On holographic complexity: warped CFT

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Integrability, Q-systems and Cluster Algebras

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Towards understanding Quantum Dynamics

- Inspirations from holography
  - Warped AdS_3 (WAdS) - generalization of AdS_3 string solutions
  - propperties of CFT dual to WAdS - WCFT

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- Dynamics on the bdy and corresponding bulk processes/reconstruction
  - Integrable or chaotic behavior?
    - in flat & curved spaces (highly nontrivial)
  - Minimal knowledge to (almost) completely describe a system

- Complexity of (quantum) integrable systems & strongly interacting compact objects
  - example: for Krylov spaces

$$b_n \sim n^{\delta}, \quad \delta \geq 1 \text{ - chaotic}, \quad 0 < \delta < 1 \text{ - integrable}$$

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- In our context - the naive notion of complexity C(t): as a correlator for some time dependent operator $A(t)$ (autocorrelation function)

$$C(t) = \langle A(t)|A\rangle, \qquad \langle A|B\rangle = \mathrm{Tr}(A^\dagger \rho_1 B \rho_2) \tag{1}$$

Complexity= Volume conjecture

- *Complexity=Volume* [see for instace: 1403.5695, 1411.0690 1509.07876, 1512.04993]

- AAdS/CFT: duality between a black hole in asymptotically anti de-Sitter spacetime and a thermal state of a CFT, in which the entropy of the black hole is dual to ordinary thermal entropy

- Susskind: the computational complexity of the state of the CFT, which continues to grow, after statistical equilibrium is reached, for a time that is exponential in the entropy

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Conjecture: Let A_H is the area of horizon and the rate of change of Complexity is $\dot{\mathcal{C}} \sim \kappa A_H/G$.
$\Longrightarrow$

$$\mathcal{C} \sim \frac{(D-3)V}{G r_H}.$$

Complexity=Action conjecture

- *Complexity=Action in holography* [see for instance 1509.07876]

- The rate of *quantum complexity for the boundary quantum state* is exactly equal to *the growth rate of the gravitational action on shell* in the bulk region in the WDW patch at the late time approximation. Then the complexity-action duality can be defined by

$$\mathcal{C} = \frac{S}{\pi\hbar},$$

- $\mathcal{C}$ is the complexity in quantum information theory, whose meaning is that the minimum numbers of quantum gates are required to produce the certain state from the reference state, and S is the total classical gravitational action in the bulk region within the WDW patch.

Geometric Complexity

Geometric Complexity

- The allowed transformations $U(\sigma)$ - as path ordered exponentials

$$V = i\frac{dU}{d\tau}U^\dagger = T_\alpha V^\alpha \quad \Longrightarrow \quad U(\sigma) = \mathcal{P}e^{-i\int_{s_i}^{\sigma} V(s)ds}$$

- s parametrizes progress along a path, starting at s_i and ending at s_f and $\sigma \in [s_i, s_f]$ is some intermediate value of s. The path-ordering $\mathcal{P}$ is required for non-commuting generators T_α, $V(s) = V^\alpha T_\alpha$.

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- bi-invariant metric

$$ds^2_{bi-inv} = \mathrm{Tr}(V^\dagger V)d\tau^2 \tag{2}$$

- The length of a path from s_i to s_f going through $|\Psi(\sigma)\rangle$

$$\ell(|\Psi(\sigma)\rangle) = \int_{s_i}^{s_f} ds(\sigma).$$

- Define the complexity $\mathcal{C}$ as the minimal length/geodesics between states driven by generators $G(s)$

$$\mathcal{C}(|\Psi(s_i)\rangle, |\Psi(s_f)\rangle) = \min_{V(s)} \ell(|\Psi(\sigma)\rangle).$$

Geometric Complexity

• Nielsen's complexity of the evolution operator corresponds to the length of the path with b.c. and velocity that minimizes the length
- penalty factors $\mu_\alpha \quad \rightarrow \quad$ the metric (for low cost directions $\mu_\alpha = 1$)

$$C_N(t) = \min_V \int_0^t d\tau \left(\sum_\alpha \mathrm{Tr}(T_\alpha V)^2 + \mu_\alpha \, \mathrm{Tr}(T_\alpha V)^2 \right)^{1/2},$$

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• Objective: geodesics connecting the identity to a target unitary $U_{target} = \exp\{-i\mathcal{H}t\}$ at a chosen moment t, with $\mathcal{H}$ being the physical Hamiltonian.
- ambiguity:

$$\mathcal{H} \;\rightarrow\; \mathcal{H} + \frac{2\pi}{t}\kappa, \qquad \kappa \in \mathbb{Z}.$$

