Integrable long range spin chains with extended symmetry

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based on: J. Lamers, V. Pasquier, D.S., arXiv:2004.13210 G.Ferrando, J. Lamers, F. Levkovich-Maslyuk, D.S., arXiv:2308.16865 A. Ben Moussa, J. Lamers, D.S. and A. Toufik, arXiv:2404.10164

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Long-range interacting integrable models

- The best studied integrable models are those with **nearest-neighbour** interaction, solved by **Algebraic Bethe Ansatz**
- Long range integrable deformations of Heisenberg-like models are important for various applications (*e.g.* AdS/CFT, 2d CFT, QHE, TTbar deformations, etc)
- The algebraic structure and general construction are not yet well understood (*e.g.* wrapping corrections, separation of variables), except for a class of models with **trigonometric interaction**

The trigonometric models I will present here have somme common characteristics:

- **no Bethe Ansatz**: the scattering phase is very simple
- **• no bound states**
- **extended symmetry:** Yangian or quantum affine symmetry

Plan

- isotropic, XXX-like model: **spin-Calogero-Sutherland model** and **Haldane-Shastry model**
	- construction of the monodromy matrix; inhomogeneous XXX model with **dynamical inhomogeneities**; diagonalisation of the **transfer matrix** in terms of an effective spin chain **arXiv:2308.16865**
- anisotropic, XXZ-like model: Uglov-Lamers model as a **quantum-deformed** version of the **Haldane-Shastry** model **arXiv:2004.13210**

• **q=i limit** of q-Haldane-Shastry, or the long-range version of the XX model

- definition and solution of the model in terms of **non-unitary fermions** as a longrange **gl(1|1) spin chain arXiv:2404.10164**

The isotropic Haldane-Shastry Hamiltonian *n n* is the property of the detailed of the details of the details

H(↵1*|*↵2*|*↵3) **[Haldane, 88; Shastry, 88]** 1 dane $88 \cdot$ Shastry 881 ^pdet(1 *^K*`) det(1 *^K*`+2)

 h ele with period *N* N su(2) spins 1/2 on a circle with periodic boundary conditions $z_j \mapsto \omega^j = e^{2\pi i j/N}$ *N* su(2) spins 1/2 on a circle with periodic boundary conditions $z_j \mapsto \omega^j = e^{2\pi i j/N}$ $p(x)$ ^{*k*} $\mu(z)$ r
Rođenja $\overline{\text{}}$ 1 $\overline{\text{}}$ $\overline{1}$ 2 on a cii r $\mathbf{P} \times \mathbf{W}$.
1⊦ *g periodic l*

$$
H_{\rm HS} = -\sum_{i \neq j} V(z_i, z_j) P_{ij}
$$

$$
V(z_i, z_j) = \frac{z_i z_j}{z_{ij} z_{ji}} = \frac{1}{4 \sin^2 \pi (i - j)/N}
$$
\n
$$
P_{jk} = \frac{1}{2} \left(\sigma_j^a \sigma_k^a + 1 \right)
$$
\nSpin

\npermutation

- XX model (idealised magnons and spinons) \sim $\frac{1}{3}$, with wave function $\frac{1}{3}$ • Simplified version of the XXX model (idealised magnons and spinons) *{z*1*, z*2*,...,z^N } ,* and *{d*1*, d*2*,...,d^N }*
- FT limit: [Haldane, Ha, Talstra, Ber · Yangian symmetry and 2dCFT limit: [Haldane, Ha, Talstra, Bernard, Pasquier, 92] Here *P*(*–*) algebraic structure: **[Bernard, Gaudin, Haldane, Pasquier, 93]** *Ki,i*+1 *d^k* = r, *d^k Ki,i*+1*, k* 6= *i, i* + 1 *, di*+1 *Ki,i*+1 *, k* = *i ,*
- $\lim_{\epsilon \to 0}$ of $\sin(2)_{1,-1}$ CFT \cdot $\frac{1}{2}$ $\frac{1}{2}$ • Yangian and spinon description of $su(2)_{k=1}$ CFT:

[Bernard, Pasquier, D.S. 94; Bouwknegt, Ludwig, Schoutens, 94]

The spectrum of the Haldane-Shastry Hamiltonian *N* The spectrum of the Haldane-Shastry Hamiltonian relates to the Yangian of gl(1*|*1). A detailed study of this (1.27a) *⁄^m* = *µM*≠*m*+1 ≠ 2 (*M* ≠ *m*)*,* 1 Æ *m* Æ *M* = *¸*(*⁄*) = *¸*(*µ*)*,* eracies (21). As we will see, these are already visible in *δ*|*z*|*,*¹ **b** m_{θ-} d**uc-5** *i***an** (*ω* → i∞)

[Haldane, Ha, Talstra, Bernard, Pasquier, 92; Bernard, Gaudin, Haldane, Pasquier, 93] Tolstre Rernard Descuier 02: Rernard Caudin Holdone Descuier 02¹ [Haldane, Ha, Talstra, Bernard, Pasquier, 92; Bernard, Gaudin, Haldane, Pasquier, 93] *"^M* := (*^M* [≠] ¹*, M* [≠] ²*, ···*) denotes the staircase partition of length *^M* [≠] ¹ and *^µ*⁺ is the *z* 02. Rornard

• the spectrum is given in terms of a collection of integers $\{\mu_m\}$ called **motifs** $\{m, \}$ called **motifs** *⁄* + 2 *"¸*(*µ*) = *µ*⁺ (1.27b) *,* parent model are known explicitly [14, 19–21]. Like for the distance function on Z*^L* := Z*/L*Z. See also the plots in Figures 1 and 2 below.

> μ_2 $\ddot{}$ \bullet *a ^j ^a ^k* + 1 *µ*¹ *µ*² *··· µ^M µ + 1 <i>, 1 6 m* + 1 *m* + 1 *m [⁄]*¯*^M [⁄]*¯*M*≠¹ M magnon motif *µm*+1 *> µ^m* + 1 *,* 1 6 *m<M.* (18) introduce a period *a* in a (suitable) function *f* via \mathcal{Q} \cdots μ_M

$$
E(\mu) - E_0 = \sum_{m=1}^{M} \varepsilon(\mu_m) = \sum_{m=1}^{M} \mu_m (N - \mu_m)
$$

each motif comes with a high degeneracy and corresponds to a Yangian representation *{µm}* ⁼ ^ÿ Figure 3. The correspondence (1.27) between a motif *µ* œ *M^N* of length *n* = α and β and given by *[⁄]^m* ⁼ *[⁄]*¯*^m* + 1, ¹ ^Æ *^m* ^Æ *^M*. Here *[⁄]*¯ characterises the extent by $\frac{1}{\sqrt{2}}$ $\frac{1}{\sqrt{2}}$ *m ^µ^m , E{µm}* ⁼ ^ÿ f comes with a high degeneracy and corresponds to a Yangian representation *N*!1 *P*
*P*₂ monds to a V_2 **zian rep** $\overline{3}$ *.*

- emtsev model interpolates betw *di*+1 *Ki,i*+1 *, k* = *i ,* ! 1 With this notation in place the (unnormalised) wave function of *|µ*Í is the following • Inozemtsev model interpolates between XXX and Haldane-Shastry; the spin interaction is (1.28) *µ*(1*, ···, M*) = ÈÈ1*, ···, M|µ*Í = ev*^Ê* ^Â *[⁄]*(*µ*)(*z*1*, ···, zM*)*.* **Inozemtsev model** interpolates between XXX and Hagiven by the Weierstrass function $\mathcal{P}(z)$ with periods *s*
<u>ketween XXX and Haldane-Shast</u> ω α *N* and $i\pi/$ *L* e Weierstrass function $\mathcal{P}(z)$ with periods *N* and $i\pi/\kappa$
- **bound states** in the XXX model evolve into descendants of Haldane-Shastry highest weight states when $\kappa \to 0$ (1.29) ^Â *[⁄]*(*z*1*, ···, zM*) := \int *n* \int *n* \int *y* \int *n* trigonometricanometricanometricanometricanometricanometricanometricanometricanometricanometricanometricanometr
Trigonometricanometricanometricanometricanometricanometricanometricanometricanometricanometricanometricanometr (*π/L*)² *e YYY mo* α *i zv*^{*i*} *x i nto* $\frac{1}{1}$ \textbf{R} XXX model evoly

