega_2)$$
$$\times \sum_{y=-[r/2]}^{[r/2]} \int_{-\infty}^{\infty} \prod_{i=1}^{8} \gamma_h(\tilde{a}_i \pm z, \tilde{u}_i \pm y; \omega_1, \omega_2) \frac{dz}{2r\sqrt{-\omega_1\omega_2}} \tag{18}$$

$$\sum_{m=-[r/2]}^{[r/2]} \int_{-\infty}^{\infty} \frac{\prod_{i=1}^{4} \gamma_h(a_i \pm x, u_i \pm m; \omega_1, \omega_2)}{\gamma_h(\pm 2ix, \pm 2m; \omega_1, \omega_2)} \frac{dx}{2r\sqrt{-\omega_1\omega_2}}$$
$$= \gamma_h\left(\sum_{i=}^{4} a_i, \sum_{i=}^{4} u_i; \omega_1, \omega_2\right) \prod_{1\le i<j\le 4} \gamma_h(a_i + a_j, u_i + u_j; \omega_1, \omega_2)$$

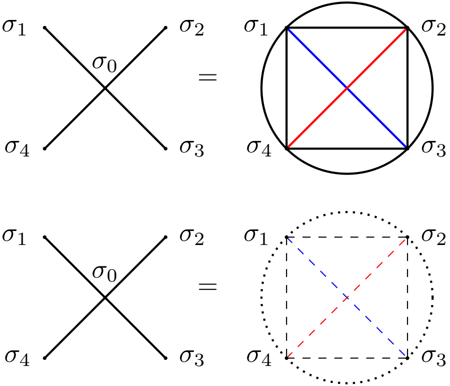
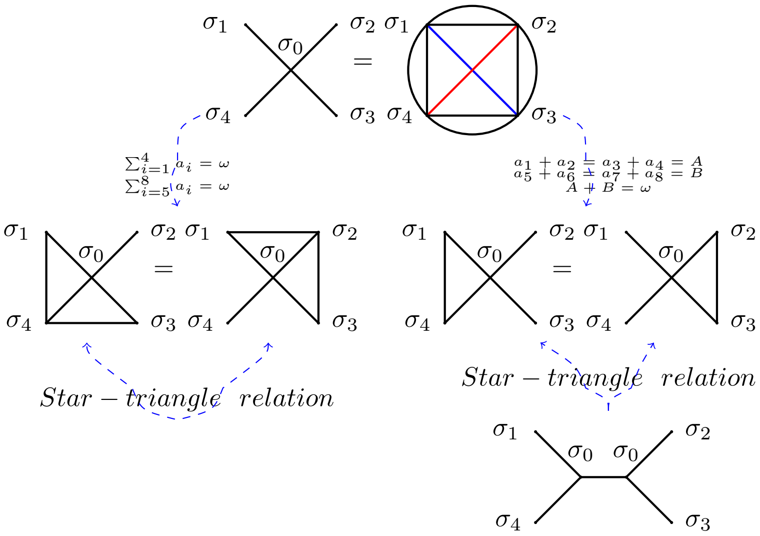



Figure: The star-square relation consists of four nearest neighbor interactions at left and seven interactions which are four nearest neighbors (dotted lines), two next nearest neighbors (dashed lines), and one quadruple interaction (broken circle) at right.

# The star-square relation [MM, 2024]



10.13 AN EQUIVALENT ISING MODEL 257

where all terms are linear in $\sigma_1, \ldots, \sigma_4$, there are 16 such terms, and $L, \ldots, L_{1234}$ are constant coefficients.

The function $W(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ is positive, so its logarithm is real and can be written in the form (10.13.6). Further, it is an even function of $\sigma_1, \ldots, \sigma_4$, so only the even terms in (10.13.6) occur. It must therefore be possible to find $L$, $L_{12}$, $L_{13}$, $L_{14}$, $L_{23}$, $L_{24}$, $L_{34}$, $L_{1234}$ such that

$$W(\sigma_1, \sigma_2, \sigma_3, \sigma_4) = \exp\left[L + \sum_{1 \leq i < j \leq 4} L_{ij}\sigma_i\sigma_j + L_{1234}\sigma_1\sigma_2\sigma_3\sigma_4\right]. \tag{10.13.7}$$

(This is known as the 'star-square' transformation: it is a generalization of the star – triangle relation of Section 6.4.)

