Two Perspectives on Elliptic Deformations of $AdS_3 \times S^3$ S-matrix

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Integrability, Q-systems and Cluster Algebras

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The success of integrability in AdS_5/CFT_4 made people ask questions about lower dimensional AdS backgrounds

AdS_4/CFT_3

AdS_3/CFT_2

AdS_2/CFT_1

Less and less is known

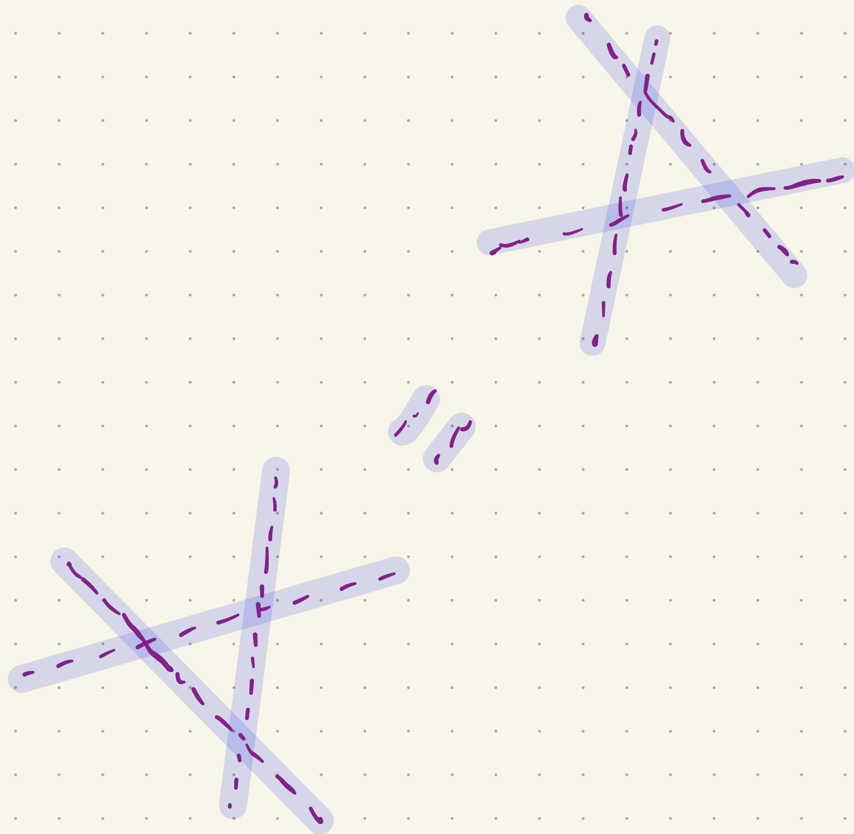
But integrability is present!

$AdS_3 \times S^3 \times T^4$

worldsheet S-matrix

is

INTEGRABLE



Cagnazzo, Zarembo, Babichenko, Stefanski, Lloyd, Sax, Sfondrini, Baggio, ...

The Green-Schwarz action is given by

$$S = -\frac{T}{2}\int d\tau\, d\sigma \left[\gamma^{\alpha\beta} G_{\mu\nu} + \epsilon^{\alpha\beta} B_{\mu\nu}\right] \partial_\alpha \psi^\mu \partial_\beta \psi^\nu$$

$$+ \text{fermions}$$

$$\underbrace{AdS_3}_{0,1,2} \times \underbrace{S^3}_{3,4,5} \times \underbrace{T^4}_{6,7,8,9}$$

$\alpha, \beta = 1, 2$

$\mu, \nu = 0, \dots, 9$

$G_{\mu\nu} \to$ metric

$B_{\mu\nu} \to$ anti-symmetric B-field

- We can only quantise this theory perturbatively
- Theory has a lot of redundancies... so the first step is fixing the gauge
- $\mathcal{L}^{g.f.} = \mathcal{L}_2 + \frac{1}{\sqrt{T}} \mathcal{L}_3 + \frac{1}{T} \mathcal{L}_4 + \cdots$
- Compute the tree-level S-matrix
- Investigate hidden symmetries
- Assume that they hold at quantum level and use them to bootstrap the quantum S-matrix

→ $$\mathcal{S} = 1 + \frac{i}{T}\,\underbrace{\mathcal{T}}_{\text{tree-level } \mathcal{S}\text{-matrix}} + \cdots$$

It satisfies the classical YBE

$$[\mathcal{T}_{12}, \mathcal{T}_{13}] + [\mathcal{T}_{12}, \mathcal{T}_{23}] + [\mathcal{T}_{13}, \mathcal{T}_{23}] = 0$$

→ Bootstrap the quantum $\mathcal{S}$-matrix using off-shell symmetries...

→ Check quantum integrability

Worldsheet S-matrix $\mathcal{S}$ satisfies the **QUANTUM YANG-BAXTER EQUATION**

but more than that

$\check{\mathcal{S}} = \check{S} \otimes \check{\tilde{S}}$ where $\check{S} = \left(\begin{array}{c|c} \check{\mathcal{S}}^{LL} & \check{\mathcal{S}}^{LR} \\ \hline \check{\mathcal{S}}^{RL} & \check{\mathcal{S}}^{RR} \end{array}\right)$

FACTORIZATION BLOCK STRUCTURE

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FACTORIZATION

BLOCK STRUCTURE

$$\check{S} = \left(\begin{array}{c|c} \check{\mathcal{S}}^{LL} & \check{\mathcal{S}}^{LR} \\ \hline \check{\mathcal{S}}^{RL} & \check{\mathcal{S}}^{RR} \end{array}\right) \qquad V^L \otimes V^L \otimes V^R \mapsto V^L \otimes V^L \otimes V^R$$

$$\mathcal{S}^{LL}_{12}(u_1, u_2)\, \mathcal{S}^{LR}_{13}(u_1, u_3)\, \mathcal{S}^{LR}_{23}(u_2, u_3)$$

$$= \mathcal{S}^{LR}_{23}(u_2, u_3)\, \mathcal{S}^{LR}_{13}(u_1, u_3)\, \mathcal{S}^{LL}_{12}(u_1, u_2)$$

