Integrable long range spin chains with extended symmetry

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based on:

J. Lamers, V. Pasquier, D.S., arXiv:2004.13210
G.Ferrando, J. Lamers, F. Levkovich-Maslyuk, D.S., arXiv:2308.16865
A. Ben Moussa, J. Lamers, D.S. and A. Toufik, arXiv:2404.10164

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Long-range interacting integrable models

- The best studied integrable models are those with **nearest-neighbour** interaction, solved by **Algebraic Bethe Ansatz**
- Long range integrable deformations of Heisenberg-like models are important for various applications (*e.g.* AdS/CFT, 2d CFT, QHE, TTbar deformations, etc)
- The algebraic structure and general construction are not yet well understood (*e.g.* wrapping corrections, separation of variables), except for a class of models with **trigonometric interaction**

The trigonometric models I will present here have somme common characteristics:

- **no Bethe Ansatz**: the scattering phase is very simple
- **no bound states**
- **extended symmetry:** Yangian or quantum affine symmetry

Plan

- isotropic, XXX-like model: **spin-Calogero-Sutherland model** and **Haldane-Shastry model**

  - construction of the monodromy matrix; inhomogeneous XXX model with **dynamical inhomogeneities**; diagonalisation of the **transfer matrix** in terms of an effective spin chain **arXiv:2308.16865**

- anisotropic, XXZ-like model: Uglov-Lamers model as a **quantum-deformed** version of the **Haldane-Shastry** model **arXiv:2004.13210**

- **q=i limit** of q-Haldane-Shastry, or the long-range version of the XX model

  - definition and solution of the model in terms of **non-unitary fermions** as a long-range **gl(1|1) spin chain** **arXiv:2404.10164**

The isotropic Haldane-Shastry Hamiltonian

[Haldane, 88; Shastry, 88]

N su(2) spins 1/2 on a circle with periodic boundary conditions $z_j \longmapsto \omega^j = \mathrm{e}^{2\pi \mathrm{i} j/N}$

$$H_{\mathrm{HS}} = -\sum_{i \neq j} V(z_i, z_j)\, P_{ij}$$

$$V(z_i, z_j) = \frac{z_i z_j}{z_{ij} z_{ji}} = \frac{1}{4 \sin^2 \pi (i-j)/N}$$

$$P_{jk} = \frac{1}{2}\left(\sigma_j^a \sigma_k^a + 1\right)$$ spin permutation

- Simplified version of the XXX model (idealised magnons and spinons)
- Yangian symmetry and 2dCFT limit: **[Haldane, Ha, Talstra, Bernard, Pasquier, 92]**

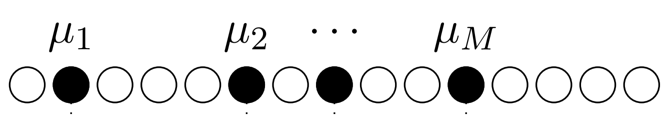
  algebraic structure: **[Bernard, Gaudin, Haldane, Pasquier, 93]**
- Yangian and spinon description of $su(2)_{k=1}$ CFT:

  [Bernard, Pasquier, D.S. 94; Bouwknegt, Ludwig, Schoutens, 94]

The spectrum of the Haldane-Shastry Hamiltonian

[Haldane, Ha, Talstra, Bernard, Pasquier, 92; Bernard, Gaudin, Haldane, Pasquier, 93]

- the spectrum is given in terms of a collection of integers $\{\mu_m\}$ called **motifs**



M magnon motif $\mu_{m+1} > \mu_m + 1\,, \qquad 1 \leqslant m < M$

$$E(\mu) - E_0 = \sum_{m=1}^{M} \varepsilon(\mu_m) = \sum_{m=1}^{M} \mu_m (N - \mu_m)$$

each motif comes with a high degeneracy and corresponds to a **Yangian representation**

- **Inozemtsev model** interpolates between XXX and Haldane-Shastry; the spin interaction is given by the Weierstrass function $\wp(z)$ with periods N and $i\pi/\kappa$

- **bound states** in the XXX model evolve into descendants of Haldane-Shastry highest weight states when $\kappa \to 0$

The solution of the Haldane Shastry Hamiltonian

- to solve the Haldane-Shastry model (and its cousins) it is useful to solve first the **spin Calogero-Sutherland** model **[Bernard, Gaudin, Haldane, Pasquier, 93]**

$$H_{B,F} = \sum_{j=1}^{N} (z_j \partial_j)^2 + \sum_{j \neq k} \beta(\beta \mp P_{jk}) \frac{z_j z_k}{(z_j - z_k)(z_k - z_j)}$$

- when $\beta \to \infty$ the positions of the particles **freeze** at their equilibrium positions and the Hamiltonian becomes that of Haldane-Shastry