The solution of the Haldane Shastry Hamiltonian *i*,j¹ + *i,j*¹ + *i,j*¹ **The solution of the Hall** ker(*T*sp *ⁱ ^T* pol *ⁱ*) >: *dⁱ Ki,i*+1 + *k* = *i* + 1 *.* ! 1

• to solve the Haldane-Shastry model (and its cousins) it is useful to solve first the spin Z *Calogero-Sutherland* model (*z^j zk*)(*z^k z^j*) \overline{y} \overline{y} = \overline{y} **[Bernard, Gaudin, Haldane, Pasquier, 93]** ! 0

$$
H_{B,F} = \sum_{j=1}^{N} (z_j \partial_j)^2 + \sum_{j \neq k} \beta(\beta \mp P_{jk}) \frac{z_j z_k}{(z_j - z_k)(z_k - z_j)}
$$

- of the particles **freeze** at their equilibrium positions and the of the particles **if eeze** at their equilibrium positions and the primitive *n*th root of units. For the particle $z_j \longmapsto \omega^j = e^{2\pi i j/N}$ • when $\beta \to \infty$ the positions of the particles freeze at their equilibrium positions and the Hamiltonian becomes that of Haldane-Shastry [Polychronakos, 93; Lyashik, Reshetikhin, Sechin, 24] $\frac{1}{2}$ e(*u*) = e0(*u*)+(*^q* 1) e(*u*) + *^O*(*^q* 1)² \overline{a} *d* \overline{b} *d* \overline{c} *d* \overline{c} X *zj k>j z^j z^k* freeze \mathbf{a} *z* \mathbf{b} *z^k z^j Kjk* 1, Sechi
- after evaluation, we can think of the *z^j* as the position of site *j* of the chain, viewed as • the model is solved using the degenerate double affine Hecke algebra (DDAHA) <u>ie double af</u> enerate double af

$$
\{z_1, z_2, \dots, z_N\}, \text{ and } \{d_1, d_2, \dots, d_N\}
$$

$$
[z_i, z_j] = [d_i, d_j] = 0
$$

 $d_j = z_j \partial_j + \beta \sum_{k>i} \frac{z_j}{z_j - z_k} K_{jk} - \beta \sum_{k$ *zj* $z_j - z_k$ $K_{jk}-\beta$ $d_i = z_i \partial_i + \beta \sum_{k} z_j^2$ *,* $K_i = \beta \sum_{k} z_k^2$ *,* Z_k *,*

*d*_{*i*}, *k* $\frac{1}{2}$ *ii*, $\frac{1}{3}$ *k* $\frac{1}{3}$

(see §1.1.3), with wave function [Hal91b,BGHP93]

 \sum

zk

Kjk

 $z_k - z_j$

k<j

Dunkl operators:

$$
K_{ij} z_i = z_j K_{ij}
$$

 $z_j \partial_j + \beta \sum_i \frac{1}{z_j - z_k} K_{jk} - \beta \sum_i \frac{1}{z_k - z_j} K_{jk}$ coordinate (106) *Ki,i*+1 *d^k* = *di*+1 *Ki,i*+1 *, k* = *i ,* permutation >< coordinate α *k* α *k* α *k* α

 \cdot higher spin symmetric su(p) representations *m<n* ! 1 ⇡*B,F* (*...Kij*) = *±*⇡*B,F* (*...Pij*) **[Dorey, Tong, Turner, 16; Gaiotto, Rapčàk, Zhou, 23; Bourgine, Matsuo, 24]**• the model is also solvable for higher spin symmetric su(p) representations \mathcal{C}'

 \sum

k>j

 \int

 \mathbb{R}

The solution of the Haldane Shastry Hamiltonian (⁺ *^Pjk*) *^zjz^k* !
|-
|- 000 | ! 1

 $\int (z_j \partial_j)^2 + \sum$ $3(A)$ *zj z^j z^k* • the spin Calogero-Sutherland model: $H_{B,F} = \sum$ *N j*=1 $(z_j\partial_j)^2 + \sum$ $j \neq k$ $\beta(\beta \mp P_{jk}) \frac{z_j z_k}{(z_j - z_k)(z_k)}$ \mathcal{N} $\frac{1}{2}$ *j*6=*k j* = *i* $\overrightarrow{j\neq k}$ *Kjk* (*z^j zk*)(*z^k z^j*) *zj* $H_{B,F} = \sum_{\cdot}$ $j=1$ *Kjk* $\epsilon_{\mu,F}=\sum^N(z_j\partial_j)^2\,\epsilon^2$ j $\frac{z_j z_k}{\beta(\beta + P_{jk})} \frac{z_j z_k}{(z_k - z_k)(z_k - z_k)}$ *^HB,F* ⁼ ^X (*zj*@*^j*) $\sum_{i=1}^{N}$ $\sum (\overline{z_i}\partial_i)^2 + \sum \beta(\beta\overline{z})$ X *zk*

is diagonalised on **functions completely (anti)symmetric** by permutations of spins and coordinates *<u><i>x*</u>*xxxxx<i>x******<i>x<i>x<i>x<i>x******<i>x******<i>x***** $=$ $\frac{1}{2}$ $\frac{1}{2}$ $\$ 11114 α **comple** *z{i*1*,i*2*,...,iM}* ⌘ *{zⁱ*¹ *, zⁱ*² *...zⁱ^M } x***i**1*tv y p*_c*in*i*u*_i*ality <i>x*_{*i*}^{*n*}</sup> *<i>x*_{*i*}^{*n*} *<i>x*_{*i*}^{*n*} *<i>x*_{*i*} *<i>n <i>x*_{*i*} *<i>n <i>n* *in*</sup> *<i>n* *i*^{*n*} *<i>n in*</sup> *<i>n* *i*^{*n*} *n*^{*n*} *i*^{*n*} *n*^{*n*} *n*^{*n*} *z<i>z***** *zk******<i>z zzz zzz zz zz z z z z z z z z z z z z*

$$
\Psi_{B,F} = \prod_{i < j} (z_i - z_j)^{\beta} \, \widetilde{\Psi}_{B,F} \qquad K_{ij} P_{ij} \widetilde{\Psi}_{B,F} = \pm \widetilde{\Psi}_{B,F}
$$

$$
\widetilde{\Psi}_{B,F} = \sum_{i_1 < i_2 < \ldots < i_M} (-1)^{\sigma_I} \Psi(z_{\{i_1, i_2, \ldots, i_M\}}, \bar{z}_{\{i_1, i_2, \ldots, i_M\}}) \quad \sigma_{i_1}^{-} \cdots \sigma_{i_M}^{-} \mid \uparrow \cdots \uparrow \rangle
$$

partially (anti)symmetric in the two groups of variables $\frac{1}{2}$ *S* O_I variables nartially (anti)symmetric in the two groups of variables *Partiary (and <i>Symmetric III the two groups of variables <u>z₁, <i>x*₁, *z*₁, *z*_{</u>}

on these spaces of functions we can define a projection $\pi_{B,F} (\dots K_{ij}) = \pm \pi_{B,F} (\dots P_{ij})$ $\therefore B, F \ (\dots \ 11)$ $\therefore B, F \ (\dots \ 11)$ ⇡*B,F* (*...Kij*) = *±*⇡*B,F* (*...Pij*) $\pi_{B,F}(\ldots K_{ij}) = \pm \pi_{B,F}(\ldots P_{ij})$ \mathcal{P} ¹ \mathcal{P} $\$ These spaces of functions we can define a projection. $\pi_{B,F}(\ldots \mathbf{A}_{ij}) = \pm \pi_{B,F}(\ldots \mathbf{P}_{ij})$ *KijPij* e *B,F* = *±* e *B,F*