$$\mathcal{S}^{AB}_{ij} = P_{ij}\, \check{\mathcal{S}}^{AB}_{ij}$$

In particular,

$\mathcal{S}^{LL}$ and $\mathcal{S}^{RR}$ are 6-vertex models

$$\begin{pmatrix} s_{11} & 0 & 0 & 0 \\ 0 & s_{22} & s_{23} & 0 \\ 0 & s_{32} & s_{33} & 0 \\ 0 & 0 & 0 & s_{44} \end{pmatrix}$$

$$s_{ij} \equiv s_{ij}(p_i, p_j) \neq s_{ij}(p_i - p_j)$$

"Similar" to XXX R-matrix

But it is of Free Fermion type:

$$s_{11}s_{44} + s_{22}s_{33} = s_{23}s_{32} + s_{14}s_{41}$$

This model has a lot of symmetry and as a consequence

$$\left[su(1|1)_L \oplus su(1|1)_R \right]^2_{ce}$$

→ FACTORISATION

→ BLOCK STRUCTURE

→ INTEGRABILITY

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→ FACTORISATION

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→ INTEGRABILITY

A lot is known about the TBA for example (Dei, Sfondrini, Baggio, Frolov, Polvara, Torrielli, Borsato, Sax, Stefański)

For more see

Exact approaches on the string worldsheet

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arXiv:2312.12930v2

There are also deformations ... (Hoare, Seibold, Tseytlin, Sfondrini, Tongeren, Vicedo, Delduc, Kameyama, Lacroix, Magro, ...)

For example, a q-deformation (trigonometric)

$$\mathcal{S}^{LL}_{q\text{-def.}} \text{ is again of form } \begin{pmatrix} s_{11} & 0 & 0 & 0 \\ 0 & s_{22} & s_{23} & 0 \\ 0 & s_{32} & s_{33} & 0 \\ 0 & 0 & 0 & s_{44} \end{pmatrix}$$

For $q \to 1$, $\Rightarrow$ undeformed $\mathcal{S}^{LL}$

"similar" to XXZ

but free-fermion

WHAT ABOUT AN ELLIPTIC

DEFORMATION OF

$AdS_3 \times S^3 \times T^4$?

From the point of view of INTEGRABILITY,

can we find an 8-vertex models such that

$$\check{S} = \left(\begin{array}{c|c} \check{\mathcal{S}}^{LL} & \check{\mathcal{S}}^{LR} \\ \hline \check{\mathcal{S}}^{RL} & \check{\mathcal{S}}^{RR} \end{array}\right)$$

under some limit $\quad S_{\text{ELLIPTIC}} \begin{array}{l} \nearrow S_{\text{TRIG.}} \text{ ? It depends!} \\ \searrow S_{\text{UNDEF.}} \text{ ? YES!} \end{array}$

BUT DOES SUCH AN S-MATRIX

COME FROM A STRING

SIGMA MODEL ?

PLAN

arXiv: 2003.04332

arXiv: 2312.14031

1. Method to construct integrable models.

   (de Leeuw, Palleta, Pribytok, ALR, Ryan, 2020)

2. Integrable and quasi-integrable elliptic deformation.

3. Do they come from a string σ-model?

   (Hoare, ALR, Seibold, 2023)

Solving the Yang-Baxter equation is quite difficult

$$R_{12}(u_1,u_2)\,R_{13}(u_1,u_3)\,R_{23}(u_2,u_3) = R_{23}(u_2,u_3)\,R_{13}(u_1,u_3)\,R_{12}(u_1,u_2)$$

Many people contributed to this problem (Baxter, Jimbo, Perk, Yang, Kuniba, Bazhanov, Batchelor, Martins, Pimenta, Vieira, Links, Doikou, Fateev, Serban, Beisert, Hoare, Torrielli, Zamolodchikov, Faddeev, Izergin, Korepin, Alcaraz, Kulish, Sklyanin, ...)

Most of them were found using strategies that make use of

SYMMETRIES

and/or are of difference form

$$R_{12}(u,v) = R_{12}(u-v)$$

But the few known non-difference form models are very interesting! Can we find more?

Can we have a systematic way to find non-difference form models $\left(R(u,v) \neq R(u-v)\right)$ without knowing in advance its algebraic structure?

$$H = \begin{bmatrix} a_{11} & 0 & a_{13} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{22} & 0 & a_{24} & 0 & 0 & 0 & 0 & 0 \\ a_{31} & 0 & a_{33} & 0 & 0 & 0 & a_{37} & 0 & 0 \\ 0 & a_{42} & 0 & a_{44} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{55} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{66} & 0 & a_{68} & 0 \\ 0 & 0 & a_{73} & 0 & 0 & 0 & a_{77} & 0 & a_{79} \\ 0 & 0 & 0 & 0 & 0 & a_{86} & 0 & a_{88} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_{97} & 0 & a_{99} \end{bmatrix}$$

For which values of a_{ij} is this integrable?

And what is the corresponding R?

de Leeuw, Palleta, Pribytok, ALR, Ryan, 2020

We can do that for

$$\boxed{\begin{array}{l} R(u,u) \sim P \\[2ex] \left.\dfrac{d R(u,v)}{du}\right|_{v=u} \sim P\, H(u) \end{array}}$$

The key ingredient is the boost operator (Links, Zhou, McKenzie, Gold, 2001)

$$B[Q_2] = -\sum_n n\, H_{n,n+1}(u) + \frac{\partial}{\partial u}$$

It is very useful because

$$Q_{r+1} = [\underbrace{\mathbb{B}[Q_2]}_{\text{Boost operator}}, Q_r] \qquad r > 1$$

$$Q_3 = [\mathbb{B}[Q_2], Q_2]$$

$$= \sum_{i=1}^{L} [H_{i-1,i}(u), H_{i,i+1}(u)] + \frac{d}{du}\mathbb{H}(u)$$

It is very useful because

$$Q_{r+1} = [\mathbb{B}[Q_2], Q_r]$$

$\downarrow$

Boost operator

For $R(u,v) = R(u-v)$

$R'(0) = PH$

$$Q_3 = [\mathbb{B}[Q_2], Q_2]$$

$$= \sum_{i=1}^{L} [H_{i-1,i}(u), H_{i,i+1}(u)] + \cancel{\frac{d}{du}\mathbb{H}(u)}$$

(Tetel'man, 1982)

STEP 1: Propose an ansatz for H:

Example: $$H(u) = \begin{pmatrix} a_{11} & 0 & 0 & a_{14} \\ 0 & a_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & 0 \\ a_{41} & 0 & 0 & a_{44} \end{pmatrix}, \quad a_{ij} = a_{ij}(u)$$

