BAPHA

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Time correlations in integrable finite temperature models

- Integrability, Q-systtems and cluster algebras

Plan

- Motivation
- I – Discrete classical integrable systems
- II – time correlators in Boxbball model
- III – Finite temperature Landau Lifshitz and critical models breaking Lorentz invariance
- Conclusion

I - Motivation

- Transport out of equilibrium

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- time correlations in integrable models

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- Transport out of equilibrium
- time correlations in integrable models
- intégrable ⇒ analytic results? difference with nonintegrable?

interesting quantity:

Spin correlations.

Charge transport.

transport

[Marco Znidaric **PRL**, 2011]

Transport of spin current in XXZ chain

Infinite temperature transport:

- Ballistic for $\Delta < 1$
- Anomalous for $\Delta = 1$
- Diffusive for $\Delta < 1$

Infinite temperature transport of magnetization in Landau Lifshitz: the same.

[Ziga Krajnick, Tomaz Prosen **PRL**, 2019]

Discrete time evolution boxball

- The vertex, time goes up:

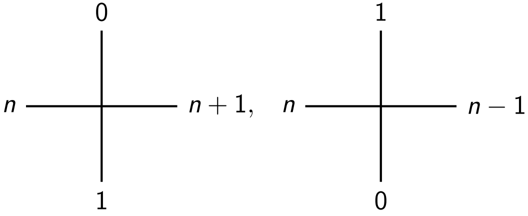

Figure: Time evolution

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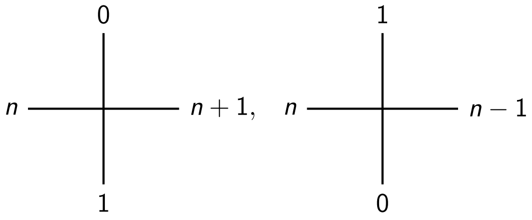

Figure: Time evolution

- The carrier has $n \geq 0$ balls. It passes through the ball configuration and picks up a ball when there is one, leaving a ball when there is none.
- Cyclicity. One must make sure that after the last step, the number of balls of the carrier coincides with its initial load.

Toda evolution

Toda evolution: N particles interact via exponential potential:

$$\frac{d^2 q_j}{dt^2} = e^{q_{j+1}-q_j} - e^{q_j - q_{j-1}}$$

Connection with many mathematica-physics topics including cluster algebras.

Discrete time evolution Toda

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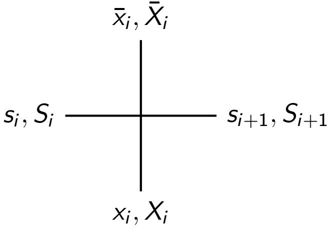

Figure: Time evolution

- The carrier has DST(discrete self trapping) variables S, s. It passes through the Toda configurations and updates the Toda variables x_i, X_i.
- Cyclicity. One must make sure that after the last step, $S_{N+1} = S_1$ and $s_{N+1} = s_1$

Discrete time evolution Toda

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- Solve for Darboux transform:

$$L_i(u) r_i(u - \tau) = r_{i+1}(u - \tau) \bar{L}_i(u)$$

Discrete time evolution Toda

- Obtain the solution (Suris, Sklyanin):

$$X_j = -\tau + \frac{x_j}{\bar{x}_j} + \frac{s_{j+1}}{x_j}$$

$$\bar{X}_j = -\tau + \frac{x_j}{\bar{x}_j} + \bar{x}_j S_j$$

$$s_j = \bar{x}_j,\ \ S_{j+1} = \frac{1}{x_j}$$

- τ is the time step.

Discrete time evolution Toda

- Newton equation:

$$\frac{x_j}{\overline{x}_j} - \frac{\underline{x}_j}{x_j} = \frac{x_j}{\underline{x}_{j-1}} - \frac{\overline{x}_{j+1}}{x_j}$$

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- $\tau \to 0$ and $x_j = e^{q_j}$:

$$\frac{d^2 q_j}{dt^2} = e^{q_{j+1}-q_j} - e^{q_j - q_{j-1}}$$

Toda chain equations.

Landau Lifshitz

- $$\frac{dS}{dt} = S \times \frac{d^2 S}{ds^2}$$

  where $S(s)$ is a unit vector $S_1^2 + S_2^2 + S_3^2 = 1$.
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- Local induction approximation for a filament in a superfluid. The filament is parametrized by its curvilinear abscissa $M(s)$, $\frac{dM}{ds} = S$, its motion is then:

  $$\frac{dM}{dt} = \frac{dM}{ds} \times \frac{d^2 M}{ds^2}$$

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- Solve for Darboux transform:

$$L_i(u)L_i(u-\tau) = L_{i+1}(u-\tau)\bar{L}_i(u)$$

This time carrier has the same lax matrix as for the dynamical variables.