^HB,F ⁼ ⇡*B,F* X con: *^ν* (*z*1*,* ···*, zM*) = # • **conserved quantities** in terms of Dunkl operators

$$
H_{B,F} = \pi_{B,F} \left(\sum_{j=1}^{N} d_j^2 \right)
$$

 $\frac{z_j z_k}{z_i - z_k(z)}$

 $(z_j - z_k)(z_k - z_j)$

 $(z_k)(z_k - z)$

 \mathbb{R}

 $z_j z_k$

 (z_j)

The algebraic structure of the Calogero-Sutherland \blacksquare **Hamiltonian** '(*p^j*) '(*pk*) *ⁱ* (108) cot *^p z*_{*i*}1 *id {i*^{*z*}} *die z*^{*i*}*Mx***i** *i*^{*m*} *p*^{*l*} *<i>m i*^{*m*} *i*^{*m*} $\bm{gero-Sutherland}$ **Executed State and Line** *p*(✓) = *m* sinh ✓

 \overline{a} <u>ly</u> namical inhe $$ *K*_{*i*} matrix with Dunk concretors as dynamical inhomogeneities moi 111011 p *Odromy matrix* $\ddot{}$ \sum ζ $\frac{1}{2}$ *.* • build a monodromy matrix with Dunkl operators as **dynamical inhomogeneities** $\sum_{i=1}^{n}$ $\rm d$ $\overline{}$ *L L* $\overline{}$ *L*

$$
T_a(u) \equiv \pi_F(\widehat{T}_a(u)), \qquad \widehat{T}_a(u) = \prod_{j=1}^N \left(1 + \frac{\beta P_{ja}}{u + d_j}\right) \qquad \qquad \pi_F(\dots K_{ij}) = -\pi_F(\dots P_{ij})
$$

• the integrals of motion of the model are generated by the **quantum determinant** *<u>p</u>*_{*x*} *a* ic gen *rated by the quantum determinal* on of the model are generated by the **quantum determinant** *l* of the moder are generated by the **quantum** \mathbf{a} *S*[*j,j*+1]

$$
q\text{Det } T_a(u) = \pi_F \left(\prod_{i=1}^N \frac{u + d_i + \beta}{u + d_i} \right) \qquad H_k = \pi_F \left(\sum_{i=1}^N d_i^k \right)
$$

and they commute with the elements of $T_a(u)$ (Yangian symmetry) the elements of $T_a(u)$ **(Yangian symme**) ney commute with the set \mathbf{v} e e ements of $T_a(u)$ (**Yangian symm**) $T_{\rm e}$ mmute with the elements of $T_{\rm e}$

• the eigenvalues of Dunkl operators are known from the theory of Macdonald (Jack)
nolynomials *e*
*il*c '(*p^j*) '(*pk*) *ⁱ* (108) polynomials *^d^j* ⁼ *^k^j* ⁺ $\delta \cdot = k$ ry of Macdona
2

$$
\delta_j = k_j + \frac{\beta}{2} (N - 2j + 1)
$$

 $\begin{array}{ccccccc} \mathbf{35} & \mathbf{1} & \$ \mathbf{r} cot *^p* k_1, k_2 *n>*0 k_N integers such that $k_1 \geq k_2 \geq \cdots \geq k_m$ with k_1, k_2, \ldots, k_N integers such that $k_1 \geq k_2 \geq \ldots \geq k_N$ $\frac{1}{2}$ $\frac{1}{2}$

Extra integrals of motion of the Calogero-Sutherland model *H* = *J* ~*i*~*i*+1 ⇠ *J i*=1 *Pi,i*+1 *q* ' *of motion of the Calogero-Su* \mathbf{L} r []] g n

 $Ferrand$ \mathbf{L} \mathbf{D} \mathbf{J} \mathbf{J} \mathbf{J} X **[Ferrando, Lamers, Levkovich-Maslyuk, D.S., 23]** $\frac{1}{5}$ $\mathbf{v}_\mathbf{y}$ *Earner* s, Lev *dk* ! \sim *L* 1 *Pi,i*+1 **[Uglov, 95]**

- the twisted trace $t_{\kappa}(u) = \kappa A(u) + \kappa^{-1} D(u)$ commutes with the integrals of motion H_k with the integration
- it commutes with the quantum determinant so **it can be diagonalised** inside each of the Yangian multiplets (labelled by the motifs) *A*(*u*) + *D*(*u*) *d^j* ! *i*✓*^j* be diagonalis *d^j* ⇠ *dj*+1 + *N contrary determinant so it can be diagonalised* d **h** t ¹
- $\frac{d}{dx}$ • the Yangian multiplets are determined by the eigenvalues of the Dunkl operators (dynamical inhomogeneities) *m*ical inhomogeneities) (dynamical inhomogeneities) ² (*^N* ²*^j* + 1)

minomogeneous

$$
\delta_j = k_j + \frac{\beta}{2} (N - 2j + 1) \qquad \delta_j = i \beta \theta_j
$$

- if $k_j = k_{j+1} \Rightarrow \theta_{j+1} \theta_j = i$ the Yangian representation is **reducible but indecomposable** (block triangular structure) $\theta_{j+1} - \theta_j = i$ the Yangian representation is reduction
- the invariant component corresponds to spins at the sites *j* and *j+1* **fusing into a singlet** μ ^{*y*} = μ μ *j* μ _{*j*} μ _{*j* μ </sup>} k ^{*j*} α ^{*j*} α ^{*j*} α ^{*j*} α ^{*j*} α ^{*j*} α ^{*j*} α *j* α ^{*j*} α *j* α *N* ! *N* 2

e effective **reduced length** of the spin chain $N \rightarrow N - 2$ \overline{y}

*d*_{*l*} \overline{f} **h**_{*j*} \overline{f} \overline{f} \overline{f} ybrid Cald **The hybrid Calogero-Sutherland model** *^d^j* ⁼ *^k^j* ⁺ ² (*^N* ²*^j* + 1) *d Caloger d*-5 *durier rand moder* 1logero-Suthe rland mode + cot ✓⇡(*^j ^k*) *^j* ⁼ *^k^j* ⁺

• the spectrum of the effective model is given by a set of Bethe Ansatz equations for the spin chain of reduced length *k*₁ *k*2 *k*² *k*¹ *k*² *... k^N Bethe Ansatz equations for* θ θ $\frac{1}{1}$ $\frac{1}{1}$ $\frac{1}{1}$ Y *u^m* ✓*^l* + i*/*2 *^u^m* ✓*^l* ⁱ*/*² ⁼ ² ^Y Y *u^m* ✓*^l* + i*/*2 *^u^m* ✓*^l* ⁱ*/*² ⁼ ^Y *n* checuve moder is given by a set of
 *u*ced length of the effective model is given by a set of Bethe Ansatz equations for the reduced length a set of Bethe Ansatz

$$
\prod_{l(\neq j_1,j_1+1)}^N \frac{u_m - \theta_l + i/2}{u_m - \theta_l - i/2} = \kappa^2 \prod_{n(\neq m)}^{M'} \frac{u_m - u_n + i}{u_m - u_n - i}, \qquad j_I = \{j_1 \dots, j_M\} \qquad N_{\text{eff}} = N - 2M
$$
\nwith

\n
$$
\theta_l = -i \left(\frac{k_l}{\beta} + \frac{1}{2} \left(N - 2l + 1 \right) \right)
$$