STEP 2: Construct Q_3 using the boost operator

$$Q_3 = -\sum_{j=1}^{L} \left[H_{j-1,j}, H_{j,j+1} \right] + \frac{dH}{du}$$

STEP 3: Require $[Q_2, Q_3] = 0$

and carefully solve all ODEs

STEP 4: Plug each ansatz on the Sutherland equation

$$\frac{\partial}{\partial u} YBE(u,v,w)\Big|_{v\to u} = 0$$

$$[R_{13} R_{23}, H_{12}(u)] = \dot{R}_{13} R_{23} - R_{13} \dot{R}_{23}$$

and solve for R using the boundary conditions

$$R(u,u) \sim P \qquad \dot{R}(u,v)\Big|_{v=u} \propto P H(u)$$

For each you will find an R-matrix

It seems that $[Q_2, Q_3] = 0$ is enough to fix $\mathcal{H}$

But transformations like

- $R(u,v) \to R(f(u), f(v))$ — reparametrisation
- $R(u,v) \to f(u,v)\, R(u,v)$ — normalisation
- $R(u,v) \to (V \otimes V)\, R(u,v)\, (V \otimes V)^{-1}$ — basis transformation
- $R(u,v) \to P\, R(u,v)\, P$
- $R(u,v) \to R(u,v)^T$ — transposition

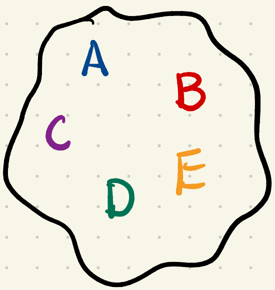
or

- $R(u,v) \to (U(u) \otimes 1)\, R(u,v)\, (1 \otimes U(v)^{-1})$ — Twist

Preserve YBE if $[R(u,v), U(u) \otimes U(v)] = 0$

Disadvantages:

Expectation:



Some interesting solutions

Disadvantages:

Normalisation

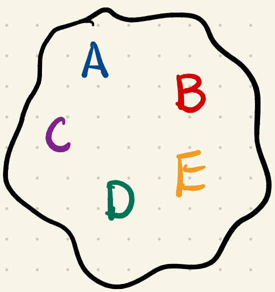
Reparametrisation

Basis transformations

Discrete transformations

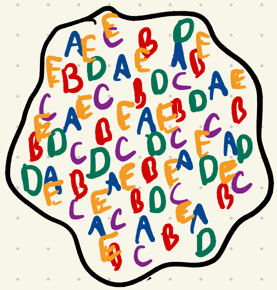
Twists

Expectation:



Some interesting solutions

Reality:



Lots of dependent solutions

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Normalisation

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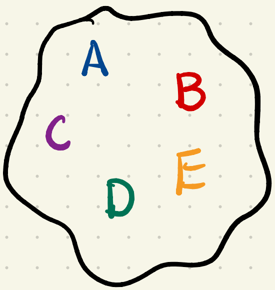
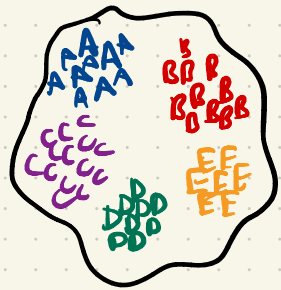


Some interesting solutions

Reality:

Lots of dependent solutions

reasonable amount of work $\Longrightarrow$



This disadvantage can be transformed into an advantage

We can use this freedom to

- Put one diagonal element of the Hamiltonian to zero;
- And one element to one;

This makes the system much easier to solve.

ANSATZ FOR H

$\Rightarrow$

USE BOOST TO CONSTRUCT Q_3

$\Rightarrow$

$[Q_2, Q_3] = 0$

Solve system of ODEs

$\Downarrow$

Solve Sutherland for each H

$\Leftarrow$

Check YBE

Main advantage:

quite efficient for non-difference form models

We applied the method to

- $su(2) \oplus su(2)$ diff. and non-diff
- 15-vertex model non-diff.

de Leeuw, Paletta, Pribytok, ALR, Ryan, 2020, 2021

- Full 4x4 non-difference form

Corcoran, de Leeuw, 2023

- Flag models

de Leeuw, Nepomechie, ALR, 2022

- Systematic construction of integrable Lindblad systems

de Leeuw, Paletta, Pozsgay, 2021

- AdS_3 and AdS_2 integrable deformations

de Leeuw, Paletta, Pribytok, ALR, Ryan, 2020

② From the point of view of INTEGRABILITY,

can we find an 8-vertex models such that

$$\check{S} = \left(\begin{array}{c|c} \check{\mathcal{S}}^{LL} & \check{\mathcal{S}}^{LR} \\ \hline \check{\mathcal{S}}^{RL} & \check{\mathcal{S}}^{RR} \end{array}\right)$$

under some limit S_{ELLIPTIC}

$\nearrow$ $S_{\text{TRIG.}}$? It depends!

$\searrow$ $S_{\text{UNDEF.}}$? YES!

Let us go back to the elliptic models $S^{LL}(u,u) \sim P$

$$r_1^{LL} = \frac{1}{\sqrt{\sin(\eta(u))\sin(\eta(v))}}\left(-\cos\eta_+ \mathrm{sn}_-^{LL} + \frac{\mathrm{cn}_-^{LL}}{\mathrm{dn}_-^{LL}}\sin\eta_+\right),$$

$$r_2^{LL} = -\frac{1}{\sqrt{\sin(\eta(u))\sin(\eta(v))}}\left(\cos\eta_- \mathrm{sn}_-^{LL} - \frac{\mathrm{cn}_-^{LL}}{\mathrm{dn}_-^{LL}}\sin\eta_-\right),$$

$$r_3^{LL} = -\frac{1}{\sqrt{\sin(\eta(u))\sin(\eta(v))}}\left(\cos\eta_- \mathrm{sn}_-^{LL} - \frac{\mathrm{cn}_-^{LL}}{\mathrm{dn}_-^{LL}}\sin\eta_-\right),$$

$$r_4^{LL} = \frac{1}{\sqrt{\sin(\eta(u))\sin(\eta(v))}}\left(\cos\eta_+ \mathrm{sn}_-^{LL} + \frac{\mathrm{cn}_-^{LL}}{\mathrm{dn}_-^{LL}}\sin\eta_+\right),$$