Discrete time evolution Landau Lifshitz

- Obtain the solution :

$$\bar{S}_j = \frac{1}{\sigma^2 + \tau^2}(\tau^2 S_j + \sigma^2 V_j - \tau S_j \wedge V_j)$$

$$V_{j+1} = \frac{1}{\sigma^2 + \tau^2}(\tau^2 V_j + \sigma^2 S_j - \tau V_j \wedge S_j)$$

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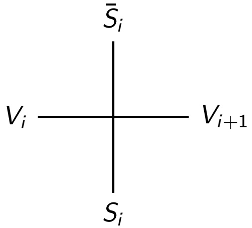

Figure: Time evolution

Integrability?

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- $$T(u) = \mathrm{trace}(L(u, S_N)L(u, S_{N-1}) \cdots L(u, S_1))$$

  is the generating function of conserves quantities.
- $$L(u, S_k)r_k = r_{k+1}\bar{L}(u, S_k)$$

  inserting this equality in the definition of $T(u)$, we get:

  $$T(u) = \bar{T}(u)$$

  Thus $T(u)$ is conserved under discrete time evolution.

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- For Toda, This amounts to solve a secon order equation which can have complex (non physical) solutions.
- For LL (and Toda) can use a perron Frobenius argument to select a physical solution: repeat

$$r_1(S_1, s_1)T_K(u) = T_{K+1}(u)r_{N+1}(S_{N+1}, s_{N+1})$$

untill $S_{N+1} = S_1$, $s_{N+1} = s_1$. For long chains once is enough.

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- For LL, and Toda configurations weighted by Bolzmann Weight:

$$P(S_1, \cdots, S_N) = \prod_j (1 + S_j . S_{j+1})^{-\beta} dS_i$$

  dS_i being the area measure.
- in particular infinte temperature state $\beta = 0$ is a bona fide stationary state (Liouville thm.). We conjecture the "non physical" configurations which cannot be periodized have a zero measure.

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- We want to compute the time dependant correlation density in the BBS model.

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  where $b(r,t)$ is one if there is a ball at position r, t and zero if not.
- In the Boxball model solitons are made of k consecutive balls and travel at a speed v_k, so a density fluctuation at the origin will split into k density peaks, each travelling at speed v_k. So, we expect the density density correlation to decompose in a sum of k delta function $c_k\delta(x - v_k t)$. Indeed this is the case and it is possible to evaluate the weights c_k analytically.

TBA for Boxball

- Thermodynamics of Boxball model can be formulated in the same way as for the delta bose gas although this is a classical model. The equation (Kirillov Reshetikin) relates the density ρ_k of soliton k to their hole density σ_k:

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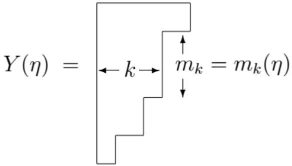
- The entropy is given by the logrithm of the binomial factors:

$$e^S = \prod_k \binom{\rho_k L}{\sigma_k L}$$

TBA for Boxball

- Conserved quantities=Action variables:



- angle variables can also be defined (Kirillov-Reshtikin)

TBA for Boxball

- Minimizing the free energy $F = \sum_k \beta_k E_k - S$ with respect to the densities ρ_k enables to obtain all the densities ρ_k.
- define:

$$Y_j = \frac{\sigma_j}{\rho_j} = e^{\epsilon}$$

- Y system:

$$\sum_j \min(k,j)\beta_j = \ln(1 + Y_k) - 2\sum_j \min(k,j)\ln(1 + Y_j^{-1})$$

- The ball fugacity is just μ_∞ all other fugacities are set to zero.

TBA for Boxball

- The soliton speed can be obtained by solving the GHD equation (also EI in the solitonic context)

$$v_k = k - 2\sum_l \rho_l (v_k - v_l)\min(k, l)$$

where as can easily be verified, k is the bare speed of soliton k and $2\min(k, l)$ the shift of the trajectories of solitons k and l after a collision.