[Polychronakos, 93; Lyashik, Reshetikhin, Sechin, 24]

$$z_j \longmapsto \omega^j = \mathrm{e}^{2\pi \mathrm{i} j/N}$$

- the model is solved using the **degenerate double affine Hecke algebra** (DDAHA)

$$\{z_1, z_2, \ldots, z_N\} \;, \;\text{ and } \{d_1, d_2, \ldots, d_N\}$$
$$[z_i, z_j] = [d_i, d_j] = 0$$

Dunkl operators:

$$d_j = z_j \partial_j + \beta \sum_{k>j} \frac{z_j}{z_j - z_k} K_{jk} - \beta \sum_{k<j} \frac{z_k}{z_k - z_j} K_{jk}$$

$$K_{ij}\, z_i = z_j\, K_{ij}$$

coordinate permutation

- the model is also solvable for higher spin symmetric su(p) representations

[Dorey, Tong, Turner, 16; Gaiotto, Rapčàk, Zhou, 23; Bourgine, Matsuo, 24]

The solution of the Haldane Shastry Hamiltonian

- the **spin Calogero-Sutherland** model:

$$H_{B,F} = \sum_{j=1}^{N} (z_j \partial_j)^2 + \sum_{j \neq k} \beta(\beta \mp P_{jk}) \frac{z_j z_k}{(z_j - z_k)(z_k - z_j)}$$

is diagonalised on **functions completely (anti)symmetric** by permutations of spins and coordinates

$$\Psi_{B,F} = \prod_{i<j} (z_i - z_j)^{\beta} \, \widetilde{\Psi}_{B,F} \qquad K_{ij} P_{ij} \widetilde{\Psi}_{B,F} = \pm \widetilde{\Psi}_{B,F}$$

$$\widetilde{\Psi}_{B,F} = \sum_{i_1 < i_2 < \ldots < i_M} (-1)^{\sigma_I} \, \Psi(z_{\{i_1, i_2, \ldots, i_M\}}, \bar{z}_{\{i_1, i_2, \ldots, i_M\}}) \quad \sigma^-_{i_1} \cdots \sigma^-_{i_M} |\uparrow \cdots \uparrow\rangle$$

partially (anti)symmetric in the two groups of variables

on these spaces of functions we can define a projection $\pi_{B,F}(\ldots K_{ij}) = \pm \pi_{B,F}(\ldots P_{ij})$

- **conserved quantities** in terms of Dunkl operators $H_{B,F} = \pi_{B,F} \left(\sum_{j=1}^{N} d_j^2 \right)$

The algebraic structure of the Calogero-Sutherland Hamiltonian

- build a monodromy matrix with Dunkl operators as **dynamical inhomogeneities**

$$T_a(u) \equiv \pi_F(\widehat{T}_a(u)) \;, \qquad \widehat{T}_a(u) = \prod_{j=1}^{N} \left(1 + \frac{\beta P_{ja}}{u + d_j}\right) \qquad \pi_F(\ldots K_{ij}) = -\pi_F(\ldots P_{ij})$$

- the integrals of motion of the model are generated by the **quantum determinant**

$$\mathrm{qDet}\; T_a(u) = \pi_F \left(\prod_{i=1}^{N} \frac{u + d_i + \beta}{u + d_i} \right) \qquad H_k = \pi_F \left(\sum_{i=1}^{N} d_i^k \right)$$

and they commute with the elements of $T_a(u)$ **(Yangian symmetry)**

- the **eigenvalues** of Dunkl operators are known from the theory of Macdonald (Jack) polynomials

$$\delta_j = k_j + \frac{\beta}{2}(N - 2j + 1)$$

with $k_1, k_2, \ldots, k_N$ integers such that $k_1 \geq k_2 \geq \ldots \geq k_N$

Extra integrals of motion of the Calogero-Sutherland model

[Ferrando, Lamers, Levkovich-Maslyuk, D.S., 23] **[Uglov, 95]**

- the twisted trace $t_\kappa(u) = \kappa A(u) + \kappa^{-1} D(u)$ commutes with the integrals of motion H_k

- it commutes with the quantum determinant so **it can be diagonalised** inside each of the Yangian multiplets (labelled by the motifs)

- the Yangian multiplets are determined by the eigenvalues of the Dunkl operators (dynamical inhomogeneities)

$$\delta_j = k_j + \frac{\beta}{2}(N - 2j + 1) \qquad \delta_j = \mathrm{i}\beta\theta_j$$

- if $k_j = k_{j+1} \quad \Rightarrow \quad \theta_{j+1} - \theta_j = i$ the Yangian representation is **reducible but indecomposable** (block triangular structure)