$$r_5^{LL} = \sqrt{\frac{g_L(v)}{g_L(u)}}, \quad r_6^{LL} = \sqrt{\frac{g_L(u)}{g_L(v)}},$$

$$r_7^{LL} = \frac{k\alpha}{\sqrt{g_L(u)g_L(v)}}\frac{\mathrm{cn}_-^{LL}\mathrm{sn}_-^{LL}}{\mathrm{dn}_-^{LL}}, \quad r_8^{LL} = \frac{k\sqrt{g_L(u)g_L(v)}}{\alpha}\frac{\mathrm{cn}_-^{LL}\mathrm{sn}_-^{LL}}{\mathrm{dn}_-^{LL}}.$$

$$S^{LL} = \begin{pmatrix} r_1 & & & r_8 \\ & r_2 & r_6 & \\ & r_5 & r_3 & \\ r_7 & & & r_4 \end{pmatrix}$$

$$\eta_\pm = \frac{\eta(u) \pm \eta(v)}{2}$$

$$\mathrm{cn}_-^{LL} = \mathrm{JacobiCN}\left(F^L(u) - F^L(v),\, k_L^2\right)$$

$$\delta^{LL}(u,u) \sim P \qquad\qquad \delta^{LR}(u,u) \neq P$$

$$\left(\begin{array}{c|c} \check{\delta}^{LL}(u,v) & \check{\delta}^{LR}(u,v) \\ \hline \check{\delta}^{RL}(u,v) & \check{\delta}^{RR}(u,v) \end{array}\right)$$

$$\delta^{RL}(u,u) \neq P \qquad\qquad \delta^{RR}(u,u) \sim P$$

$$k_R = k_L \equiv k$$

$$r_1^{\mathrm{LR}} = \frac{1}{\sqrt{\sin(\eta(u))\sin(\eta(v))}}\sqrt{\frac{g_{\mathrm{R}}(v)}{g_{\mathrm{L}}(u)}}\left(\cos\eta_-\mathrm{sn}_+^{\mathrm{LR}} + \frac{\mathrm{cn}_+^{\mathrm{LR}}}{\mathrm{dn}_+^{\mathrm{LR}}}\sin\eta_-\right),$$

$$r_2^{\mathrm{LR}} = \frac{1}{\sqrt{\sin(\eta(u))\sin(\eta(v))}}\sqrt{\frac{g_{\mathrm{R}}(v)}{g_{\mathrm{L}}(u)}}\left(\cos\eta_+\mathrm{sn}_+^{\mathrm{LR}} - \frac{\mathrm{cn}_+^{\mathrm{LR}}}{\mathrm{dn}_+^{\mathrm{LR}}}\sin\eta_+\right),$$

$$r_3^{\mathrm{LR}} = \frac{1}{\sqrt{\sin(\eta(u))\sin(\eta(v))}}\sqrt{\frac{g_{\mathrm{R}}(v)}{g_{\mathrm{L}}(u)}}\left(\cos\eta_+\mathrm{sn}_+^{\mathrm{LR}} + \frac{\mathrm{cn}_+^{\mathrm{LR}}}{\mathrm{dn}_+^{\mathrm{LR}}}\sin\eta_+\right),$$

$$r_4^{\mathrm{LR}} = \frac{1}{\sqrt{\sin(\eta(u))\sin(\eta(v))}}\sqrt{\frac{g_{\mathrm{R}}(v)}{g_{\mathrm{L}}(u)}}\left(-\cos\eta_-\mathrm{sn}_+^{\mathrm{LR}} + \frac{\mathrm{cn}_+^{\mathrm{LR}}}{\mathrm{dn}_+^{\mathrm{LR}}}\sin\eta_-\right)$$

$$r_5^{\mathrm{LR}} = -\frac{k\alpha}{g_{\mathrm{L}}(u)}\frac{\mathrm{cn}_+^{\mathrm{LR}}\mathrm{sn}_+^{\mathrm{LR}}}{\mathrm{dn}_+^{\mathrm{LR}}}, \quad r_6^{\mathrm{LR}} = \frac{k g_{\mathrm{R}}(v)}{\alpha}\frac{\mathrm{cn}_+^{\mathrm{LR}}\mathrm{sn}_+^{\mathrm{LR}}}{\mathrm{dn}_+^{\mathrm{LR}}},$$

$$r_7^{\mathrm{LR}} = -\frac{g_{\mathrm{R}}(u)}{g_{\mathrm{L}}(v)}, \quad r_8^{\mathrm{LR}} = 1.$$

$$\mathrm{cn}_{\pm}^{LR} = \mathrm{JacobiCN}\left[F^L(u) \pm F^R(v),\ k^2\right]$$

$$\eta_\pm = \frac{\eta(u) \pm \eta(v)}{2}$$

$$S_{\text{ELLIPTIC}}(u,v): \quad \left(\begin{array}{c|c} \check{\mathcal{S}}^{LL} & \check{\mathcal{S}}^{LR} \\ \hline \check{\mathcal{S}}^{RL} & \check{\mathcal{S}}^{RR} \end{array}\right)$$

BLOCK STRUCTURE !

YBE !

CROSSING SYMMETRY !

→ $k \to 0$, we can define F, η, g such that S_{ELLIPTIC} reduces to $AdS_3 \times S^3 \times T^4$ S-matrix

→ But does NOT reduce to the q-deformed $AdS_3 \times S^3 \times T^4$ S-matrix unless we use a twist that breaks integrability.

BUT DOES THIS COMES FROM

A STRING SIGMA MODEL ?