TBA for Boxball

- An important tool of TBA is the dressing which in BBS takes the form:

$$A_i^{\mathrm{dr}} = A_i - \sum_k 2\min(i,k)\frac{\rho_k}{\sigma_k}A_k^{\mathrm{dr}}$$

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$$A_i^{\mathrm{dr}} = A_i - \sum_k 2\min(i,k)\frac{\rho_k}{\sigma_k}A_k^{\mathrm{dr}}$$

- Then it can be verified that the equation for the effective speed of the soliton writes:

$$v_k = \frac{k^{\mathrm{dr}}}{1^{\mathrm{dr}}}$$

Correlations Boxball

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- Thus correlations of normal modes can be obtained:

$$\langle \delta\epsilon_k(0,0)\delta\epsilon_j(x,t)\rangle = \delta_{j,k}\frac{1+e^{\epsilon_k}}{\sigma_k}\delta(x - v_k t)$$

Correlations Boxball

- Density correlations are then obtained

$$\langle \rho(0,0)\delta\rho(x,t)\rangle = \sum_k (\partial_{\epsilon_k}\rho)^2 \langle \delta\epsilon_k(0,0)\delta\epsilon_j(x,t)\rangle$$

- With the expression of $\partial_{\epsilon_k}\rho = -\rho_k\sigma_k v_k$ we finally get:

$$c_k = \sigma_k\rho_k(\sigma_k + \rho_k)v_k^2$$

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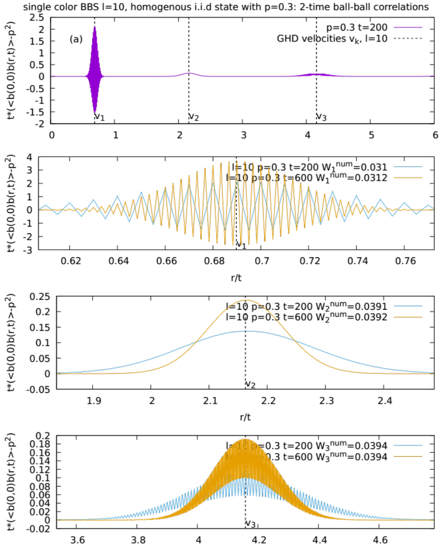
- by construcion:

$$\sum_k c_k = p(1-p)$$

correlation



correlation

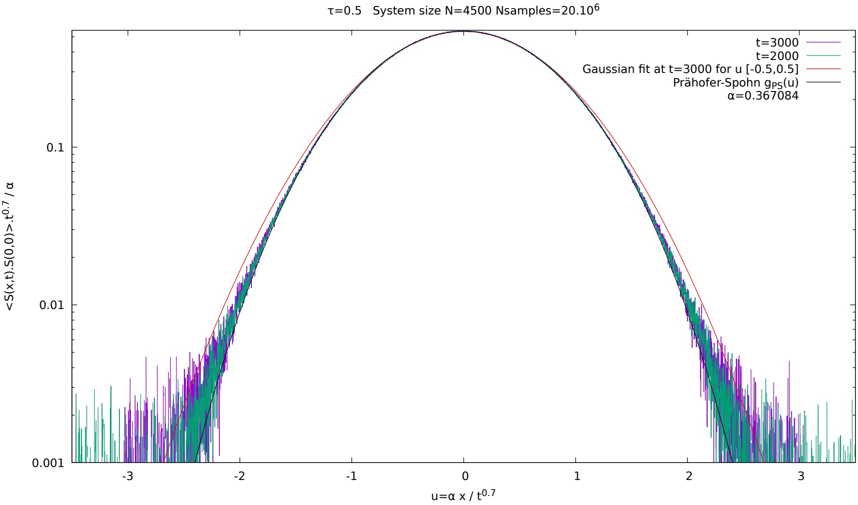
k	theory	numerics
1	0.055	0.055
2	0.049	0.049
3	0.0305	0.031
4	0.015	0.015

  Table: W_k for $p = .2$

k	theory	numerics
1	0.0312	0.0312
2	0.0392	0.0392
3	0.0394	0.0394
4	0.0334	0.0334

  Table: W_k for $p = .3$

Spin correlations in Landau Lifshitz



Conclusion and perspective

- We have succeeded in obtaining time correlations in a simple model: BBS. The tool was GHD and explicit expression of the TBA kernel: $2\min(i,j)$ which is the skeleton of the XXZ spin chain kernel for TBA.

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- We have succeeded in obtaining time correlations in a simple model: BBS. The tool was GHD and explicit expression of the TBA kernel: $2\min(i,j)$ which is the skeleton of the XXZ spin chain kernel for TBA.
- we have not succeeded in computing correlation function in Landau Lifshitz, not even understood 2/3 ! nevertheless, there are strong indications they are related to correlations of KPZ: Prähofer Spohn curve. pause
- one possible direction of research is the equivalence between Landau Lifshitz and attractive nonlinear Schrödinger equation. or in the discretized case with Ablowitz-Ladik.