- ambiguity in the spectrum

$$E_n \;\rightarrow\; E_n - \frac{2\pi}{t}\kappa_n \equiv 2\pi y_n/t.$$

Geometric Complexity

- accounting for penalties in the metric

$$\sqrt{\sum_{\alpha}[\mathrm{Tr}(T_\alpha V)^2 + \mu_\alpha \, \mathrm{Tr}(T_\alpha V)^2]} = \sqrt{y_n Q_{nm} y_m}$$

$$\implies \quad Q_{nm} = \sum_{\alpha} \mu_\alpha \langle n|T_\alpha|n\rangle\langle m|T_\alpha^\dagger|m\rangle, \quad (3)$$

where $\mu_\alpha = 1$ for low cost directions and $\mathrm{Tr}(T_\alpha T_\beta) = \delta_{\alpha\beta}$.

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- a pure state in theory with gauge symmetry $\rightarrow$ "generalized length" : curve $\gamma(t)$ on the group manifold (A_i is the gauge connection):

$$C_\gamma = \int_0^1 d\tau ||\dot{\gamma}(t)|| - \int_0^1 d\tau A_i(\gamma(t))\dot{\gamma}^i.$$

$\rightarrow$ the state complexity of $|\psi_T\rangle$: the equivalence class of some Gaussian transformation $M \in G$ (group manifold) $\rightarrow$ the length of the geodesic connecting 1 to the point where the equivalence class $[M]$ intersects $\exp(stab_\perp(N))$.

Spread states and Operator growth

Spread states and Operator growth

• Unitary evolution mixes the initial state $|\psi\rangle$ with other quantum states as time evolves

$$|\psi(t)\rangle = e^{-i\mathcal{H}t}|\psi(0)\rangle = \sum_{n=0}^{\infty} \frac{(-i\mathcal{H}t)^n}{n!}|\psi\rangle = \sum_{n=0}^{\infty} \frac{(-it)^n}{n!}|\psi_n\rangle. \qquad (4)$$

$\Rightarrow$ understanding the states $|\psi_n\rangle \equiv \mathcal{H}^n|\psi\rangle$.

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• The Gram–Schmidt procedure applied to generate an ordered, orthonormal basis $K = \{|K_0\rangle, |K_1\rangle, \dots\}$.

- consider a basis $B = \{|B_i\rangle\, i = 0, 1, \dots\}$ and def *cost finction*

$$C_B(t) = \sum_n c_n |\langle\psi_n|B_n\rangle|^2, \qquad c_n \textit{ positive increasing}, \;\; |B_0\rangle = |\psi(t_0)\rangle$$

- def *Complexity*

$$\boxed{C(t) = \min_B C_B(t)}$$

Spread states and Operator growth

- Operator growth

$$\mathcal{O}(t) = e^{i\mathcal{H}t}\,\mathcal{O}(0)\,e^{-i\mathcal{H}t} = \sum_{n=0}^{\infty} \frac{(it)^n}{n!}\tilde{\mathcal{O}}_n, \tag{5}$$

where

$$\tilde{\mathcal{O}}_0 = \mathcal{O}, \quad \tilde{\mathcal{O}}_1 = [\mathcal{H}, \mathcal{O}], \quad \tilde{\mathcal{O}}_2 = [\mathcal{H}, [\mathcal{H}, \mathcal{O}]] \ldots \tag{6}$$

As time progresses, a simple operator $\mathcal{O}(t)$ "grows" in the space of operators of the theory becoming more "complex".
- the idea: use $\tilde{\mathcal{O}}_n$ to construct states of the basis $\{|\mathcal{O}_n(0))\}$

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- Notion of Liouvillian (superoperator)

$$\mathcal{L} := [\mathcal{H}, *] \quad \Longrightarrow \quad \tilde{\mathcal{O}}_n = \mathcal{L}^n \mathcal{O}(0) \quad \Longrightarrow \quad \mathcal{O}(t) = e^{i\mathcal{L}t}\mathcal{O}(0). \tag{7}$$

- Subtlety: the states $|\mathcal{O}_n(0)) = \mathcal{O}_n|0\rangle$ may not be orthogonal (and the set $\{|\mathcal{O}_n(0))\}$ may not define a basis)