✓*^l* = *i k* is characterised by the p $j_l = \{j_1, \ldots, j_M\}$ ϵ aled κ *l*(6=*j,j*+1) *n*(6=*m*) $\partial f \times S$, $j_I = \{j_1, ..., j_M\}$ α rised by the positic ¹
d by the positions • the multiplet is characterised by the positions of the repeated k 's, $\frac{1}{2}$ $j_I = \{j_1, \ldots, j_M\}$

 \overline{X}

• the **reference state** $|\Omega\rangle$ is replaced with the **Yangian highest weight** state $|k_1, k_2, \dots, k_N\rangle$ $\left| \right|$ = $\left| \right|$ $\left|$ weight state | *|k*₁, *k*₂, *k*₁, *k P* aced with the **Yan** \mathbf{h} ighest weigh

example for N=2 highest weight :
$$
k_1 > k_2
$$
, $|k_1, k_2\rangle = (z_1 - z_2) P_{k_1-1, k_2}^{\beta}(z_1, z_2) | \uparrow \uparrow \rangle$
\n $k_1 = k_2$, $|k_1, k_2\rangle = (z_1 z_2)^{k_1} (|\uparrow \downarrow \rangle - |\downarrow \uparrow \rangle)$

 \mathcal{L} • the Bethe states are built using the B operator $B(u_1) \dots B(u_{M'}) \mid k_1, k_2, \dots, k_N$ $P(x) = P(x) - P(x) - \frac{1}{2}h$

The Haldane-Shastry limit he Ha *z^k z^j* a _{ne}-Shastry limit *^d^j* ⁼ *^k^j* ⁺ *B*(*u*1)*...B*(*u^M*0) *|k*1*, k*2*,...,k^N* i *B*(*u*1)*...B*(*u^M*0) *|k*1*, k*2*,...,k^N* i*^M frv* lii *d^j dj*+1 = *k^j kj*+1 + (1.1) *^H*hs = ev*^Ê ^H*Âhs *, ^H*Âhs ⁼ [≠]^ÿ The overall sign ensures that (1.1) is positive: (≠)*H*hs is (anti)ferromagnetic. Let *Ê* :=

• In the limit $\beta \to \infty$ we get back a Haldane-Shastry-like spin chain, *k*¹ *k*² *... k^N k*^{*j*} + *k*^{*y*} + *k*^{*j*} + *k k*¹ *k*² *... k^N* $\frac{1}{2}$ back a Haldane-Shastry-like spin cha $(2i)$ $z_j \mapsto \omega^j = e^{2\pi i j}$ *k*1*, k*2*,...,k^N* k **c** *k* a Haldane-Shastry-like spin chain, $z_j \mapsto \omega^j = e^{2\pi i j/N}$

a (new) hamiltonian:
\n
$$
t_3 = \frac{1}{2} \sum_{i < j < k} [P_{ij} P_{jk} + P_{jk} P_{ij}] + \frac{1}{2} \sum_{i < j < k} \left[\cot \left(\frac{\pi(i-j)}{N} \right) + \cot \left(\frac{\pi(j-k)}{N} \right) + \cot \left(\frac{\pi(k-i)}{N} \right) \right] (P_{ij} P_{jk} - P_{jk} P_{ij})
$$

• effective BAEs:

$$
\sum_{l(\neq j_1,j_1+1)}^N \frac{u_m - \theta_l + i/2}{u_m - \theta_l - i/2} = \prod_{n(\neq m)}^{M'} \frac{u_m - u_n + i}{u_m - u_n - i}, \qquad j_I = \{j_1, \ldots, j_M\}
$$

*|k*1*, k*2*,...,k^N* i

- *^k*¹ *> k*² *, [|]k*1*, k*2ⁱ = (*z*¹ *^z*2) *^P* • inhomogeneities: $\theta_l = -\frac{i}{2}(N - 2l + 1)$
- each pair of repeated *k*'s corresponds to a reversed spin (magnon) *^k*11*,k*² (*z*1*, z*2) *|*""i responds to a reversed spin (magnon) *l* $\frac{1}{2}$ *l* $\frac{1}{2}$ + 100 $\frac{1}{2}$ + \mathbf{u} ² *⁄ t*o a reversed spin (magnon) $\frac{1}{2}$
- *n*/_{**ighest weight states**} *x* α *b*₂ α *k*₂ β *<i>k*₂ β *k*₂ β *k*₂ • the reference states are the Yangian highest weight states Sutherland coupling (§A.1). The special case *P*(1*/*2) gnest weight states \overline{a}

$$
|k_1, k_2, \ldots, k_N\rangle_M = \sum_{i_1 < i_2 < \ldots < i_M} \prod_{m < n} (z_{i_m} - z_{i_n})^2 P_{\lambda}^{\beta}(z_{i_1}, \ldots, z_{i_M}) |i_1, i_2, \ldots, i_M\rangle
$$

• again, the Bethe states are built using the B operator $B(u_1) \dots B(u_{M'}) \, | k_1, k_2, \dots, k_N \rangle_M$ $\mathbf{H} \cap \mathbf{H} \cap \mathbf{$ the B operator $B(u_1) \ldots B(u_{M'}) |k_1, k_2, \ldots, k_N\rangle_M$

> *[|]k*1*, k*2*,...,k^N* ⁱ*^M* ⁼ ^X a total of *M+M'* magnons $(1 + M)$ magnons

Conclusions and open questions (particle) Conclusions and open questions (part 1) (*zⁱ^m zⁱⁿ*) (*zⁱ*¹ *,...,zⁱ^M*) *|i*1*, i*2*,...,iM*ii **Conclusions and open questions (part 1)**

- new integrable long-range spin with dynamical inhomogeneities μ ¹ - with dynamical innomogenemes - new *N* d
tegrable long-rar *N* ϵ spin - with dyr *N* ◆ (*Pij ^Pjk ^Pjk ^Pij*)*, ,*
	- solvable by (effective) Bethe Ansatz (*z*) \overline{e} itve) detile Alisa
	- the twist κ interpolates between the Gelfand-Tsetlin bases for $\kappa \to 0, \infty$ and the isotropic states for $\kappa = 1$
	- **F**_{*f*} the anti errom *dk i* - the antiferromagnetic vacuum is the same as for Haldane-Shastry
	- what are the excitations?
	- *t*(*u*) = *A*(*u*) + ¹ - how do the generic solutions of the BAE look like?

The q-Haldane-Shastry Hamiltonian (Uglov-Lamers) Ugl95,Lam18]. do we have new results \mathbf{s} ¹⁰⁷ Eamiltonians. Here the ⁴◊⁴ matrix is with respect to the standard basis *[|]*øøÍ*, [|]*ø¿Í*, [|]*¿øÍ*, [|]*¿¿Í of ^C2¢C2. -Shastry Hamiltonian (Uglov-Lamers) six-vertex model's local weights. The properties of (1.7) will be reviewed in §2.2.2. *zj* \mathbf{v} *z* \mathbf{v} \mathbf{v} \mathbf{v} \mathbf{v} *rv* Hamiltonian (Uglov-L *k*1*<k*² *F*
*P Hamiltonian (Uglov-I) k*1*<k*² *F*² *M*¹ *K*²) + R(ˇ *u/v*; q) =

 $(\mathbf{C} \cdot \mathbf{C})$ of deformed $\mathbf{D} \mathbf{C}$ and $\mathbf{D} \mathbf{C}$ [Bernard, Gaudin, Haldane, Pasquier, 93; Uglov 95; Lamers 18; Lamers, Pasquier, D.S., 22] Pasquier, 93; Uglov 95; Lamers 18; Lamers, Pasquier, D.S., 22] sin² ⇡(*ⁱ ^j*)*/N* Lamers 18; Lamers, Pasquier, D.S., 22]