(Ben Hoare, ALR, Fiona Seibold, 2023)

Principal Chiral Model (basic ideas)

$g(\tau, \sigma)$ valued in a Lie Group G (Plays the role of the target space)

$$j = g^{-1} dg \in \mathfrak{g} = \mathrm{Lie}(G)$$

$$k = -dg\, g^{-1} \in \mathfrak{g} = \mathrm{Lie}(G)$$

$$S_{PCM} = \frac{T}{2} \int d\tau\, d\sigma\, \mathrm{tr}\left(g^{-1} \partial_\alpha g\, g^{-1} \partial^\alpha g \right)$$

j, k satisfy the zero curvature equation

$$\partial_\alpha j_\beta - \partial_\beta j_\alpha + [j_\alpha, j_\beta] = 0$$

Global symmetry : $G_L \times G_R$

$$g \to g_L\, g\, g_R \qquad g_L, g_R \in G$$

$$x^{\pm} = \frac{\tau \pm \sigma}{2} ,$$

$$\mathcal{L}_{\pm}(u) = \frac{j_{\pm}}{1 \mp u} \to \text{Lax}$$

$$S_{PCM} = \frac{T}{2}\int d\tau\, d\sigma\, \mathrm{tr}\left(g^{-1}\partial_\alpha g\, g^{-1}\partial^\alpha g\right) = \frac{T}{2}\int d\tau\, d\sigma\, \mathrm{tr}\, j_\mu j^\mu$$

$$= \frac{T}{2}\int d\tau\, d\sigma \left(\gamma^{\alpha\beta} G_{\mu\nu} + \epsilon^{\alpha\beta} B_{\mu\nu}\right)\partial_\alpha \varphi^\mu \partial_\beta \varphi^\nu$$

Focus on the bosonic strings on $AdS_3 \times S^3$

$$AdS_3 = \frac{SO(2,2)}{SO(1,2)} \cong \frac{SL(2;\mathbb{R}) \times SL(2;\mathbb{R})}{SL(2;\mathbb{R})}$$

$$S^3 = \frac{SO(4)}{SO(3)} \cong \frac{SU(2) \times SU(2)}{SU(2)}$$

One way to think about it is to describe this as a Principal Chiral Model with

$$G = SL(2;\mathbb{R}) \times SU(2)$$

from AdS_3 $\quad$ from S^3

$$G = SL(2;\mathbb{R}) \times SU(2)$$

$\mathfrak{sl}(2;\mathbb{R})$ algebra $\quad$ $\mathfrak{su}(2)$ algebra

$$L_1 = \begin{pmatrix} \sigma_1 & 0 \\ 0 & 0 \end{pmatrix} \qquad L_3 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & 0 \end{pmatrix} \qquad J_j = \begin{pmatrix} 0 & 0 \\ 0 & i\sigma_j \end{pmatrix}$$

$$L_2 = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & 0 \end{pmatrix}$$

$$g = e^{T L_3} e^{U L_3} e^{V L_1} e^{\Phi J_2} e^{X J_3} e^{Y J_1}$$

$\{T, U, V\}$ coordinates in AdS_3

$\{\Phi, X, Y\}$ coordinates in S^3

Work with Fiona Seibold and Ben Hoare (arxiv 2312.14031)

For now focus on $AdS_3 \times S^3$

STEP 1: Deform the PCM for $G = SL(2;\mathbb{R}) \times SU(2)$

$$S = \frac{1}{4}\int d\tau d\sigma \, (\gamma^{\alpha\beta} + \epsilon^{\alpha\beta}) \, \mathrm{Tr}\left[g^{-1}\partial_\alpha g \, \Theta \, g^{-1}\partial_\beta g\right]$$

$\Theta(L_j) = -\alpha_j L_j$

$\Theta(J_j) = \beta_j J_j$

$j = 1, 2, 3$

$\alpha_1 \neq \alpha_2 \neq \alpha_3$ elliptic

$\alpha_1 = \alpha_3 \neq \alpha_2$ trigonometric

$\alpha_1 = \alpha_2 = \alpha_3$ rational / undeformed

STEP 2: Compare it with Green-Schwarz action

$B_{\mu\nu} = 0$

and the metric $ds^2 = ds_a^2 + ds_b^2$

$$ds_a^2 = \left(\alpha_1 \sinh^2 2U + \cosh^2 2U \left(-\alpha_2 \cosh^2 2V + \alpha_3 \sinh^2 2V\right)\right) dT^2 + \alpha_1 dV^2 + \\ + \left(\alpha_3 \cosh^2 2V - \alpha_2 \sinh^2 2V\right) dU^2 - 2\alpha_1 \sinh 2U \, dT dV + (\alpha_3 - \alpha_2) \cosh 2U \sinh 4V \, dT dU$$

$$ds_b^2 = \left(\beta_1 \sin^2 2X + \cos^2 2X \left(\beta_2 \cos^2 2Y + \beta_3 \sin^2 2Y\right)\right) d\Phi^2 + \beta_1 dY^2 + \\ + \left(\beta_3 \cos^2 2Y + \beta_2 \sin^2 2Y\right) dX^2 - 2\beta_1 \sin 2X \, d\Phi dY + (\beta_3 - \beta_2) \cos 2X \sin 4Y \, d\Phi dX \, .$$

$\Downarrow$

$$ds_a^2 = \left(\alpha_1 \sinh^2 2U + \cosh^2 2U\left(-\alpha_2 \cosh^2 2V + \alpha_3 \sinh^2 2V\right)\right) dT^2 + \alpha_1 dV^2 + \\ + \left(\alpha_3 \cosh^2 2V - \alpha_2 \sinh^2 2V\right) dU^2 - 2\alpha_1 \sinh 2U\, dT\, dV + (\alpha_3 - \alpha_2)\cosh 2U \sinh 4V\, dT\, dU$$

$$ds_b^2 = \left(\beta_1 \sin^2 2X + \cos^2 2X\left(\beta_2 \cos^2 2Y + \beta_3 \sin^2 2Y\right)\right) d\Phi^2 + \beta_1 dY^2 + \\ + \left(\beta_3 \cos^2 2Y + \beta_2 \sin^2 2Y\right) dX^2 - 2\beta_1 \sin 2X\, d\Phi\, dY + (\beta_3 - \beta_2)\cos 2X \sin 4Y\, d\Phi\, dX\ .$$

STEP 3: FIX LIGHT-CONE GAUGE

$\Downarrow$

STEP 4: DECOMPACTIFY AND COMPUTE THE SCATTERING MATRIX

Perturbatively!