Constructing Krylov spaces

Constructing Krylov spaces

- The algorithm of orthogonalization (Arnoldi iteration)
  1. set $b_0 \equiv 0$ and $|\mathcal{O}_{-1}) \equiv 0$
  2. Define $|\mathcal{O})_0 = \frac{1}{\sqrt{(\mathcal{O}|\mathcal{O})}}\mathcal{O})$
  3. For $n = 1$:
     - $|A_1) = \mathcal{L}|\mathcal{O}_0)$
     - $b_1 = ||A_1||$
     - If $b_1 \neq 0$ define$|\mathcal{O}_1) = \frac{1}{b_1}|A_1)$
  4. For $n > 1$:
     - $|A_n) = \mathcal{L}|\mathcal{O}_{n-1}) - b_{n-1}|\mathcal{O}_{n-2})$
     - $b_n = ||A_n|| \equiv \sqrt{(A_n|A_n)}$
     - If $b_n = 0$ stop the procedure; if not, define $|\mathcal{O}_n) = \frac{1}{b_n}|A_n)$ and go to step 4.

- The Krylov subspace: spanned by $\{P_n(\mathcal{L})|\hat{\mathcal{O}})\}$; Krylov basis is

$$|\hat{\mathcal{O}}_n) := |P_n(\mathcal{L})\hat{\mathcal{O}}), \qquad n = 0, 1, \ldots$$

- If $(\hat{\mathcal{O}}_m|\mathcal{L}|\hat{\mathcal{O}}_n)$ is a Hermitian matrix

$$\mathcal{L}_{nm} \equiv \begin{pmatrix} 0 & b_1 & 0 & 0 & \cdots \\ b_1 & 0 & b_2 & 0 & \cdots \\ 0 & b_2 & 0 & b_3 & \cdots \\ 0 & 0 & b_3 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{8}$$

$\Longrightarrow$ a three-term recurrence relation

$$\mathcal{L}P_n(\mathcal{L}) = b_{n+1}P_{n+1}(\mathcal{L}) + b_nP_{n-1}(\mathcal{L}) \tag{9}$$

$\Longrightarrow$ by Favard's theorem $\exists$ measure wrw $P_n(\mathcal{L})$ are orthogonal.

Moments and Hankel determinant

• A key quantity containing equivalent information is the moment matrix $\mathfrak{M}$ defined by

$$\mathfrak{M}_0 = \begin{pmatrix} \int x^0 d\omega & \int x d\omega & \cdots & \int x^n d\omega \\ \int x d\omega & \int x^2 d\omega & \cdots & \int x^{n+1} d\omega \\ . & . & \cdots & \\ \int x^n d\omega & \int x^{n+1} d\omega & \cdots & \int x^{2n} d\omega \end{pmatrix} = \begin{pmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ . & . & \cdots & \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{pmatrix}$$

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- Hankel determinant D_n

$$D_n = \det_{1 \le i,j \le N} (\mu_{i+j}) = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ . & . & \cdots & \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{vmatrix} \tag{10}$$

Orthogonal polynomials

- Moments, Hankel and orthogonal polynomial $D_n(x)$

$$D_n(x) = \begin{vmatrix} \int x^0 d\omega & \int x d\omega & \cdots & \int x^n d\omega \\ \int x d\omega & \int x^2 d\omega & \cdots & \int x^{n+1} d\omega \\ . & . & \cdots & \\ \int x^{n-1} d\omega & \int x^n d\omega & \cdots & \int x^{2n-1} d\omega \\ 1 & x & \cdots & x^n \end{vmatrix}. \tag{11}$$

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- Using D_n and $D_(x)$ $\Longrightarrow$ define an orthogonal polynomial

$$P_n(x) = \frac{D_n(x)}{\sqrt{D_{n-1} D_n}} \tag{12}$$

- Using recurent relations one finds the relations to Lanczos coefficients

$$b_n^2 = \frac{D_{n-1} D_{n+1}}{D_n^2}, \qquad a_n = \ln \frac{D_n}{D_{n-1}}. \tag{13}$$

Krylov complexity

- Decomposition of $\mathcal{O}(t)$ in terms of the Krylov elements:

$$|\mathcal{O}(t)) = \sum_{n=0}^{K-1} \phi_n(t)|\mathcal{O}_n). \tag{14}$$

- The Liouvillian in Krylov basis

$$\mathcal{L} = \sum_{n=0}^{K-1} b_{n+1} \left[|\mathcal{O}_n)(\mathcal{O}_{n+1}| + |\mathcal{O}_{n+1})(\mathcal{O}_n| \right] \tag{15}$$

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- *Krylov Complexity and K-entropy (Shannon)*

$$\mathcal{K}(t) = \sum n|\phi_n(t)|^2, \qquad S(t) = \sum |\phi_n(t)|^2 \log |\phi_n(t)|^2 \tag{16}$$

Warped geometry and warped CFT

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- The symmetry (for left/right movers) under

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  $+$ unitarity, locality & a bounded below spectrum of the dilatation operator - translations and dilatations are enhanced to an infinite-dimensional symmetries.
- For every value of $\mu\ell \neq 3$: $\exists$ other solutions - $SL(2,R) \times U(1)$ $WAdS_3$ geometries. It is achieved by multiplying the fiber metric with a constant warp factor.
  $\Longrightarrow$ breaks $SL(2,R)_L \times SL(2,R)_R$ to $SL(2,R) \times U(1)$.