- the invariant component corresponds to spins at the sites *j* and *j+1* **fusing into a singlet**

⟶ effective **reduced length** of the spin chain $N \to N-2$

The hybrid Calogero-Sutherland model

- the spectrum of the effective model is given by a set of Bethe Ansatz equations for the spin chain of reduced length

$$\prod_{l(\neq j_I, j_I+1)}^{N} \frac{u_m - \theta_l + \mathrm{i}/2}{u_m - \theta_l - \mathrm{i}/2} = \kappa^2 \prod_{n(\neq m)}^{M'} \frac{u_m - u_n + \mathrm{i}}{u_m - u_n - \mathrm{i}}\,, \qquad j_I = \{j_1 \ldots, j_M\} \qquad N_{\text{eff}} = N - 2M$$

with $$\theta_l = -\mathrm{i}\left(\frac{k_l}{\beta} + \frac{1}{2}(N - 2l + 1)\right)$$

- the multiplet is characterised by the positions of the repeated *k*'s, $j_I = \{j_1 \ldots, j_M\}$

- the **reference state** $|\Omega\rangle$ is replaced with the **Yangian highest weight** state $|k_1, k_2, \ldots, k_N\rangle$

example for N=2 highest weight :

$$k_1 > k_2\,, \qquad |k_1, k_2\rangle = (z_1 - z_2)\, P^{\beta}_{k_1-1,k_2}(z_1, z_2)\, |\uparrow\uparrow\rangle$$

$$k_1 = k_2\,, \qquad |k_1, k_2\rangle = (z_1 z_2)^{k_1}\, (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$$

- the Bethe states are built using the B operator $B(u_1)\ldots B(u_{M'})\,|k_1, k_2, \ldots, k_N\rangle$

The Haldane-Shastry limit

- In the limit $\beta \to \infty$ we get back a Haldane-Shastry-like spin chain, $z_j \longmapsto \omega^j = \mathrm{e}^{2\pi \mathrm{i} j/N}$

  a (new) hamiltonian:

$$t_3 = \frac{1}{2} \sum_{i<j<k} [P_{ij}\, P_{jk} + P_{jk}\, P_{ij}]$$
$$+ \frac{\mathrm{i}}{2} \sum_{i<j<k} \left[\cot\left(\frac{\pi(i-j)}{N}\right) + \cot\left(\frac{\pi(j-k)}{N}\right) + \cot\left(\frac{\pi(k-i)}{N}\right) \right] (P_{ij}\, P_{jk} - P_{jk}\, P_{ij})$$

- effective BAEs: $$\prod_{l(\neq j_I, j_I+1)}^{N} \frac{u_m - \theta_l + \mathrm{i}/2}{u_m - \theta_l - \mathrm{i}/2} = \prod_{n(\neq m)}^{M'} \frac{u_m - u_n + \mathrm{i}}{u_m - u_n - \mathrm{i}}\,, \qquad j_I = \{j_1 \ldots, j_M\}$$

- inhomogeneities: $$\theta_l = -\frac{\mathrm{i}}{2}\,(N - 2l + 1)$$

- each pair of repeated *k*’s corresponds to a reversed spin (magnon)

- the reference states are the Yangian highest weight states

$$|k_1, k_2, \ldots, k_N\rangle_M = \sum_{i_1 < i_2 < \ldots < l_M} \prod_{m<n} (z_{i_m} - z_{i_n})^2\, P_\lambda^\beta(z_{i_1}, \ldots, z_{i_M})\, |i_1, i_2, \ldots, i_M\rangle\rangle$$

- again, the Bethe states are built using the B operator $B(u_1) \ldots B(u_{M'})\, |k_1, k_2, \ldots, k_N\rangle_M$

⟶ a total of *M*+*M’* magnons

Conclusions and open questions (part 1)

- new integrable long-range spin - with dynamical inhomogeneities

- solvable by (effective) Bethe Ansatz

- the twist κ interpolates between the Gelfand-Tsetlin bases for $\kappa \to 0, \infty$ and the isotropic states for $\kappa = 1$

- the antiferromagnetic vacuum is the same as for Haldane-Shastry

- what are the excitations?

- how do the generic solutions of the BAE look like?