The XXZ model can also be deformed to accommodate for long-range interaction,
at the price of introducing **multi-spin interaction** at the price of introducing **multi-spin interaction** The same structure, cf. it is perhaps most arrows at the top indicate that the diagrams most control of the top indicate that the diagrams are read from both top indicate that the diagrams are read from both top (time stru $\overline{\textbf{a}}$ ction commodate f *^k* + *f* ⁺ \bm{p} in \bm{f} \bm{f} \bm{f} \bm{f} \bm{f} \bm{f} \bm{f} \bm{f} The final ingredients that we need are the 'quantum integers' $\mathcal{L}_{\mathcal{A}}$

$$
H^{L} = \frac{[N]_{q}}{N} \sum_{i < j}^{N} V(i - j) S_{[i,j]}^{L} \qquad [N] := \frac{q^{N} - q^{-N}}{q - q^{-1}}
$$

$$
V(k) = \frac{1}{(q\omega^k - q^{-1})(q\omega^{-k} - q^{-1})} = \frac{1}{4\sin(\pi k/N + \eta)\sin(\pi k/N - \eta)}
$$
 q = e^{i η}

 $T = 10$ **[Lamers 18]**

$$
\begin{pmatrix}\nz_j & z_{j+1} & z_N \\
\nearrow & \uparrow & \uparrow \\
& \searrow & \uparrow \\
& \searrow & \downarrow\n\end{pmatrix} \qquad := \begin{pmatrix}\n0 & 0 & 0 & 0 \\
0 & q^{-1} & -1 & 0 \\
0 & -1 & q & 0 \\
0 & 0 & 0 & 0\n\end{pmatrix} = e_i
$$

 z_{j-1} \cdots a ortigues up in the role of T_{α} in the lines. $\vert \vert$ \vert **q**-and *symmethiser* (*remperty-Lieb*) This q-antisymmetriser (up to normalisation) is the local Hamiltonian of the quantum-sl² invariant Heisenberg spin chain [PS90], see §2.2.3. *KEK*¹ = q² *E , KFK*¹ = q² *F ,* [*E,F*] = *^K ^K*¹ q-antisymmetriser (Temperley-Lieb)

$$
\bigvee_{u}^{v} := \check{R}(u/v)
$$

[Lamers 18]

$$
\tilde{R}_{k,k+1}(u) = 1 - f(u) e_k, \qquad f(u) = \frac{u-1}{q u - q^{-1}}
$$

The Uglov-Lamers Hamiltonian *ⁱ* . Unlike for the Heisenberg xxz chain no particle ever really wraps around \blacksquare the chain. This periodicity breaking is required by the coproduct (\blacksquare \overline{J} omove \overline{H} *zj zj <u>iltonian</u>* indicated. The nearest-neighbour transport is accounted for by the *R*-matrix, [*i,j*] *,* ^H^r ⁼ [*N*]^q *N e^k* = (*f* ⁺ *^k* + *f* ⁺ *^k*+1)(*f^k* + *fk*+1)

Several **new features** compared to the case $q=1$: transparent. For $\phi = \phi \circ \phi$ will show some $\phi = \phi \circ \phi$ and $\phi = \phi \circ \phi$ (§1.2.2). $\frac{1}{\sqrt{2}}$ *N ^k |*0i **S**everal i *I* **new features compared** V(*ⁱ ^j*) S^r

• the model is not translationally invariant, but there is a **q-translation operator**, G $T_{\rm eff}$ is diagrams at the top indicate that the diagrams are read from both σ invariant, out there is a q-transiation operator, σ *i* variant, but there is a **q-translation operator**, G *N* $\frac{1}{2}$ *k*=1 **·** the model is not translationally invariant, but there is a q-transl: e model is not translationally invariant, but the (*k* is not translationally invariant, but there is a c *-translation*

! *^G*! := *^R*ˇ*N*−1*,N* (*z*1*/z^N*)··· *^R*ˇ12(*z*1*/z*2) = *z*2 *z*² ··· ··· *z^N z*¹ *z^N z*1 *z*1 *z*1 *z*1 *.* (1.10) *v v u u* := *R*ˇ(*u/v*)*,* (1.18) *u u v v* := *e*sp = −(q − q−1) *R*ˇ" := BB@ 00 00 0 q¹ 1 0 0 1q0 00 00 1 CCA ⁼ *^eⁱ |{n*1*, n*2*}*ⁱ ⁼ ^X *^V* (*zi, z^j*) = *^ziz^j* (*zⁱ ^z^j*)² ⁼ ⁴ *,* !*^k*²) ⁺ (*zⁱ ^z^j*)² ⁼ ⁴ sin² ⇡(*ⁱ ^j*)*/N |{n}*ⁱ ⁼ ^X *k*=1 *|{n*1*, n*2*}*ⁱ ⁼ ^X *,* G] = [G*,* H^r] *,* (140) G G*^N* = 1 *,* G] = [G*,* H^r] *,* (140) G G*^N* = 1 References

• there exists another Hamiltonian with the **opposite "chirality"** In [Lam18] it was conjectured that *H*^l is q-homogeneous. The stronger statement from Pro**there exists another Hamiltonian.** Some the Exists and the Yang-Baxter v^{99} **Keking Pt-3** *E , KFK*¹ = q² *miltonian with the opp*

nonabelian symmetries (§1.2.5), cf. [HS96]. As q → 1 the 'wall' between sites *N* and 1 becomes

• there exists another Hamiltonian with the **opposite "chirality"**
\n
$$
H^{L} = \frac{[N]_{q}}{N} \sum_{i\n
$$
S_{[i,j]}^{L} := \begin{bmatrix} z_{i-1} & z_{i} & z_{i+1} & \cdots & z_{j-1} & z_{j} & z_{j+1} & z_{N} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \
$$
$$

sense to define the *full* Hamiltonian of the q-deformed Haldane–Shastry spin chain as (1.21) *^H*full := ¹ $[H^L, H^R] = [H^L, G] = [G, H^R]$

The Uglov-Lamers Hamiltonian *^k |*0i *Partitional The Uploy-Lamers H* integers. The *length ¸*(*⁄*) of *⁄* is the number of nonzero parts of *⁄*. Then² *Extended symmetry.* The parent model has quantum- α symmetry, which underpins the large degeneration under the large degeneration of α

• Both Hamiltonians can be diagonalised simultaneously and the spectrum can be written in terms of **motifs**, with eigenvalues (not real, for $|q|=1$) **M** \blacksquare **M** \blacksquare \blacks ly and the spectrum can be ry and the spectrum can be *The spectrum.* The spectrum and degeneracies of the

i i i j^{*//ii*} *//*
• The Yangian symmetry gets deformed to quantum affine symmetry the HS chain, the quantum numbers are 'motifs' [15] • The Yangian symmetry gets deformed to **quantum affine symmetry**

$$
\bigcup_{\substack{\mu_1 \ \sigma \in \mathcal{L}, \mathbb{R}(\mu) = \sum_{m=1}^{M} \epsilon^{\mathbf{L}, \mathbf{R}}(\mu_m)}} \mathbf{M} \text{ magnon motif} \qquad \mu_{m+1} > \mu_m + 1, \qquad 1 \leq m < M
$$

 one *magno* on dispersion relat \overline{a} m dispersion relations: *m*=1 1 one -magnon disper \blacksquare One-magnon dispersion relations: Figure 3. The correspondence (1.27) between a motif *µ* œ *M^N* of length given by *[⁄]^m* ⁼ *[⁄]*¯*^m* + 1, ¹ ^Æ *^m* ^Æ *^M*. Here *[⁄]*¯ characterises the extent by one-magnon dispersion relations:

$$
\varepsilon^{\mathcal{L}}(n) = \frac{1}{\mathcal{Q} - \mathcal{Q}^{-1}} \left(\mathcal{Q}^{N-n}[n] - \frac{n}{N}[N] \right) , \qquad \varepsilon^{\mathcal{R}}(n) = \frac{-1}{\mathcal{Q} - \mathcal{Q}^{-1}} \left(\mathcal{Q}^{n-N}[n] - \frac{n}{N}[N] \right)
$$

$$
\mathcal{L}^{\mathcal{L}}(n) = \frac{1}{\mathcal{Q} - \mathcal{Q}^{-1}} \left(\mathcal{Q}^{N-n}[n] - \frac{n}{N}[N] \right)
$$