- AdS_3 deformation

$$ds^2 = \frac{\ell^2}{4}[-\cosh^2\sigma d\tau^2 + d\sigma^2 + (du + \sinh\sigma d\tau)^2] \quad \Longrightarrow$$
$$ds^2 = \frac{\ell^2}{\nu^2+3}\left[-\cosh^2\sigma d\tau^2 + d\sigma^2 + \frac{4\nu^2}{\nu^2+3}(du + \sinh\sigma d\tau)^2\right], \tag{17}$$

$\{u,\tau,\sigma\} \in [-\infty,\infty]$, $\nu^2 \geq 1$ - spacelike stretched AdS_3; $\nu^2 \leq 1$ - spacelike squashed AdS_3.

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- Detournay, Hartman and Hofman [1210.0539]: transl. inv. only + chiral scaling symmetry $\Longrightarrow$ one Vir and a U(1) current algebra.
- Holographically: a WCFT can be described as a $SL(2,R) \times U(1)$ Chern-Simons theory in 3d [Castro, Hofman, Iqbal] .

Comments: Recently: the Kerr BH background a hidden $SL(2,R) \times U(1)$ ("Love") symmetry in the near zone approximation.

Warped Conformal Symmetry

- The BH solutions, asymptotic to warped AdS_3

$$
\begin{aligned}
ds^2 =& dt^2 + \frac{l^2}{3+\nu^2}\frac{dr^2}{(r-r_-)(r-r_+)} - 2(\nu r + \frac{1}{2}\sqrt{r_+ r_-(3+\nu^2)})dt d\phi \\
&+ \frac{r}{4}[3(\nu^2-1)r + (3+\nu^2)(r_+ + r_-) + 4\nu\sqrt{r_+ r_-(3+\nu^2)}]d\phi^2
\end{aligned}
$$

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$$ds^2 = dt^2 + \frac{l^2}{3+\nu^2}\frac{dr^2}{(r-r_-)(r-r_+)} - 2(\nu r + \frac{1}{2}\sqrt{r_+ r_-(3+\nu^2)})dtd\phi$$
$$+ \frac{r}{4}[3(\nu^2-1)r + (3+\nu^2)(r_+ + r_-) + 4\nu\sqrt{r_+ r_-(3+\nu^2)}]d\phi^2$$

- The asymptotic algebra

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c_V}{12}m^3\delta_{n+m,0} \\
[L_m, J_n] &= -nJ_{m+n}, \\
[J_m, J_n] &= \frac{c_J}{12}m\delta_{m+n,0} = \frac{k}{2}m\delta_{m+n,0},
\end{aligned} \tag{18}$$

$$c_V = \frac{5\nu^2+3}{\nu(\nu^2+3)}\frac{l}{G}, \qquad c_J = \frac{\nu^2+3}{\nu}\frac{l}{G} = k/6. \tag{19}$$

Symmetries and operators

- Transformations of local operators under global scaling symmetry $x \to \lambda x$ and translational symmetry $x \to x + a$, $y \to y + b$,

$$\Phi_i(\lambda x + a, y + b) = \lambda^{-h_i} \Phi_i(x, y)\,, \tag{20}$$

- Infinitesumally

$$[L_n, \mathcal{O}(x,y)] = [x^{n+1}\partial_x + (n+1)x^n h]\mathcal{O}(x,y)\,, \tag{21}$$

$$[J_n, \mathcal{O}(x,y)] = i x^n \partial_y \mathcal{O}(x,y) \tag{22}$$

$$= -x^n Q \mathcal{O}(x,y)\,, \tag{23}$$

- The standard basis

$$|\mathcal{O}^{\{\vec{N},\vec{M}\}}\rangle = L_{-1}^{N_1} L_{-2}^{N_2} \ldots J_{-1}^{M_1} J_{-2}^{M_2} \ldots |\Delta, Q\rangle$$

A new basis of operators

- $U(1)$ Sugawara

$$T^{\text{sug}}(z) = \sum_n \frac{L_n^{\text{sug}}}{z^{n+2}}, \qquad L_n^{\text{sug}} = \frac{1}{2k}\left(\sum_{m\leq -1} J_m J_{n-m} + \sum_{m\geq 0} J_{n-m} J_m\right), \tag{24}$$