The q-Haldane-Shastry Hamiltonian (Uglov-Lamers)

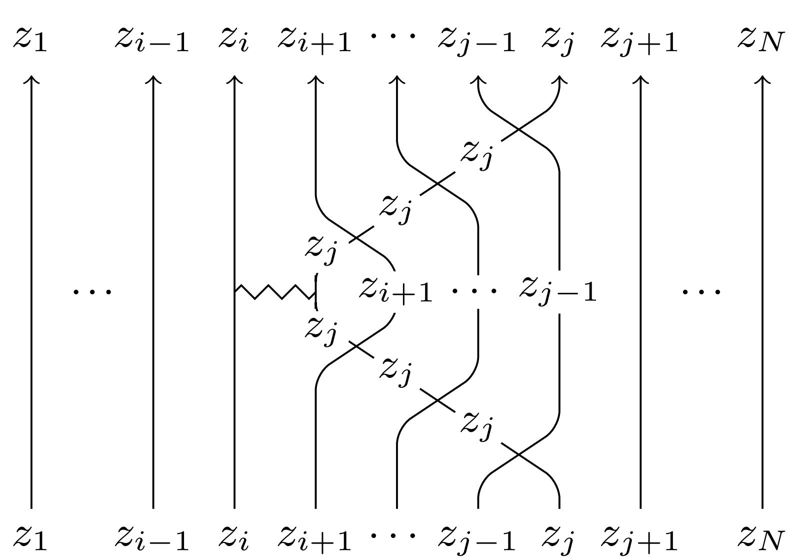
[Bernard, Gaudin, Haldane, Pasquier, 93; Uglov 95; Lamers 18; Lamers, Pasquier, D.S., 22]

The XXZ model can also be deformed to accommodate for long-range interaction, at the price of introducing **multi-spin interaction**

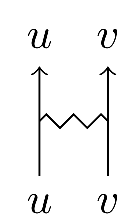
$$\mathrm{H}^{\mathrm{L}} = \frac{[N]_{\mathrm{q}}}{N} \sum_{i<j}^{N} \mathrm{V}(i-j)\, \mathrm{S}^{\mathrm{L}}_{[i,j]} \qquad [N] := \frac{\mathsf{q}^{N} - \mathsf{q}^{-N}}{\mathsf{q} - \mathsf{q}^{-1}}$$

$$\mathrm{V}(k) \equiv \frac{1}{(\mathrm{q}\,\omega^{k} - \mathrm{q}^{-1})(\mathrm{q}\,\omega^{-k} - \mathrm{q}^{-1})} = \frac{1}{4\,\sin(\pi k/N + \eta)\,\sin(\pi k/N - \eta)} \qquad \mathrm{q} = \mathrm{e}^{\mathrm{i}\eta}$$

$S^{\mathrm{L}}_{[i,j]} :=$

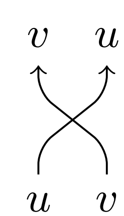


[Lamers 18]



$$:= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \mathrm{q}^{-1} & -1 & 0 \\ 0 & -1 & \mathrm{q} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = e_i$$

q-antisymmetriser (Temperley-Lieb)



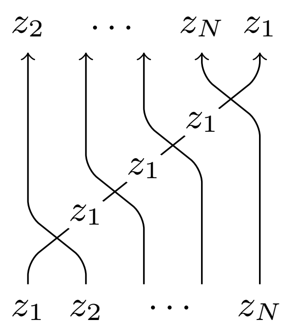
$$:= \check{R}(u/v)$$

$$\check{\mathrm{R}}_{k,k+1}(u) = 1 - f(u)\, e_k\,, \qquad f(u) = \frac{u-1}{\mathrm{q}\,u - \mathrm{q}^{-1}}$$

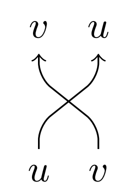
The Uglov-Lamers Hamiltonian

Several **new features** compared to the case q=1:

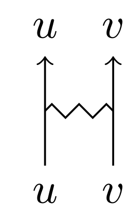
- the model is not translationally invariant, but there is a **q-translation operator**, G