$$
\begin{aligned}
ds_a^2 = &\left(\alpha_1 \sinh^2 2U + \cosh^2 2U \left(-\alpha_2 \cosh^2 2V + \alpha_3 \sinh^2 2V\right)\right) dT^2 + \alpha_1 dV^2 + \\
&+ \left(\alpha_3 \cosh^2 2V - \alpha_2 \sinh^2 2V\right) dU^2 - 2\alpha_1 \sinh 2U\, dT\, dV + (\alpha_3 - \alpha_2) \cosh 2U \sinh 4V\, dT\, dU \\
ds_b^2 = &\left(\beta_1 \sin^2 2X + \cos^2 2X \left(\beta_2 \cos^2 2Y + \beta_3 \sin^2 2Y\right)\right) d\Phi^2 + \beta_1 dY^2 + \\
&+ \left(\beta_3 \cos^2 2Y + \beta_2 \sin^2 2Y\right) dX^2 - 2\beta_1 \sin 2X\, d\Phi\, dY + (\beta_3 - \beta_2) \cos 2X \sin 4Y\, d\Phi\, dX \ .
\end{aligned}
$$

$$
\boxed{
\begin{aligned}
\sqrt{\omega^a_\pm(p)^2 + \tilde{\gamma}_2^2} &= \sqrt{p^2 + \tilde{\gamma}_1^2} \pm \tilde{\gamma}_3 \\
\sqrt{\omega^b_\pm(p)^2 + \gamma_2^2} &= \sqrt{p^2 + \gamma_1^2} \pm \gamma_3
\end{aligned}
}
$$

$\longleftarrow$ AdS_3

$\longleftarrow$ S^3

DISPERSION RELATIONS

$$
\begin{aligned}
ds_a^2 = &\left(\alpha_1 \sinh^2 2U + \cosh^2 2U \left(-\alpha_2 \cosh^2 2V + \alpha_3 \sinh^2 2V\right)\right) dT^2 + \alpha_1 dV^2 + \\
&+ \left(\alpha_3 \cosh^2 2V - \alpha_2 \sinh^2 2V\right) dU^2 - 2\alpha_1 \sinh 2U \, dT dV + (\alpha_3 - \alpha_2) \cosh 2U \sinh 4V \, dT dU \\
ds_b^2 = &\left(\beta_1 \sin^2 2X + \cos^2 2X \left(\beta_2 \cos^2 2Y + \beta_3 \sin^2 2Y\right)\right) d\Phi^2 + \beta_1 dY^2 + \\
&+ \left(\beta_3 \cos^2 2Y + \beta_2 \sin^2 2Y\right) dX^2 - 2\beta_1 \sin 2X \, d\Phi dY + (\beta_3 - \beta_2) \cos 2X \sin 4Y \, d\Phi dX \ .
\end{aligned}
$$

But the tree-level (bosonic) S-MATRIX is DIAGONAL

↓

IT HAS AN UNEXPECTED

U(1) SYMMETRY

It automatically satisfies the classical YBE!

$$J = J^{LL} = \begin{pmatrix} r_1 & 0 & 0 & s_4 \\ 0 & r_2 & s_3 & 0 \\ 0 & s_2 & r_3 & 0 \\ s_1 & 0 & 0 & r_4 \end{pmatrix}$$

Tried to find $(s_i,\ i=1,\dots,4)$ such that

$$[T_{12}, T_{13}] + [T_{12}, T_{23}] + [T_{13}, T_{23}] = 0$$

IT IS IMPOSSIBLE!

When we include the fermions, one of the following will happen

1) It breaks $\check{S} = \check{\dot{S}} \otimes \check{\tilde{S}}$

FACTORIZATION

2) It breaks $\left(\begin{array}{c|c} \check{S}^{LL} & \check{S}^{LR} \\ \hline \check{S}^{RL} & \check{S}^{RR} \end{array}\right)$

BLOCK STRUCTURE

3) IT BREAKS INTEGRABILITY!

4) IT "WEAKLY" BREAKS INTEGRABILITY!

CONCLUSIONS

I presented a method to construct R-matrices

Showed one elliptic integrable deformation coming from integrability and another from the sigma model.

NEXT STEPS:

(in progress with B. Hoare and F. Seibold)

- Compute the tree-level S-matrix including fermions
- Investigate the existence of hidden symmetries.
- Bootstrap the exact S-matrix
- Connection to $AdS_2 \times S^2$

THANK YOU!

EXTRA SLIDES

$$\mathcal{L}^{gf} = \mathcal{L}_2 + \mathcal{L}_4 + \dots$$

$$W = \frac{1}{\sqrt{2}}(U + iV)\,, \qquad \overline{W} = \frac{1}{\sqrt{2}}(U - iV)\,, \qquad Z = \frac{1}{\sqrt{2}}(X + iY)\,, \qquad \overline{Z} = \frac{1}{\sqrt{2}}(X - iY)\,.$$

$$\tilde{\gamma}_1 = \frac{\alpha_2}{\sqrt{\alpha_1\alpha_2\alpha_3}}\,, \qquad \tilde{\gamma}_2 = \frac{\alpha_1 - \alpha_3}{\sqrt{\alpha_1\alpha_2\alpha_3}}\,, \qquad \tilde{\gamma}_3 = \frac{\alpha_1 - \alpha_2 + \alpha_3}{\sqrt{\alpha_1\alpha_2\alpha_3}}\,,$$

$$\gamma_1 = \frac{\beta_2}{\sqrt{\beta_1\beta_2\beta_3}}\,, \qquad \gamma_2 = \frac{\beta_1 - \beta_3}{\sqrt{\beta_1\beta_2\beta_3}}\,, \qquad \gamma_3 = \frac{\beta_1 - \beta_2 + \beta_3}{\sqrt{\beta_1\beta_2\beta_3}}\,.$$

$$\begin{aligned}\mathcal{L}_2 =& |\dot{W}|^2 - |\acute{W}|^2 - i\tilde{\gamma}_3\left(W\dot{\overline{W}} - \dot{W}\overline{W}\right) - (\tilde{\gamma}_1^2 - \tilde{\gamma}_2^2 - \tilde{\gamma}_3^3)|W|^2 + \tilde{\gamma}_2\tilde{\gamma}_3(W^2 + \overline{W}^2) \\ &+ |\dot{Z}|^2 - |\acute{Z}|^2 - i\gamma_3\left(Z\dot{\overline{Z}} - \dot{Z}\overline{Z}\right) - (\gamma_1^2 - \gamma_2^2 - \gamma_3^3)|Z|^2 + \gamma_2\gamma_3(Z^2 + \overline{Z}^2)\,,\end{aligned}$$