$\Longrightarrow$

$$\begin{aligned} [L_n^{\text{sug}}, L_m^{\text{sug}}] &= (n-m)L_{n+m}^{\text{sug}} + \frac{1}{12}n(n^2-1)\delta_{n+m,0}, \\ [L_n^{\text{sug}}, J_m] &= -mJ_{n+m} \end{aligned} \tag{25}$$

&

$$[L_n, L_m^{\text{sug}}] = (n-m)L_{n+m}^{\text{sug}} + \frac{1}{12}n(n^2-1)\delta_{n+m,0}. \tag{26}$$

A new basis

A new basis

- Define spectral flow invariant Virasoro generators

$$\mathcal{L}_n \equiv L_n - L_n^{\text{sug}} = L_n - \frac{1}{k}\Big(\sum_{m\le -1} J_m J_{n-m} + \sum_{m\ge 0} J_{n-m} J_m\Big)\,.. \qquad (27)$$

The key point: $\mathcal{L}_n$ and J_n generators provide a basis that factors the algebra into separate Virasoro and $U(1)$ sectors:

$$[\mathcal{L}_n, \mathcal{L}_m] = (n-m)\mathcal{L}_{n+m} + \frac{c-1}{12}n(n^2-1)\delta_{n+m,0},$$
$$[\mathcal{L}_n, J_m] = 0. \qquad (28)$$

$\Longrightarrow$ states $|\phi\rangle$ that are primary with respect to the L_n's and J_n's, with weight h and charge q_ϕ, are primary under $\mathcal{L}_n$ as well, with weight

$$h^{(0)} = h - \frac{Q_\phi^2}{2k}. \qquad (29)$$

Primaries

- The primary state $|\Delta, Q\rangle$ under $\mathcal{L}_n$ and $\mathcal{J}_n$,

$$\mathcal{L}_0|\Delta, Q\rangle = \Delta^{inv}|\Delta, Q\rangle, \quad \mathcal{J}_0|\Delta, Q\rangle = -Q|\Delta, Q\rangle\,, \tag{30}$$

$$\mathcal{L}_n|\Delta, Q\rangle = 0, \quad \mathcal{J}_n|\Delta, Q\rangle = 0, \quad \forall n > 0\,, \tag{31}$$

- The conformal weight

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- *Remark*: The advantage of using $\{\mathcal{L}_n, \mathcal{J}_m\}$ basis:

- orthogonality of the corresponding descendant states
- factorization of the norm of mixed states including both, Vir & $U(1)$ descendants

Descendants

- A descendant operators for $|\Delta, Q\rangle$

$$|\mathcal{O}^{\{\vec{N},\vec{M}\}}\rangle = \mathcal{L}_{-1}^{N_1}\mathcal{L}_{-2}^{N_2}...\mathcal{J}_{-1}^{M_1}\mathcal{J}_{-2}^{M_2}\ldots|\Delta, Q\rangle\,, \quad \vec{N} = N_1, \cdots \& \vec{M} = M_1, \cdots.$$

- The spectral invariant conformal weight and charge

$$\mathcal{L}_0|\mathcal{O}^{\{\vec{N},\vec{M}\}}\rangle = \left(\Delta^{inv} + \sum_{n>0} nN_n\right)|\mathcal{O}^{\{\vec{N},\vec{M}\}}\rangle, \tag{33}$$

$$\mathcal{J}_0|\mathcal{O}^{\{\vec{N},\vec{M}\}}\rangle = -Q|\mathcal{O}^{\{\vec{N},\vec{M}\}}\rangle\,. \tag{34}$$

- The conformal weight

$$h = \Delta + \sum_n nN_n + \sum_m mN_m - \frac{Q^2}{k}\,. \tag{35}$$

$SL(2,R)$ subsector of Virasoro

- The action of $SL(2,R)$ on a Fock state

$$L_0|h,n\rangle = (h+n)|h,n\rangle, \quad L_{-1}|h,n\rangle = \sqrt{(n+1)2h+n}|h,n+1\rangle \tag{36}$$

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- Perelomov construction

$$e^{zL_{-1}}|h\rangle = \sum_{n=0}^{\infty} \frac{z^n}{n!} L_{-1}^n|h\rangle = \sum_{n=0}^{\infty} \frac{z^n}{n!} \sqrt{\frac{n!\Gamma(2h+n)}{\Gamma(2h)}}|h,n\rangle. \quad (38)$$