$\mathrm{G} =$ 

$$\mathrm{G}^N = 1$$



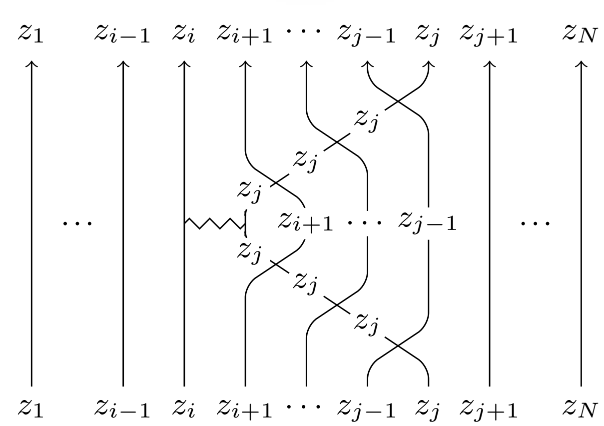
$:= \check{R}(u/v)$

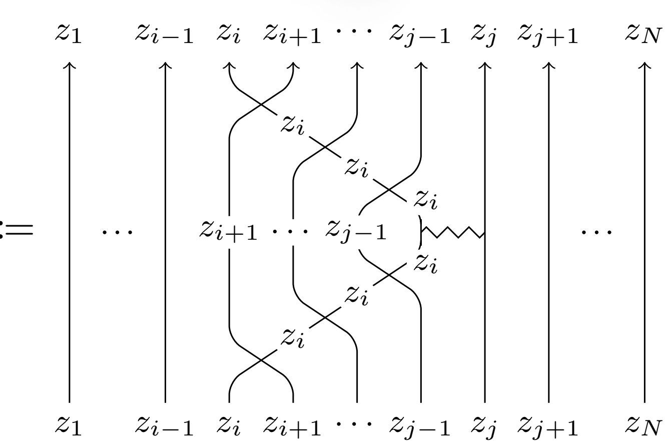


$= e_i$

- there exists another Hamiltonian with the **opposite "chirality"**

$$\mathrm{H}^{\mathrm{L}} = \frac{[N]_{\mathrm{q}}}{N} \sum_{i<j}^{N} \mathrm{V}(i-j)\, \mathrm{S}^{\mathrm{L}}_{[i,j]} \qquad\qquad \mathrm{H}^{\mathrm{R}} = \frac{[N]_{\mathrm{q}}}{N} \sum_{i<j}^{N} \mathrm{V}(i-j)\, \mathrm{S}^{\mathrm{R}}_{[i,j]}$$

$S^{\mathrm{L}}_{[i,j]} :=$ 

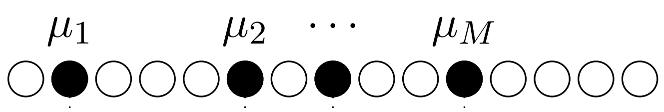
$S^{\mathrm{R}}_{[i,j]} :=$ 

.

$$[\mathrm{H}^{\mathrm{L}}, \mathrm{H}^{\mathrm{R}}] = [\mathrm{H}^{\mathrm{L}}, \mathrm{G}] = [\mathrm{G}, \mathrm{H}^{\mathrm{R}}]$$

The Uglov-Lamers Hamiltonian

- Both Hamiltonians can be diagonalised simultaneously and the spectrum can be written in terms of **motifs**, with eigenvalues (not real, for |q|=1)
- The Yangian symmetry gets deformed to **quantum affine symmetry**



M magnon motif $\mu_{m+1} > \mu_m + 1\,, \qquad 1 \leqslant m < M$

$$\varepsilon^{\mathrm{L,R}}(\mu) = \sum_{m=1}^{M} \epsilon^{\mathrm{L,R}}(\mu_m)$$

one-magnon dispersion relations:

$$\varepsilon^{\mathrm{L}}(n) = \frac{1}{\mathrm{q}-\mathrm{q}^{-1}}\left(\mathrm{q}^{N-n}[n] - \frac{n}{N}[N]\right)\,, \qquad \varepsilon^{\mathrm{R}}(n) = \frac{-1}{\mathrm{q}-\mathrm{q}^{-1}}\left(\mathrm{q}^{n-N}[n] - \frac{n}{N}[N]\right)$$

$\mathrm{H} = \frac{1}{2}\left(\mathrm{H}^{\mathrm{L}} + \mathrm{H}^{\mathrm{R}}\right)$ has real spectrum both for q real and $|\mathrm{q}| = 1$

$$\varepsilon(n) = \frac{1}{2}\left(\varepsilon^{\mathrm{L}}(n) + \varepsilon^{\mathrm{R}}(n)\right) = \frac{1}{2}[n][N-n]$$

q-number generalisation of HS spectrum

q=i, non-unitary fermions and gl(1|1)

- consider a 1-dimensional lattice with N sites and the **fermionic degrees of freedom**

$$\{f_i, f_j^+\} = (-1)^i\, \delta_{ij}\,, \quad \{f_i, f_j\} = \{f_i^+, f_j^+\} = 0$$

$$f_j = (-\mathrm{i})^j\, c_j\,, \; f_j^+ = (-\mathrm{i})^j\, c_j^\dagger.$$

Jordan-Wigner fermions

- they generate a **global gl(1|1) algebra** with (anti-)commutation relations

$$[\mathrm{N}, \mathrm{F}_1] = -\mathrm{F}_1\,, \quad [\mathrm{N}, \mathrm{F}_1^+] = \mathrm{F}_1^+\,, \quad \{\mathrm{F}_1, \mathrm{F}_1^+\} = \mathrm{E}$$

$$\mathrm{F}_1 = \sum_{i=1}^{N} f_i\,, \quad \mathrm{F}_1^+ = \sum_{i=1}^{N} f_i^+\,, \qquad \mathrm{N} = \sum_{i=1}^{N} (-1)^i\, f_i^+ f_i\,, \quad \mathrm{E} = \sum_{i=1}^{N} (-1)^i$$

difference between N even and odd !