Do we have more solvable models

from supersymmetric gauge theories?

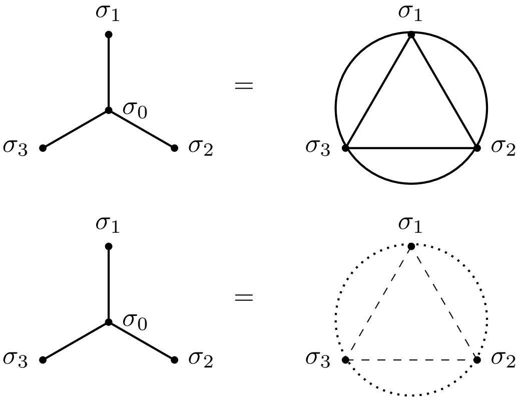



Figure: The generalized star-triangle relation consists of three nearest neighbor interactions at the left and four interactions which are three nearest neighbors (dashed lines) and one triple interaction (broken circle) at the right.

Can we introduce one-dimensional relations

in the gauge/YBE correspondence?

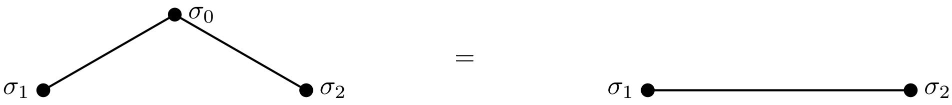



Figure: The decoration transformation

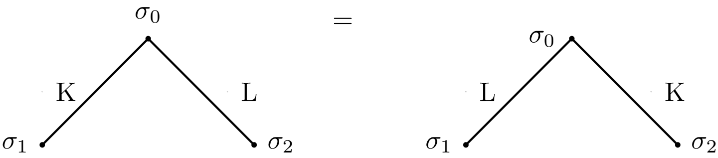



Figure: The flipping relation.

The decoration transformation for IRF-type models

$$\sum_{m_0} \int dx_0 \, R\begin{pmatrix} \sigma & \sigma \\ \sigma_0 & \sigma \end{pmatrix} R\begin{pmatrix} \sigma & \sigma \\ \sigma_0 & \sigma \end{pmatrix} = R\begin{pmatrix} \sigma & \sigma \\ \sigma & \sigma \end{pmatrix} \tag{19}$$

The flipping relation for IRF-type models

$$\sum_{m_0} \int dx_0 \, R\begin{pmatrix} \sigma & \sigma \\ \sigma_0 & \sigma \end{pmatrix} R\begin{pmatrix} \sigma & \sigma \\ \sigma_0 & \sigma \end{pmatrix} = \sum_{m_0} \int dx_0 \, R\begin{pmatrix} \sigma & \sigma_0 \\ \sigma & \sigma \end{pmatrix} R\begin{pmatrix} \sigma & \sigma_0 \\ \sigma & \sigma \end{pmatrix} \tag{20}$$

The decoration transformation for vertex-type models

$$M(s,q)_{w,k;z,m} M(t,p)_{z,m;x,j} = M(s+t, q+p)_{w,k;x,j} \; . \tag{21}$$

The flipping relation for vertex-type models

$$M(c,d)_{w,k;z,m} M(s+t, q+p)_{w,k;x,j} = M(s+c, q+d)_{w,k;x,j} M(t,p)_{w,k;x,j} \; . \tag{22}$$

$$M(t,p)_{z,m;x,j} = \frac{1}{C(t,p)} \sum_{j=0}^{[r/2]} \int_{-\infty}^{\infty} \gamma_h(-t \pm z \pm x, \pm m - p \pm j; \omega_1, \omega_2) \frac{[\mathrm{d}_j x]}{2r\sqrt{-\omega_1\omega_2}} \,, \tag{23}$$

$$D(t,p;x,j;z,m) = I(t,p)$$

The operators satisfy both the decoration transformation and the flipping relation.