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- The explicit form of a state

$$\boxed{|z,h\rangle = (1-|z|^2)^h \sum_{n=0}^{\infty} z^n \sqrt{\frac{\Gamma(2h+n)}{n!\Gamma(2h)}}\,|h,n\rangle.} \tag{39}$$

$SL(2,R)$ subsector of Virasoro

- The state generated by Liouvillian $\mathcal{L} = L_{-1} + L_1$

$$|\mathcal{O}(t)) = e^{i\alpha(L_{-1}+L_1)t}|h\rangle = |z = i\tanh(\alpha t); h = \eta/2\rangle \tag{40}$$

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- Identification between the Krylov basis and the basis vectors

$$|\mathcal{O}(t)) = |h\rangle, \qquad |\mathcal{O}_n) = |h,n\rangle.$$

- The Lanczos coeffcients (from (37)):

$$b_n = \alpha\sqrt{n(2h+n-1)}. \tag{41}$$

$\Longrightarrow$ the wavefunctions are just coefficients of the coherent state.

- Krylov Complexity for $SL(2,R)$

$$K_{\mathcal{O}} = \langle \mathcal{O}(t)|\mathcal{O}(t)\rangle = 2h\sinh^2(\alpha t). \tag{42}$$

Other subsectors of Virasoro

- Virasoro algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{m+n,0}, \tag{43}$$

- construct $SL(2,\mathbb{R})$ from L_0 and $L_k = L_{-k}^\dagger$ using

$$[L_k, L_{-k}] = 2kL_0 + \frac{c}{12}k(k^2-1), \qquad [L_0, L_{\pm k}] = \mp k L_{\pm k}. \tag{44}$$

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$$\tilde{L}_\pm = \frac{1}{k}L_{\pm k}, \qquad \tilde{L}_0 = \frac{1}{k}\left(L_0 + \frac{c}{12}k(k^2-1)\right). \tag{45}$$

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$$\begin{aligned} \Longrightarrow \quad D_k(\xi) &= e^{\xi L_{-k} - \bar{\xi} L_k} \\ &= e^{i\phi \frac{\tanh(kr)}{k} L_{-k}} e^{-\frac{2}{k}\log(\cosh(kr))\left(L_0 + \frac{c}{12}k(k^2-1)\right)} e^{-i\phi\frac{\tanh(kr)}{k}L_k}. \end{aligned} \tag{46}$$

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- Autocorrelation function for SL case

$$C(t) = (1|\psi_{\mathcal{O}}(t)) = \frac{1}{\cosh^{2h}(\alpha t)}$$

• In oscillator basis $\alpha_n = \frac{i}{\sqrt{2}} \frac{\partial}{\partial u_n},\ \alpha_{-n} = -i\sqrt{2} n u_n,\ n > 0$

$$\langle f|L_n|u\rangle = \langle u|L_{-n}|f\rangle = l_{-n} f(u) = l_{-n} f(u). \quad (47)$$

• A generic descendant state at level $N = \sum_j j m_j$ is a sum of monomials $u_1^{m_1} u_2^{m_2} u_3^{m_3} \ldots$.

• Operators ($c = 1 + 24\mu^2,\ h = \mu^2 + \lambda^2$)

$$l_0 = h + \sum_{n=1}^{\infty} n\, u_n \frac{\partial}{\partial u_n},$$

$$l_k = \sum_{n=1}^{\infty} n\, u_n \frac{\partial}{\partial u_{n+k}} - \frac{1}{4} \sum_{n=1}^{k-1} \frac{\partial^2}{\partial u_n \partial u_{k-n}} + (\mu + i\lambda) \frac{\partial}{\partial u_k}, \qquad k > 0 \quad (48)$$

$$l_{-k} = \sum_{n=1}^{\infty} (n+k)\, u_{n+k} \frac{\partial}{\partial u_n} - \sum_{n=1}^{k-1} n(k-n) u_n u_{k-n} + 2k(\mu - i\lambda) u_k, \quad k > 0$$

The action on descendants

- A generic descendant in oscillator basis is

$$\Phi_{\{m\}}(u) \equiv \frac{u_1^{m_1} u_2^{m_2} \cdots}{N_{\{m\}}}, \qquad N_{\{m\}} = \sqrt{\prod_{j=1}^{\infty} \frac{m_j!}{(2j)^{m_j}}}. \tag{49}$$

$$\langle \Phi_{\{m\}}, \Phi_{\{m\}} \rangle = 1, \qquad (\Phi_{\{m\}}, l_0 \Phi_{\{m\}}) = h + \sum_j j m_j = h + N. \tag{50}$$

$\Longrightarrow$ the orthogonal descendants are labeled by integer partitions of the descendant level N.