- **gl(1|1)** has simple (anti-)commutation relations with the **two-site operators**

$$g_i \equiv f_i + f_{i+1}\,, \quad g_i^+ = f_i^+ + f_{i+1}^+\,, \qquad 1 \leqslant i < N$$

Non-unitary fermions and Temperley-Lieb

- the **two-site operators** $g_i \equiv f_i + f_{i+1}\,, \quad g_i^+ = f_i^+ + f_{i+1}^+\,, \qquad 1 \leqslant i < N$

can be used to generate the free-fermion **Temperley-Lieb** algebra

$$e_i^2 = 0\,, \quad e_i\, e_{i\pm 1}\, e_i = e_i\,, \quad [e_i, e_j] = 0 \text{ if } |i-j| > 1 \qquad e_i \equiv g_i^+ g_i$$

- **TL** generators commute with the global gl(1|1) and with $\mathrm{F}_2 = \sum_{i<j}^{N} f_i\, f_j\,, \quad \mathrm{F}_2^+ = \sum_{i<j}^{N} f_i^+\, f_j^+$

[Gainutdinov, Read, Saleur, 11]

$U_{\mathrm{q}}\,\mathfrak{sl}(2)|_{\mathrm{q}=\mathrm{i}}$

The Hamiltonians at N odd

- the interaction can be defined in terms of **commutators of TL generators**

$$e_{[i,j]} \equiv [[\cdots[e_i, e_{i+1}], \cdots], e_j] \qquad\qquad e_{[i,i]} \equiv e_i$$

- the two **chiral Hamiltonians** $\mathrm{H}^{\mathrm{L,R}} = \frac{\mathrm{i}}{2} \sum_{1 \leqslant i \leqslant j < N} h^{\mathrm{L,R}}_{ij}\, e_{[i,j]}$

with explicit coefficients $h^{\mathrm{R}}_{ij} = (-1)^{j-i}\, h^{\mathrm{L}}_{N-j,N-i} = -h^{\mathrm{L}}_{ij} \longrightarrow \mathrm{H}^{\mathrm{L}} = -\mathrm{H}^{\mathrm{R}}$

- **the non-chiral Hamiltonian** $\mathrm{H} = -\frac{1}{4N} \sum_{1 \leqslant i \leqslant j < k \leqslant l < N} \left(h^{\mathrm{L}}_{ij;kl} + h^{\mathrm{R}}_{ij;kl}\right) \left\{e_{[i,j]}, e_{[k,l]}\right\}$

- the **quasi-translation operator** $\mathrm{G} = (1 + t_{N-1}\, e_{N-1}) \cdots (1 + t_1\, e_1)\,, \quad \mathrm{G}^N = 1$ $\qquad t_k \equiv \tan\frac{\pi k}{N}$

$$\left[\mathrm{G}, \mathrm{H}^{\mathrm{L}}\right] = \left[\mathrm{G}, \mathrm{H}\right] = \left[\mathrm{H}^{\mathrm{L}}, \mathrm{H}\right] = 0$$

generalises the XXZ Hamiltonian $H^{\mathrm{open}}_{\mathrm{XXZ}} = -\sum_{j=1}^{N-1} e_j$ at $\Delta = \frac{\mathrm{q} + \mathrm{q}^{-1}}{2} = 0$

[Pasquier, Saleur, 90]

Discrete symmetries

The model has real spectrum in spite of being non-unitary, due to **PT symmetry**

- **Parity P**: $\mathrm{P}(f_i) = f_{N+1-i}$

$$\mathrm{P}(\mathrm{H}^{\mathrm{L}}) = -\mathrm{H}^{\mathrm{L}}, \quad \mathrm{P}(\mathrm{H}) = \mathrm{H}, \quad \mathrm{P}(\mathrm{G}) = \mathrm{G}^{-1}$$

- **Time reversal T** (anti-linear): $\mathrm{T}(e_i) = e_i$

$$\mathrm{T}(\mathrm{H}^{\mathrm{L}}) = -\mathrm{H}^{\mathrm{L}}, \quad \mathrm{T}(\mathrm{H}) = \mathrm{H}, \quad \mathrm{T}(\mathrm{G}) = \mathrm{G}$$

$$\mathrm{p} = -\mathrm{i} \log \mathrm{G}$$

- **Charge conjugation C** (anti-linear):

$$\mathrm{C}(\mathrm{H}^{\mathrm{L}}) = -\mathrm{H}^{\mathrm{L}}, \quad \mathrm{C}(\mathrm{H}) = \mathrm{H}, \quad \mathrm{C}(\mathrm{G}) = \mathrm{G}$$