$$\begin{aligned} M(t,p)_{z,m;x,j} =& \frac{1}{C(t,p)} \sum_{j=0}^{[r/2]} \int_{-\infty}^{\infty} \gamma_h(-t+z+x, m-p+j; \omega_1, \omega_2) \\ & \times \gamma_h(-t-z-x, -m-p-j; \omega_1, \omega_2) \frac{[\mathrm{d}_j x]}{2r\sqrt{-\omega_1\omega_2}} \,, \end{aligned} \tag{24}$$

$$D(t,p;x,j;z,m) = I(t,p)$$

## Kagome lattice [Fisher, 1959] and [M., 2023]

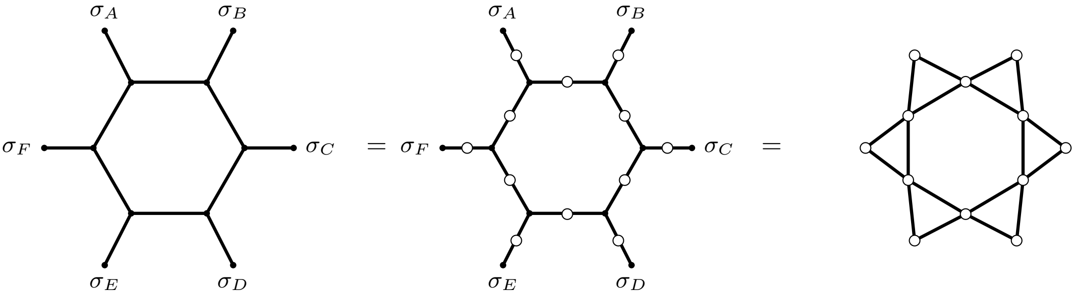



Figure: Constructing Kagome lattice from the hexagonal lattice by the use of decoration transformation and the star-triangle relation, respectively.

The infinite product representation is

$$\gamma^{(2)}(z;\omega_1,\omega_2) = e^{\frac{\pi i}{2}B_{2,2}(z;\omega_1,\omega_2)}\frac{(e^{2\pi i\frac{z}{\omega_2}}\tilde{q};\tilde{q})_\infty}{(e^{2\pi i\frac{z}{\omega_1}};q)_\infty} = e^{\frac{\pi i}{2}B_{2,2}(z;\omega_1,\omega_2)}\prod_{i=0}^{\infty}\frac{(1-e^{2\pi i\frac{z}{\omega_2}}\tilde{q}\tilde{q}^i)}{(1-e^{2\pi i\frac{z}{\omega_1}}q^i)}\,, \quad (25)$$

where parameters are $\tilde{q} = e^{2\pi i\omega_1/\omega_2}$ and $q = e^{-2\pi i\omega_2/\omega_1}$ and the Bernoulli polynomial is

$$B_{2,2}(z;\omega_1,\omega_2) = \frac{z^2 - z(\omega_1+\omega_2)}{\omega_1\omega_2} + \frac{\omega_1^2 + 3\omega_1\omega_2 + \omega_2^2}{6\omega_1\omega_2}. \quad (26)$$

One of the several representations of this special function is the following

$$\gamma^{(2)}(z;\omega_1,\omega_2) = \exp\left(-\int_0^\infty \frac{dx}{x}\left[\frac{\sinh x(2z-\omega_1-\omega_2)}{2\sinh(x\omega_1)\sinh(x\omega_2)} - \frac{2z-\omega_1-\omega_2}{2x\omega_1\omega_2}\right]\right), \quad (27)$$

where $Re(\omega_1), Re(\omega_2) > 0$ and $Re(\omega_1+\omega_2) > Re(z) > 0$.