$$
\begin{aligned}
\mathcal{L}_4 = {} & \check{\mathcal{L}}_4(W,\overline{W}) + \hat{\mathcal{L}}_4(Z,\overline{Z}) + \tilde{\mathcal{L}}_4(W,\overline{W},Z,\overline{Z}) - \left(a - \frac{1}{2}\right) O_{T\bar{T}} \\
& + \frac{i(a-1)}{2}\,(\tilde{\gamma}_1 + \tilde{\gamma}_2)(W^2 - \overline{W}^2)\left(\dot{Z}\mathcal{E}_{\overline{Z}} + \dot{\overline{Z}}\mathcal{E}_Z\right) + \frac{ia}{2}\,(\gamma_1 + \gamma_2)(Z^2 - \overline{Z}^2)\left(\dot{W}\mathcal{E}_{\overline{W}} + \dot{\overline{W}}\mathcal{E}_W\right) \\
& + \frac{i(a-1)}{2}\,(\tilde{\gamma}_1 + \tilde{\gamma}_2)(W^2 - \overline{W}^2)\left(\dot{W}\mathcal{E}_{\bar{W}} + \dot{\bar{W}}\mathcal{E}_W\right) + \frac{ia}{2}\,(\gamma_1 + \gamma_2)(Z^2 - \overline{Z}^2)\left(\dot{Z}\mathcal{E}_{\overline{Z}} + \dot{\overline{Z}}\mathcal{E}_Z\right)
\end{aligned}
$$

$$
\begin{aligned}
\check{\mathcal{L}}_4 = {} & -2(\tilde{\gamma}_1^2 - \tilde{\gamma}_2^2 - \tilde{\gamma}_3^2)|W|^2|\dot{W}|^2 + \frac{i}{2}\,\xi_1|W|^2\left(W\dot{\overline{W}} - \dot{W}\overline{W}\right) + \xi_2|W|^4 \\
& - \frac{1}{2}\,\tilde{\gamma}_3\overline{W}\left((\tilde{\gamma}_1 + \tilde{\gamma}_3)\overline{W} - i\dot{\overline{W}}\right)\left(\dot{W}^2 - \acute{W}^2\right) - \frac{1}{2}\,\tilde{\gamma}_3 W\left((\tilde{\gamma}_1 + \tilde{\gamma}_3)W + i\dot{W}\right)\left(\dot{\overline{W}}^2 - \acute{\overline{W}}^2\right) \\
& - \tilde{\gamma}_2\tilde{\gamma}_3|W|^2\left(\dot{W}^2 - \acute{W}^2 + \dot{\overline{W}}^2 - \acute{\overline{W}}^2\right) + 2\tilde{\gamma}_2\tilde{\gamma}_3|\dot{W}|^2\left(W^2 + \overline{W}^2\right) - \frac{1}{3}\,\tilde{\gamma}_2\xi_3\left(W^3\overline{W} + W\overline{W}^3\right) \\
& - 2i\tilde{\gamma}_2\xi_6|W|^2\left(W\dot{W} - \overline{W}\dot{\overline{W}}\right) - \frac{1}{6}\,\tilde{\gamma}_2^2\xi_4(W^4 + \overline{W}^4) + \frac{1}{12}\,\xi_0\left(W^3\mathcal{E}_W + \overline{W}^3\mathcal{E}_{\overline{W}}\right) \\
& + \frac{1}{4}\,\xi_5\left(\mathcal{E}_W\overline{W}\left(W^2 - \overline{W}^2 - |W|^2\right) + \mathcal{E}_{\overline{W}}W\left(\overline{W}^2 - W^2 - |W|^2\right)\right) ,
\end{aligned}
$$

$$
\begin{aligned}
\xi_0 &= \tilde{\gamma}_1^2 + 2\tilde{\gamma}_2^2 + 3\tilde{\gamma}_1\tilde{\gamma}_2 + \tilde{\gamma}_1\tilde{\gamma}_3 , \\
\xi_1 &= 2\tilde{\gamma}_1^3 + 3\tilde{\gamma}_1^2\tilde{\gamma}_3 - 7\tilde{\gamma}_2^2\tilde{\gamma}_3 - 3\tilde{\gamma}_3^3 - 2\tilde{\gamma}_1(\tilde{\gamma}_2^2 + \tilde{\gamma}_3^2) \\
\xi_2 &= \tilde{\gamma}_2^4 - \tilde{\gamma}_1^3\tilde{\gamma}_3 + 4\tilde{\gamma}_2^2\tilde{\gamma}_3^2 + \tilde{\gamma}_3^4 - \tilde{\gamma}_1^2(\tilde{\gamma}_2^2 + \tilde{\gamma}_3^2) + \tilde{\gamma}_1\tilde{\gamma}_3(3\tilde{\gamma}_2^2 + \tilde{\gamma}_3^2) \\
\xi_3 &= 2\tilde{\gamma}_1^3 + 3\tilde{\gamma}_1^2\tilde{\gamma}_3 - 7\tilde{\gamma}_2^2\tilde{\gamma}_3 - 2\tilde{\gamma}_1\tilde{\gamma}_2^2 - 5\tilde{\gamma}_1\tilde{\gamma}_3^2 - 6\tilde{\gamma}_3^3 , \\
\xi_4 &= \tilde{\gamma}_1^2 - \tilde{\gamma}_2^2 - 2\tilde{\gamma}_1\tilde{\gamma}_3 - 7\tilde{\gamma}_3^2 , \\
\xi_5 &= \tilde{\gamma}_1(\tilde{\gamma}_1 + \tilde{\gamma}_2 + \tilde{\gamma}_3) , \\
\xi_6 &= \tilde{\gamma}_1^2 - \tilde{\gamma}_2^2 - 2\tilde{\gamma}_1\tilde{\gamma}_3 - 4\tilde{\gamma}_3^2 .
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\mathcal{L}}_4 = &\left((\gamma_1^2-\gamma_2^2-\gamma_3^2)|Z|^2-\gamma_2\gamma_3(Z^2+\overline{Z}^2)\right)\left(|\dot{W}|^2+|\acute{W}|^2\right)\\
&-\left((\tilde{\gamma}_1^2-\tilde{\gamma}_2^2-\tilde{\gamma}_3^2)|W|^2-\tilde{\gamma}_2\tilde{\gamma}_3(W^2+\overline{W}^2)\right)\left(|\dot{Z}|^2+|\acute{Z}|^2\right)\\
&+\frac{i}{2}\,\gamma_3\left((\tilde{\gamma}_1^2-\tilde{\gamma}_2^2-\tilde{\gamma}_3^2)|W|^2-\tilde{\gamma}_2\tilde{\gamma}_3(W^2+\overline{W}^2)\right)(Z\dot{\overline{Z}}-\dot{Z}\overline{Z})\\
&-\frac{i}{2}\,\tilde{\gamma}_3\left((\gamma_1^2-\gamma_2^2-\gamma_3^2)|Z|^2-\gamma_2\gamma_3(Z^2+\overline{Z}^2)\right)(W\dot{\overline{W}}-\dot{W}\overline{W})\\
&+\frac{i}{2}\,\gamma_3\left(Z\dot{\overline{Z}}-\dot{Z}\overline{Z}\right)\left(\dot{W}\dot{\overline{W}}+\acute{W}\acute{\overline{W}}\right)-\frac{i}{2}\,\gamma_3\left(Z\acute{\overline{Z}}-\acute{Z}\overline{Z}\right)\left(\dot{W}\acute{\overline{W}}+\acute{W}\dot{\overline{W}}\right)\\
&-\frac{i}{2}\,\tilde{\gamma}_3\left(W\dot{\overline{W}}-\dot{W}\overline{W}\right)\left(\dot{Z}\dot{\overline{Z}}+\acute{Z}\acute{\overline{Z}}\right)-\frac{i}{2}\,\tilde{\gamma}_3\left(W\acute{\overline{W}}-\acute{W}\overline{W}\right)\left(\dot{Z}\acute{\overline{Z}}+\acute{Z}\dot{\overline{Z}}\right)
\end{aligned}
$$