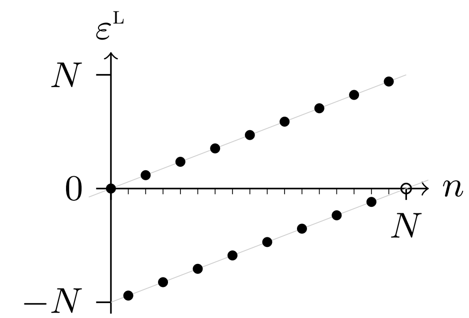
The spectrum for N odd

The spectrum becomes **very simple** at q=i and **N odd**

- quasi-momentum $\quad \mathrm{p} = -\mathrm{i} \log \mathrm{G} \quad\quad p = \frac{2\pi}{N} \sum_m \mu_m \bmod 2\pi$

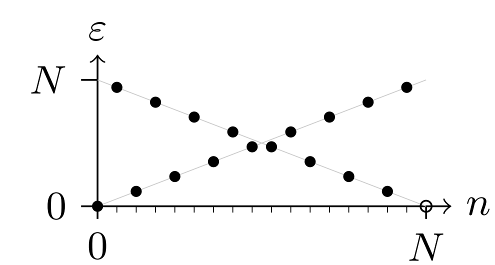
- chiral Hamiltonian

$$E^{\mathrm{L}}_{\{\mu_m\}} = \sum_{m=1}^{M} \varepsilon^{\mathrm{L}}_{\mu_m} \qquad \varepsilon^{\mathrm{L}}_n = \begin{cases} n\,, & n \text{ even}\,, \\ n - N\,, & n \text{ odd}\,, \end{cases}$$



- non-chiral Hamiltonian

$$E_{\{\mu_m\}} = \sum_{m=1}^{M} \varepsilon_{\mu_m} \qquad \varepsilon_n = |\varepsilon^{\mathrm{L}}_n|$$



How to solve the model for N odd

We want to solve it in terms of (non-unitary) fermions ⟶ **use quasi-translations**

- start with the first site and **translate the fermions** via

$$\Phi_i \equiv \mathrm{G}^{1-i}\, f_1\, \mathrm{G}^{i-1}\,, \qquad \Phi_i^+ \equiv \mathrm{G}^{1-i}\, f_1^+\, \mathrm{G}^{i-1}$$

- the transformation is **periodic** due to $\mathrm{G}^N = 1$

$$\Phi_{i+N} = \Phi_i\,, \qquad \Phi_{i+N}^+ = \Phi_i^+$$

- the price to pay is that the commutation relations are **non-local** (but **translationally invariant**)

$$\{\Phi_i, \Phi_j^+\} = -(1 + t_{j-i})\,, \quad \{\Phi_i, \Phi_j\} = \{\Phi_i^+, \Phi_j^+\} = 0$$

How to solve the model for N odd

- next we use the **Fourier modes** of the quasi-translated fermions

$$\tilde{\Psi}_n \equiv \frac{a_n}{N} \sum_{j=1}^{N} \mathrm{e}^{-2\mathrm{i}\pi n j/N}\, \Phi_j\,, \quad \tilde{\Psi}_n^+ \equiv \frac{a_n}{N} \sum_{j=1}^{N} \mathrm{e}^{2\mathrm{i}\pi n j/N}\, \Phi_j^+$$

$$a_0 \equiv \mathrm{i} \text{ and } a_n \equiv \mathrm{i}^{n+1/2} \text{ else}$$

- to get canonical commutation relations we **rescaled** the Fourier modes

$$\{\tilde{\Psi}_n, \tilde{\Psi}_m^+\} = \delta_{nm}\,, \quad \{\tilde{\Psi}_n, \tilde{\Psi}_m\} = \{\tilde{\Psi}_n^+, \tilde{\Psi}_m^+\} = 0$$

- the zero modes are **generators of the gl(1|1) algebra**

$$\frac{1}{a_0}\,\tilde{\Psi}_0 = \sum_{i=1}^{N} f_i = \mathrm{F}_1\,, \quad \frac{1}{a_0}\,\tilde{\Psi}_0^+ = \sum_{i=1}^{N} f_i^+ = \mathrm{F}_1^+$$

- the other modes are **linear combinations of the two-site operators**

$$\frac{1}{a_n}\,\tilde{\Psi}_n = \sum_{i=1}^{N-1} M_{ni}\, g_i\,, \quad \frac{1}{a_n}\,\tilde{\Psi}_n^+ = \sum_{i=1}^{N-1} \bar{M}_{ni}\, g_i^+$$

$$g_i \equiv f_i + f_{i+1}\,, \quad g_i^+ = f_i^+ + f_{i+1}^+$$

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- in these variables **the chiral Hamiltonian** becomes purely quadratic

$$\mathrm{H}^{\mathrm{L}} = \sum_{n=1}^{N-1} \varepsilon_n^{\mathrm{L}} \, \tilde{\Psi}_n^+ \, \tilde{\Psi}_n$$