Reflection property

$$\gamma^{(2)}(z;\omega_1,\omega_2)\gamma^{(2)}(\omega_1+\omega_2-z;\omega_1,\omega_2)=1\ . \tag{28}$$

The difference equation

$$\frac{\gamma^{(2)}(z+\omega_1;\omega_1,\omega_2)}{\gamma^{(2)}(z;\omega_1,\omega_2)}=2\sin\left(\frac{\pi z}{\omega_2}\right)\ . \tag{29}$$

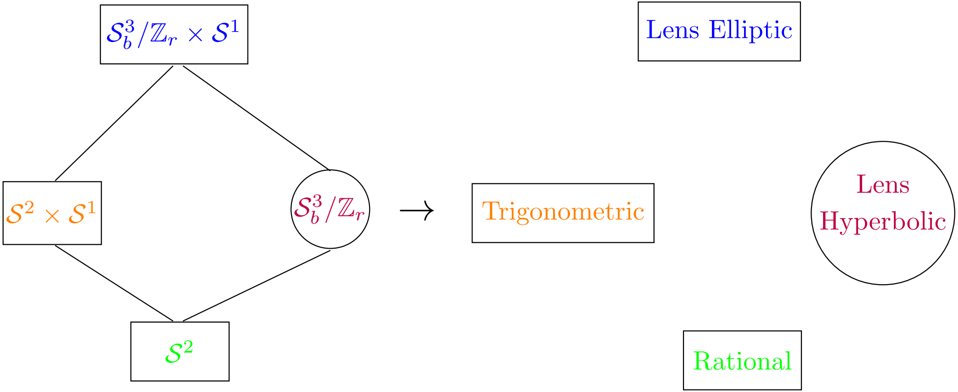
The asymptotic behaviours allows the gauge symmetry breaking

$$\lim_{z\to\infty} e^{\frac{\pi i}{2}B_{2,2}(z;\omega_1,\omega_2)}\gamma^{(2)}(z;\omega_1,\omega_2)=1 \ \text{ for } \ \arg\omega_2+\pi>\arg z>\arg\omega_1 \tag{30}$$

$$\lim_{z\to\infty} e^{-\frac{\pi i}{2}B_{2,2}(z;\omega_1,\omega_2)}\gamma^{(2)}(z;\omega_1,\omega_2)=1 \ \text{ for } \ \arg\omega_2>\arg z>\arg\omega_1-\pi\ , \tag{31}$$

where $\mathrm{Im}(\frac{\omega_1}{\omega_2})>0$. We use the following notation

$$\begin{aligned}\gamma_h(a_i\pm x,u_i\pm m;\omega_1,\omega_2)&=\gamma^{(2)}(-i(a_i\pm z)-i\omega_1(u_i\pm y);-i\omega_1 r,-i\omega_1-i\omega_2)\\ &\times\gamma^{(2)}(-i(a_i\pm z)-i\omega_2(r-(u_i\pm y));-i\omega_2 r,-i\omega_1-i\omega_2)\end{aligned} \tag{32}$$



$\mathcal{S}_b^3/\mathbb{Z}_r \times \mathcal{S}^1$
$\mathcal{S}^2 \times \mathcal{S}^1$
$\mathcal{S}_b^3/\mathbb{Z}_r$
$\mathcal{S}^2$
$\to$
Lens Elliptic
Trigonometric
Lens
Hyperbolic
Rational

Pentagon identity
[Bozkurt et al. 2020]

Knot invariant
[Hikami, 2007]

Bailey pairs
[Gahramanov et al. 2022]

Quantum groups
[Bozkurt et al. 2020]

gauge/YBE correspondence
[Gahramanov and Kels, 2016]

Painleve equations
[Kels and Yamazaki, 2017]

SUSY gauge theories
[Imamura and Yokoyama, 2012]

Special functions
[van de Bult, 2007]

- We found integrability properties of the supersymmetric gauge theories on the squashed lens space.
- What is the Hamiltonian of 1D spin chain via these 2D classical integrable models?
- What is the corresponding 8-vertex model of Ising-like models by the use of the star-square relation?
- Does the generalized Faddeev-Volkov model correspond to the representation of quantum groups $U_q(osp(1|r))$?
- $\cdots$

Thank You!