 $\Pi = \frac{1}{2} (\Pi^+ + \Pi^+)$ has it at spin chain at Π has real spectrum both for q real and $|q| = 1$ $\epsilon = \frac{1}{2}$ (F) h for a such that the combined Hamiltonian has a real spectrum when q is real or *|*q*|* = 1 , $H = \frac{1}{2} (H^L + H^R)$ has real spectrum both for q real and $|q| = 1$ $H = \frac{1}{2} (H^L + H^R)$ has real spectrum both for q re

$$
\varepsilon(n) = \frac{1}{2} \left(\varepsilon^{\text{L}}(n) + \varepsilon^{\text{R}}(n) \right) = \frac{1}{2} [n][N - n]
$$

The cases of even and odd length, *N* = 2*L* or *N* = 2*L* + 1 are qualitatively different and [2*k*] = 0 and [2*^k* + 1] = (1)*^k .* (3.13) q-number generalisation of HS spectrum q -*i*, and the spectrum of the long range chain simplifies dramatically when the intervalse of q - \mathbf{q}_{-1}

q=i, non-unitary fermions and gl(1|1) \mathbf{q} - \mathbf{q} decomposition dierent motifs. *h*l *ij*;*kl* © (≠1)*^k*≠*^j* ^ÿ

• consider a 1-dimensional lattice with N sites and the fermionic degrees of freedom $\overline{}$ **EXAMPLE IN STREET IN THE LETTIN ONLY LEGATE AS A PROPERTY OF THE COOLD IN** μ implements a statistical setting μ is the selection μ principle successive model is the successive from the internet we served charges that commute with each other and (7). \mathbf{w} of recubing

$$
\{f_i, f_j^+\} = (-1)^i \delta_{ij}, \quad \{f_i, f_j\} = \{f_i^+, f_j^+\} = 0 \qquad f_j = (-i)^j c_j, \ f_j^+ = (-i)^j c_j^{\dagger},
$$

Jordan-Wigner fermions

as *^f^j* = (≠i) *^j ^c^j* , *^f* ⁺ *^j* = (≠i) *^j ^c ^j* . The *f*s will avoid a • they generate a **global gl(1|1) algebra** with (anti-)commutation relations \mathbf{f}_max transparent. From the two-site fermionic operators \mathbf{f}_max *Symmetries.* The commuting charges (7)–(9) have varrepresentative a grow of graph or $f(x)$ and $f(x)$, $f(x)$ ≠*N* \mathbf{r}_1 is a long-range grading gloring-separation \mathbf{r}_2 a global gl(1*|*1) algebra with (anti-)commutation relations \mathcal{L} is the free-fermion TL algebra \mathcal{L} .

$$
[N, F_1] = -F_1, \quad [N, F_1^+] = F_1^+, \quad \{F_1, F_1^+\} = E
$$

$$
F_1 = \sum_{i=1}^N f_i, \quad F_1^+ = \sum_{i=1}^N f_i^+, \quad N = \sum_{i=1}^N (-1)^i f_i^+ f_i, \quad E = \sum_{i=1}^N (-1)^i
$$

ⁱ = 0 *, eⁱ eⁱ±*¹ *eⁱ* = *eⁱ ,* [*ei, e^j*] = 0 if *|i*≠*j| >* 1 *.* (4) difference between N even and odd ! tive, (7)–(9) is a long-range gl(1*|*1) super-spin chain. For odd length the alternating central charge breaks periodic Given the extremely simple dispersion, further ('acciden-They are related to canonical Jordan–Wigner fermions

. From this perspec-

a linear combination of anticommutators of \mathcal{A}

as simple (anti-)commutation relations with the **two-site operators** npie (anu-*j*commutation relations *i*=1 *i*=1 Indeed, these operators anticommute with all *gi, g*⁺ *m*=1 imple (anti-)commutation relations with the two-site operators \mathbf{p} and make the symmetries more than \mathbf{p} • **gl(1|1)** has simple (anti-)commutation relations with the **two-site operators**

$$
g_i \equiv f_i + f_{i+1} \,, \quad g_i^+ = f_i^+ + f_{i+1}^+ \,, \qquad 1 \leqslant i < N
$$

Non-unitary fermions and Temperley-Lieb *{fi, f* ⁺ *^j }* = (≠1)*ⁱ "ij , {fi, fj}* ⁼ *{^f* ⁺ *ⁱ , f* ⁺ *^j }* = 0 *.* (1) \mathcal{N} are related to canonical Jordan–Wigner fermions in the canonical Jordan–Wigner fermions in the canonical Jordan \mathcal{N} mions and Temperley-Lieb dimensional lattice with an *odd* number *N* sites. The simplest definition of our model uses *non-unitary* the usual spin algebra. Each site *i* carries a gl(1*|*1) and E is central. This is just a fermionic version of representation generated by \mathbf{f} physics. In 2d such systems exhibit rich collective be-*Main results.* We introduce and solve a new integrable nitary lermions and temperiey-lied.

• the two-site operators $g_i \equiv f_i + f_{i+1}$, $g_i^+ = f_i^+ + f_{i+1}^+$, $1 \le i \le N$ *Parity.* Parity acts by reversitive sites \mathcal{P} $u_{\rm eff}$ interesting critical phenomena that are driven by α discrete the two-site of \cdot the two-site of **iii** and $g_i \equiv f_i + f_{i+1}, \quad g_i' = f_i' + f_{i+1}, \quad 1 \leq i$

can be used to generate the free-fermion **Temperley-Lieb** algebra ree-fermion Temperiey-Lieb algebra *^gⁱ* © *^fⁱ* ⁺ *^fi*+1 *, g*⁺ *N* (≠1)*ⁱ ^f* ⁺ can be used to in \mathcal{Z} . It generalises the isotropic HS chain by break-by break erate the free-fermion Temperley-Lieb algebra

$$
e_i^2 = 0
$$
, $e_i e_{i\pm 1} e_i = e_i$, $[e_i, e_j] = 0$ if $|i - j| > 1$ $e_i \equiv g_i^+ g_i$

= *sij* ! *g*+ *^j ^gⁱ* + (≠1)*ⁱ*≠*^jg*⁺ *ⁱ g^j* " $\frac{1}{\sqrt{2}}$ **j** $\frac{1}{\sqrt{2}}$, $\frac{1}{\sqrt{$ where *^sij* © (≠1)(*i*≠*j*)(*i*+*j*≠1)*/*², and we set *^e*[*i,i*] © *^ei*. th the global gl(1|1) and with $F_c = \sum^N f_i f_j$ $F^+ = \sum^N f^+ f^+$ $i < j$ In the global gi(1|1) and with $F_2 = \sum f_i f_j$, $F_2 =$ T(H^l $\frac{1}{i}$ *j* $\frac{j}{j}$ *i*, $\frac{j}{j}$ *,* $\frac{j}{j}$
i $\leq j$ $F_2 = \sum$ *N i<j* $f_i f_j$, $F_2^+ = \sum$ *N i<j* • **TL** generators commute with the global gl(1|1) and with $F_2 = \sum f_i f_j$, $F_2^+ = \sum f_i^+ f_j^+$ chains **chains in the chain are deep to matter the matter of the matter vertice** to matter \mathbf{TL} is expected to matter the matter els, exclusion statistics and 2d CFT [10, 11]. Long-range $\frac{N}{\sqrt{N}}$ mmute with the global gl(1|1) and with $F_2 = \sum f_i f_j$, $\frac{1}{i < j}$