- it can be diagonalised on the Fourier Fock space spanned by $|n_1, \ldots, n_M\rangle \equiv \tilde{\Psi}_{n_1}^+ \ldots \tilde{\Psi}_{n_M}^+ |\varnothing\rangle$

$$0 \leqslant n_1 < \cdots < n_M < N$$

- compatibility with the motif rule thanks to $\varepsilon_n^{\mathrm{L}} + \varepsilon_{n+1}^{\mathrm{L}} = \varepsilon_{2n+1 \bmod N}^{\mathrm{L}}$

- the **non-chiral Hamiltonian** is quartic

$$\mathrm{H} = \sum_{n=1}^{N-1} \varepsilon_n \, \tilde{\Psi}_n^+ \, \tilde{\Psi}_n + \sum_{\substack{1 \leqslant m < n < N \\ 1 \leqslant r < s < N}} \tilde{V}_{mn;rs} \, \tilde{\Psi}_m^+ \, \tilde{\Psi}_n^+ \, \tilde{\Psi}_r \, \tilde{\Psi}_s$$

$$\tilde{V}_{mn;m+k,n-k} = (-1)^{k+1} 4 \, \delta_{m \, \mathrm{odd}}$$

$$0 \leqslant 2k < n-m$$

eigenvalues

statistical repulsion

selection rule: $\varepsilon_m^{\mathrm{L}} + \varepsilon_n^{\mathrm{L}} = \varepsilon_r^{\mathrm{L}} + \varepsilon_s^{\mathrm{L}}$

N even

- for **N even**, $N = 2L$ there are **singularities** (poles) in the coefficients of the conserved quantities, which can be regularised as

$$t_k \equiv \tan\frac{\pi k}{N}, \quad k \neq L \quad \text{and} \quad t_L = t_L(\alpha) \equiv \tan\frac{\pi(L+\alpha)}{N} = \cot\frac{\pi\alpha}{N}$$

- the **residues of these poles** give a collection of conserved quantities

$$\left[\mathscr{G}, \mathscr{H}^C\right] = \left[\mathscr{G}, \mathscr{H}\right] = \left[\mathscr{H}^C, \mathscr{H}\right] = 0$$

$$\mathscr{G} \equiv (1+t_{N-1}\,e_{N-1})\cdots(1+t_{L+1}\,e_{L+1})\,e_L\,(1+t_{L-1}\,e_{L-1})\cdots(1+t_1\,e_1)\,,$$
$$\bar{\mathscr{G}} \equiv (1-t_1\,e_1)\cdots(1-t_{L-1}\,e_{L-1})\,e_L\,(1-t_{L+1}\,e_{L+1})\cdots(1-t_{N-1}\,e_{N-1})\,.$$

$$\mathscr{G}\bar{\mathscr{G}} = \bar{\mathscr{G}}\mathscr{G} = 0$$

$$\mathscr{H} = \sum_{1\leqslant i\leqslant j<N} \mathrm{h}_{ij}\; e_{[i,j]}\,,$$
$$\mathscr{H}^C = \sum_{1\leqslant i\leqslant j<k\leqslant l<N} \mathrm{h}_{ij;kl}\left\{e_{[i,j]}, e_{[k,l]}\right\}$$

$$\mathrm{h}_{ij} = \lim_{\alpha\to 0}\frac{\pi^2\alpha^2}{2N^2}\left(h^{\mathrm{L}}_{ij}(\alpha) + h^{\mathrm{R}}_{ij}(\alpha)\right)$$
$$\mathrm{h}_{ij;kl} = \lim_{\alpha\to 0}\frac{\pi^2\alpha^2}{2N^2}\left(h^{\mathrm{L}}_{ij;kl}(\alpha) - h^{\mathrm{R}}_{ij;kl}(\alpha)\right)$$

the roles the chiral and non-chiral hamiltonians is interchanged

- the eigenvalues of these conserved quantities are all **zero** (with **Jordan blocks** of size up to L+1)

To do list

- Study the system for **even length:** spectrum identically zero; Jordan blocks
- Identify the **extended symmetry:** gl(1|1) Yangian?
- Interpret the **staggering** & the **linear dispersion relations** in the odd case
- **CFT limit:** gl(1|1) Kac-Moody algebra?
- **Free field** realisation and **vertex operators algebra**
- **Wave functions** in the fermionic representation & **Macdonald polynomials**
- **Other roots of unity**: q^3=1 and gl(2|1) symmetry