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$\Longrightarrow$ the orthogonal descendants are labeled by integer partitions of the descendant level N.

- The action of $\mathcal{L}$ on an arbitrary descendant

$$\langle u|\mathcal{L}|\Phi_{\{m\}}\rangle = \xi(l_{-1} + l_1)\Phi_{\{m\}} = \sum_{\sum j r_j = N+1} b_{\{m\}\to\{r_j\}} \Phi_{\{r_j\}}(u) + \sum_{\sum j s_j = N-1} b_{\{m\}\to\{s_j\}} \Phi_{\{s_j\}}(u) \tag{51}$$

Lanczos coefficients

- Elements of the Lanczos matrix

$$b_{\{m\}\to\{r_j\}} = (\Phi_{\{m\}}(u), \xi l_{-1}\Phi_{\{r_j\}}(u))$$

$$\Longrightarrow \; l_{-1}\Phi_{\{m_k\}} = \sum_{n=1}^{N} \sqrt{n(n+1)m_n(m_{n+1}+1)}\Phi_{\dots,m_n-1,m_{n+1}+1,\dots}(u) + (\mu - i\lambda)\sqrt{2(m_1+1)}\Phi_{m_1+1,m_2,\dots}(u). \quad (52)$$

$\Longrightarrow$ two types Lanczos coefficients (Caputa & Datta 2021')

Type 1: $b^{(1)}_{\{m_k\}\to\{\dots,m_n-1,m_{n+1}+1,\dots\}} = \alpha\sqrt{n(n+1)m_n(m_{n+1}+1)}$ (53)

Type 2: $b^{(2)}_{\{m+k\}\to\{m_1+1,m_2,\dots\}} = \alpha(\mu - i\lambda)\sqrt{2(m_1+1)}.$ (54)

- Dimensions

$$\dim_{Lanczos}\left[b_{\{m\}\to\{r_j\}}\right] = p(N)\times p(N+1) \overset{N\to\infty}{\sim} \frac{e^{2\pi\sqrt{2N/3}}}{N^2}$$

$$\dim_{\text{links}} \sim \int_0^\infty dn\, p(n) \overset{N\to\infty}{\sim} \frac{e^{\pi\sqrt{2N/3}}}{\sqrt{2N}} \;\; : \text{ suppression by } \sim e^{-\pi\sqrt{2N/3}}$$

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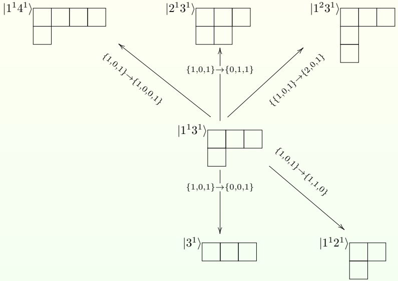
$$\dim_{\text{links}} \sim \int_0^\infty dn\, p(n) \overset{N\to\infty}{\sim} \frac{e^{\pi\sqrt{2N/3}}}{\sqrt{2N}}$$: suppression by $\sim e^{-\pi\sqrt{2N/3}}$

- An example: descendants resulting from the action of $L_{\pm 1}$ on $|1^1 3^1\rangle$



Lanczos coefficients for typical descendants

⋆ Lanczos coefficients for typical high-level descendants of a heavy primary

- states with (c, h) dependence, $n \ll N$

$$b_{\{m_i\}\to\{\dots,m_n-1,m_n+1,\dots\}} \quad \Longrightarrow \quad b_n \sim \sqrt{N}$$

- states without (c, h) dependence, $n \ll N$

$$b_{\{m_i\}\to\{m_1+1,m_2,\dots\}} \quad \Longrightarrow \quad b_n \sim \sqrt[4]{n}$$

Expansion over normalized descendants

- $$\Psi_{\mathcal{O}}(t) := \langle u|e^{i\alpha t(l_1+l_{-1})}\mathcal{O}(0)|0\rangle$$
$$= e^{\alpha_0 h}\left[1+\sum_{N=1}^{\infty}\sum_{\sum im_i=N}\varphi_{\{m_i\}}(t)\Phi_{\{m_i\}}(u)\right], \qquad (55)$$

- 'wavefunctions', $\varphi_{\{m_i\}}(t)$, of the primary operator are given by

$$\varphi_{\{m_i\}}(t) = \frac{z^N}{\cosh^{2h}(\alpha t)}\frac{[2(\mu-i\lambda)]^{\sum m_j}}{\sqrt{\prod_i T_{i,m_i}}}, \qquad \sum_j jm_j = N\,. \qquad (56)$$

with $z = i\tanh(\alpha t)$, $\alpha_0 = -2h\log\cosh(\alpha t)$, $T_{j,m} = (2j)^m m_j!$