Then the chiral hamiltonian reads

equivalent to free fermions via the Jordan–Wigner trans-

 $\sum_{i=1}^n$ \mathbf{y} , Kau, Sarui, 11 *<i><u>i* $\left| \begin{array}{c} \mathbf{f} \\ \hline \mathbf{f} \end{array} \right|$ and \mathbf{f} and \mathbf{f} </u> \ldots 111. The spectrum is real \ldots \mathbf{u}, \mathbf{u} *g*^{*i*} *g***₂ ***g*_{*g***¹ ***g*_{*g***¹ ***g***_{***i***}^{***g***₁} ***g*_{*g*^{*n*} *g*_{*g*}^{*g*}_{*n*} *g_{<i>g*}^{*g*}_{*g*}^{*g*}_{*g*}^{*g*}_{*g*}^{*g*}_{*g*}^{*g*}_{*g*}^{*g*}_{*g*}^{*g*}_{*g*}^{*g*}_{*g*}^{*g*}_{*g*}^{*g*}_{*g*}^{*g*}_{*g*}^{}}}} **Example 2 e** *i* Gainutdinov, Read, Saleur, 11] gether, (16)–(17) generate the full global-symmetry alge-**[Gainutdinov, Read, Saleur, 11]** we consider the simple but important case $\mathcal F$ $\overline{}$ ainutdinov, R

 \overline{N} and \overline{N}

*M r*_∤ $\int f(x) dx$ \leq $\int f(x) dx$ $\$ $U_{\bf q} \, \mathfrak{sl}(2)|_{{\bf q}={\bf i}}$

The Hamiltonians at N odd *i*^e is *Ilemians* of *N* add which is useful formal computations. This proves (7) from the main text once we show (B.16). In §E3a we show (B *e*[*i,j*] © [[*···* [*ei, ei*+1]*, ···*]*, e^j*] = *sij* ! *g*+ *^j ^gⁱ* + (≠1)*ⁱ*≠*^jg*⁺ L *ⁱ* = 0 *, eⁱ eⁱ±*¹ *eⁱ* = *eⁱ ,* [*ei, e^j*] = 0 if *|i*≠*j| >* 1 *.* (4) Furthermann at 11 due *^Tk*(*x*) = ¹ $\overline{\mathbf{a}}$ 1*a*115 a

• the interaction can be defined in terms of **commutators of TL generators** where the last relation uses \mathbf{r} = \mathbf{r} [*i,i*+*k*] ⁼ ^ÿ ction ca *l,m,k*≠¹ *^e*[*i*+*l,i*+*m*] *, ^Î*^l *l,m,k*≠¹ © (≠1)*^l ^t^k*≠*l,k ^t^k*≠*m,k , ^Î*^r *l,m,k* © *^Î*^l \mathcal{F} this can be more explicitly factorised as \mathcal{F} ρ *in the fermions of* \mathbf{T} *concretors* which means that (B.10) holds coefficient by coefficient in terms of the nested TL commutators. This is the origin of the o Editors of the generators can be defined in terms of **commutators of TL gener** nmutators of TL generat $\frac{1}{2}$ det(1 *^Kk*) (144)

$$
e_{[i,j]} \equiv [[\cdots [e_i, e_{i+1}], \cdots], e_j]
$$

$$
e_{[i,i]} \equiv e_i
$$

 $2\sqrt{2}$ ₁ $\mathrm{H}^{\text{{\tiny L,R}}}=\frac{\mathrm{i}}{\mathrm{a}}$ 2 \sum $1 \leqslant i \leqslant j \leqslant N$ • the two chiral Hamiltonians $H^{L,R} = \frac{1}{2} \sum h_{ij}^{L,R} e_{[i,j]}$ $2\frac{1}{1} \leq i \leq j < N$ $k\leq i \leq j < N$ spin operators. As we will outline in § B 3, they can be written as anticommutators of the nested commutators (B.15): $\sum_{1 \leq i \leq j \leq N} h_{ij}^{i} \left[c_{i,j} \right]$ $=$ $\frac{1}{1}$ \mathbb{R} $\mathbf{L}^{\mathbf{L}},$ $\leq i < N$ $e_{[i,j]}$ ϵ ₂ \leq *y* \leq pied successive mode numbers, that is inherited from the \cdot and the two chain We can explicitly down the next charge. The next charge \mathbf{i} is not charge, which is not charge. a manifoliation of ${\bf h}^{n,m} = \frac{1}{2}$ and h_{ij}^{n} and $e_{[i,j]}$ $1 \leqslant i \leqslant j \leqslant N$ *thiral Hamiltonians* $H^{L,R} = \frac{1}{2} \sum_{1 \le i \le i \le N} h_{ij}^{L,R} e_{[i,j]}$ $[i,j]$

> *M* $n_{ij} = (-1)^{j}$ *r*_{*i*} $n_{ij} = (-1)^{j}$ *r*_{*k*} $S_{N-i} = -h_{ii}^L$ \longrightarrow $H^L = -H^R$ invariant, the spectrum is real function in \mathcal{G} . The same is real function in \mathcal{G} with explicit coefficients $h_{ij}^{\text{R}} = (-1)^{j-i} h_{N-j,N-i}^{\text{L}} = -h_{ij}^{\text{L}} \longrightarrow H^{\text{L}} = -H^{\text{R}}$ $\frac{1}{\sqrt{2}}$ $\frac{1}{\sqrt{2}}$ $\frac{1}{\sqrt{2}}$ $\frac{1}{\sqrt{2}}$ $\frac{1}{\sqrt{2}}$ 06*j*6*l<m*6*n<k* s $h^{\text{R}} = (-1)^{j-i} h^{\text{L}}$ $\cdots = -h^{\text{L}}$ \cdots \cdots n_{ij} = $\begin{bmatrix} 1 & i & i \\ 1 & i & i \end{bmatrix}$ $\mu_{\mu} = -\mu_{\mu}$ $\text{F} \cdot \text{H} = -\text{H}$ with explicit coefficients $h_{ij}^{\text{R}} = (-1)^{j-i} h_{N-j,N-i}^{\text{L}} = -h_{ij}^{\text{L}} \longrightarrow H^{\text{L}} = -H^{\text{R}}$ with explicit coefficients $h_{ij}^{\text{R}} = (-1)^{j-i} h_{N-j,N-i}^{\text{L}} = -h_{ij}^{\text{L}} \longrightarrow H^{\text{L}} = -H^{\text{R}}$ $h^{\text{R}} = (-1)^{j-i} h^{\text{L}}$ $\cdots = -h^{\text{L}}$ $H^{\text{L}} = -H^{\text{R}}$ $h_{ij}^{\rm R} = (-1)^{j-i} h_{N-j,N-i}^{\rm L} = -h$ $(b)^{j-i} h^{\text{L}}_{N-j,N-i} = -h^{\text{L}}_{ij}$ -