$$
\hat{\mathcal{L}}_4 = -\check{\mathcal{L}}_4\Big|_{W\to Z,\tilde{\gamma}_i\to\gamma_i}.
$$

$\{\alpha_1,\alpha_2,\alpha_3\}$	$\{\tilde{\gamma}_1,\tilde{\gamma}_2,\tilde{\gamma}_3\}$	Deformation type
$\alpha_1\neq\alpha_2\neq\alpha_3$	$\tilde{\gamma}_1\neq\tilde{\gamma}_2\neq\tilde{\gamma}_3$	elliptic
$\alpha_1=\alpha_3\neq\alpha_2$	$\tilde{\gamma}_1\neq\tilde{\gamma}_3,\ \tilde{\gamma}_2=0$	trigonometric
$\alpha_1=\alpha_2=\alpha_3$	$\tilde{\gamma}_1=\tilde{\gamma}_3,\ \tilde{\gamma}_2=0$	rational (undeformed)

$$
\begin{aligned}
\mathcal{T}\big|a^\dagger_{\mu_1}(p_1)a^\dagger_{\mu_2}(p_2)\big\rangle &= (-2\mathcal{A}^{aa}_{\mu_1\mu_2} - \mathcal{B}^{aa}_{\mu_1\mu_2} - \mathcal{D}^{aa}_{\mu_1\mu_2})\big|a^\dagger_{\mu_1}(p_1)a^\dagger_{\mu_2}(p_2)\big\rangle\ , \\
\mathcal{T}\big|b^\dagger_{\mu_1}(p_1)b^\dagger_{\mu_2}(p_2)\big\rangle &= (+2\mathcal{A}^{bb}_{\mu_1\mu_2} + \mathcal{B}^{bb}_{\mu_1\mu_2} - \mathcal{D}^{bb}_{\mu_1\mu_2})\big|b^\dagger_{\mu_1}(p_1)b^\dagger_{\mu_2}(p_2)\big\rangle\ , \\
\mathcal{T}\big|a^\dagger_{\mu_1}(p_1)b^\dagger_{\mu_2}(p_2)\big\rangle &= (+2\mathcal{G}^{ab}_{\mu_1\mu_2} - \mathcal{D}^{ab}_{\mu_1\mu_2})\big|a^\dagger_{\mu_1}(p_1)b^\dagger_{\mu_2}(p_2)\big\rangle\ , \\
\mathcal{T}\big|b^\dagger_{\mu_1}(p_1)a^\dagger_{\mu_2}(p_2)\big\rangle &= (-2\mathcal{G}^{ba}_{\mu_1\mu_2} - \mathcal{D}^{ba}_{\mu_1\mu_2})\big|b^\dagger_{\mu_1}(p_1)a^\dagger_{\mu_2}(p_2)\big\rangle\ ,
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{A}^{c_1c_2}_{\mu_1\mu_2} &= \frac{1}{4}\,\frac{p_1^2\omega^{c_2}_{\mu_2}(p_2)^2 + p_2^2\omega^{c_1}_{\mu_1}(p_1)^2 - 2p_1^2p_2^2}{p_1\omega^{c_2}_{\mu_2}(p_2) - p_2\omega^{c_1}_{\mu_1}(p_1)}\ , \\
\mathcal{G}^{c_1c_2}_{\mu_1\mu_2} &= \frac{1}{4}\left(p_1\omega^{c_2}_{\mu_2}(p_2) + p_2\omega^{c_1}_{\mu_1}(p_1)\right)\ , \\
\mathcal{D}^{c_1c_2}_{\mu_1\mu_2} &= \left(a - \frac{1}{2}\right)\left(p_1\omega^{c_2}_{\mu_2}(p_2) - p_2\omega^{c_1}_{\mu_1}(p_1)\right)\ ,
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{B}^{aa}_{\mu_1\mu_2} &= \mu_1\mu_2\left((\tilde\gamma_1-\tilde\gamma_3)^2-\tilde\gamma_2^2\right)\frac{(\sqrt{p_1^2+\tilde\gamma_1^2}-\mu_1\tilde\gamma_1)(\sqrt{p_2^2+\tilde\gamma_1^2}-\mu_2\tilde\gamma_1)}{p_1\omega^a_{\mu_2}(p_2)-p_2\omega^a_{\mu_1}(p_1)}\ , \\
\mathcal{B}^{bb}_{\mu_1\mu_2} &= \mu_1\mu_2\left((\gamma_1-\gamma_3)^2-\gamma_2^2\right)\frac{(\sqrt{p_1^2+\gamma_1^2}-\mu_1\gamma_1)(\sqrt{p_2^2+\gamma_1^2}-\mu_2\gamma_1)}{p_1\omega^b_{\mu_2}(p_2)-p_2\omega^b_{\mu_1}(p_1)}\ .
\end{aligned}
$$