- The probabilities

$$p_{\{m_j\}}(t) = |\varphi_{\{m_j\}}(t)|^2 = \frac{\tanh^{2N}(\alpha t)}{\cosh^{4h}(\alpha t)}\frac{[4h]^{\sum m_j}}{\prod_i(2i)^{m_i}m_i!}\,.$$

Vir contribution

- Krylov complexity (see also Caputa,Datta 21')

$$K_{\mathcal{O}}(t) = \sum_{N=0}^{\infty} N \sum_{\sum i m_i = N} |\phi_{m_i}|^2(t) = 2h\sinh^2(\alpha t) \tag{57}$$

$\Longrightarrow$ exponential growth of $K_{\mathcal{O}}(t)$ at late times

$$K_{\mathcal{O}}(t \to \infty) \sim \frac{h}{2} e^{2\alpha t}.$$

- Normalized variance

$$\delta^2_{\mathcal{O}}(t) = \frac{\langle \hat{N}^2 \rangle - \langle \hat{N} \rangle^2}{\langle \hat{N} \rangle^2} \quad \Longrightarrow \quad \delta_{\mathcal{O}}(t \to \infty) \sim \frac{1}{\sqrt{2h}}.$$

$U(1)$ contribution

- Rescaling of J_n:

$$J_n \longrightarrow \mathcal{J}_n = \sqrt{\frac{2}{k}} J_n$$

$\Longrightarrow$ the algebra

$$[\mathcal{J}_n, \mathcal{J}_m] = n\delta_{n+m}.$$

- States $\quad |k_n\rangle = \frac{e^{\beta J_{-1}}}{\sqrt{\langle 0|J_1^n J_{-1}^n|0\rangle}}|0\rangle$

- Autocorrelation function

$$C^U(t) \sim \frac{1}{2\cosh^{2Q}(\beta \frac{k}{2} t)}$$

Virasoro-Kac-Moody Character

- Virasoro-Kac-Moody character - product of $U(1)$ and Vir conttributions

Virasoro-Kac-Moody Character

- Virasoro-Kac-Moody character - product of $U(1)$ and Vir conttributions
  - the contribution of the $U(1)$ descendants

$$\prod_{n=1}^{\infty} \frac{1}{1+q^n} = q^{1/24} \frac{\eta(\tau)}{\eta(2\tau)}.$$

  - the contribution of the Vir descendants

$$(1 - \delta^{(0)} q) \prod_{n=1}^{\infty} \frac{1}{1-q^n} = q^{1/24} \frac{1}{\eta(\tau)} (1 - \delta^{(0)} q).$$

  - the full Virasoro-Kac-Moody character

$$\chi_{h,n}(\tau, \kappa) = q^{h+2/24-c/24} \frac{1}{\eta(2\tau)} r^n (1 - \delta^{(0)} q).$$

- This character is independent of the basis used for the Vir descendants!

The warped system

- Autocorrelation functions

$$C^W(t) \sim \frac{1}{\cosh^{2h}(\alpha t)} \frac{1}{\cosh^{2Q}(\beta \frac{k}{2} t)}$$

- Krylov complexity

$$K_W(t) \simeq 2hQ \cosh^2(\alpha t) \cosh^2(\beta \frac{k}{2} t)$$

- Operator growth

$$K_W(t) \sim e^{(2\alpha + \beta k)t}$$

- Normalized variance

$$\delta_W(t \to \infty) \sim \frac{1}{\sqrt{2hQ}}.$$

- Information metric

$$ds^2 = \frac{Q}{1 - |z_2|^2} dz_1 d\bar{z}_1 + \frac{Q|z_1|^2 + 2h(1 - |z_2|^2)}{(1 - |z_2|^2)^3} dz_2 d\bar{z}_2$$

Conclusions

⋆ Considerations of the operator growth in 2d WCFT's show:

- Lanczos coefficients essentially depend on the details of descendant states
- a subset of them does saturate the upper bound of linear growth (as conjectured)
- K-complexity: universal but is not sensitive enough to distinguish WCFT from $SL(2,R) \times U(1)$ case
- K-complexity defined for subclasses of vertices (as in Caputa,Datta'21)

⋆ Future directions:

- Lanczos coefficients for W_2; doo they still obey the maximal bound?
- relations to dipole deformations?
- embedding in higher dimensional cases
- study complexity of multi-gluonic compound states in QCD?
- ...

END

THANK YOU!