- walihitu.
... *Particle-hole transformation.* Another simple opera- $\left[\begin{smallmatrix} \mathcal{C}^{\mathcal{C}}[t,j], \mathcal{C}^{\mathcal{C}}[k,l] \end{smallmatrix}\right]$ $\overline{1}$ $\leq i \leq j <$ *n n n i*,*m*_{*i*}, *m*_{*i}* **hold hamiltonian.** The find and $\sum_{i} (h_{ij;kl}^{\text{L}} + h_{ij;kl}^{\text{R}}) \{e_{[i,j]}, e_{[k,l]}\}$ spin $s \leq i \leq j < k \leq l < N$ **•** the non-chiral Hamiltonian $H = -\frac{1}{4N}$ $\overline{}$ 16*i*6*j<k*6*l<N* $\left(h_{ij;kl}^{\text{L}} + h_{ij;kl}^{\text{R}} \right) \left\{ e_{[i,j]}, e_{[k,l]} \right\}$ long-range hopping. It is not translationally invariant: $H = -\frac{1}{4N}$ $\sum (h_{i,j,kl}^{\text{L}} + h_{i,j,kl}^{\text{R}})$ $\{e_{[i,j]}, e_{[k,l]}\}$ $4N$ $\sum_{1 \leqslant i \leqslant j \leqslant k \leqslant l \leqslant N}$ neighbours *i, i* + 1. Instead, the standard translation op-**H** $H = -\frac{1}{4N} \sum_{1 \leq i \leq i \leq k \leq l \leq N} \left(h_{ij;kl}^{\text{L}} + h_{ij;kl}^{\text{R}} \right) \left\{ e_{[i,j]}, e_{[k,l]} \right\}$ and is the commutant of the free-fermion TL algebra [2]. $a_i \leq j \leq k \leq l \leq N$ *N t*
 $H = -\frac{1}{4N} \sum_{1 \le i \le j \le k \le l \le N} (h_{ij,i}^{\text{L}})$ $\frac{1}{\sqrt{2\pi}}$ $H = -\frac{1}{\sqrt{2}}$ $\sum (h_i^L)$ *g*@*^g* ln *G^T*` 1 + 2*p g*@*^g* ln *G^T^p ,* (145) $\overleftrightarrow{\leq} i \leq j \leq k \leq$ ^p2*^N* ϵ ^{\sim 1} r
S 1
- *hation operator* $G = (1 + t_{N-1} e_{N-1})$. *franslation operator* $G = (1 + t_{N-1} e_{N-1}) \cdots (1 + t_1 e_1), G^N = 1$ $t_k \equiv \tan \frac{\pi k}{N}$ e_{N-1} , \cdots (1 + $t_1 e_1$), G = 1 • the quasi-translation operator $G = (1 + t_{N-1} e_{N-1}) \cdots (1 + t_1 e_1), G^N = 1$ $t_k \equiv \tan \frac{\pi k}{N}$ $r₁$ $\bigcap_{i=1}^n$ V *x*_{$f_x = \tan \pi k$} $G^N=1$ *ej* t _{*j*} • the quasi-translation operator $G = (1 + t_{N-1} e_{N-1}) \cdots (1 + t_1 e_1)$, $G^N = 1$ $t_k \equiv \tan \frac{\pi k}{N}$ dimensional lattice with an *odd* number *N* sites. $\mathbf{G}^N = 1 \cdot \mathbf{G}^N = 1 \qquad \qquad t_k \equiv \text{tan} \mathbf{G}$

$$
\left[G,H^{L}\right] =\left[G,H\right] =\left[H^{L},H\right] =0
$$

the amplitudes *h*^l $\frac{1}{\sqrt{2}}$ depend on the distance of $\frac{1}{\sqrt{2}}$ depend on the distribution of $\frac{1}{\sqrt{2}}$ depend on generalises the XXZ Hamiltonian $H_{\text{XXZ}}^{\text{open}} = -\sum_{i=1} e_i$ at neighbours *i, i* + 1. Instead, the standard translation op*h*r $\chi_{\rm Z} = -\sum_{i}$ $H_{\text{XXZ}}^{\text{open}} = \frac{a}{Z} =$ = (≠1)*^l*≠*j*+*k*≠*ⁱ ^h*^l $\overline{\text{Pascal}}$ Supermote the main $\left[1 \text{ avg}(0.0, \text{Sate}(0.9))\right]$ $\left[1 \text{ erg}(0.9) \right]$ **[Pasquier, Saleur, 90]** *j*=1 *ej*

generalises the XXZ Hamiltonian [Pasquier, Saleur, 90]
$$
H_{\text{XXZ}}^{\text{open}} = -\sum_{j=1}^{N-1} e_j \quad \text{at} \quad \Delta = \frac{q+q^{-1}}{2} = 0
$$

 $\frac{\pi k}{N}$

"l*,*r(*µm*) (3.13)

Discrete symmetries \mathcal{S} **Parity. Parity and Discrete symmetries.** The commuting charges (7)–(9) have several of the commuting charges (7)–(9) μ invariant under particular in the international nontrivial result that P(H) $\frac{1}{2}$

The model has real spectrum in spite of being non-unitary, due to PT symmetry eral transformation properties and symmetries. proliferation of factors of i and make the symmetries more \mathbf{f} the two-site fermionic operators from the two-site fermionic operators \mathbf{f} *Parity.* Parity acts by reversal of the lattice sites has real spectrum in spite of being non-unitary, due *i* The model has real spectrum in spite of being non-unitary, due to PT s icar speculum in spite of being non-unitary, due to **i i** s $\mathbf T$ P(*fi*) = *fN*+1≠*ⁱ*. This preserves the anticommutation relations in the Special *n* spice of being non-unitary, the e *l* has real spectrum in spite of being

i Parity P: $P(f_i) = f_{N+1-i}$ relations (1) since *N* is odd. The TL generators transform as P(*ei*) = *e^N*≠*ⁱ*. The chiral hamiltonian (7) is not $P(f_i) = f_{N+1-i}$ nontrivial result that P(H^l) = [≠]H^l [22]. We have $\Gamma(\ell)$ is name. It is name. It is name. It is name. It is a highly interesting it is a highly interesting in $\Gamma(\ell)$ **P**: $P(f_i) = f_{N+1-i}$ $\mathcal{F}_{\mathcal{F}}$ $P(f_i) = f_{N+1-i}$

$$
P(H^{\rm\scriptscriptstyle L})=-H^{\rm\scriptscriptstyle L} \,,\quad P(H)=H \,,\quad P(G)=G^{-1}
$$

• Time reversal T (anti-linear): $T(e_i) = e_i$ where the last relation uses *tⁱ* = ≠*t^N*≠*ⁱ*. • Time reversal T (anti-linear): $T(e_i) = e_i$ reversal T (anti-linear): $T(e_i) = e_i$ where *^sij* © (≠1)(*i*≠*j*)(*i*+*j*≠1)*/*², and we set *^e*[*i,i*] © *^ei*. $\mathbf{1}$ is bilinear in the fermions (1).

$$
T(H^L) = -H^L, \quad T(H) = H, \quad T(G) = G
$$

e[*i,j*] © [[*···* [*ei, ei*+1]*, ···*]*, e^j*] \mathbf{u} *i* \mathbf{g} \mathbf{v} \mathbf{v} \mathbf{u} \mathbf{g} \mathbf{u} \mathbf{v} \cdot **Charge conjugation C** (anti-linear): Ω in eatiest Ω (anti-linear). t unjugation t (and -mical). \overline{S} is bilinear in the fermions (1). Finally set \overline{S} true for the 'quasi-momentum' p = ≠i log G. • Charge conjugation C (anti-linear): where *^sij* © (≠1)(*i*≠*j*)(*i*+*j*≠1)*/*², and we set *^e*[*i,i*] © *^ei*. $\mathbf{C}_{\mathbf{H}}$ c conjugation C (and inicar).

$$
C(H^{\textrm{L}})=-H^{\textrm{L}}\,,\quad C(H)=H\,,\quad C(G)=G
$$

The spectrum for N odd extended symmetry will be performed elsewhere. **The spectrum and degree spectrum and degree of the spectrum** where the last relation uses \textbf{The} eracies (21). As we will see, these are already visible in the spectrum for N odd relates to the Yangian of gl(1*|*1). A detailed study of this *Global symmetry.* The model can be seen as a longwhich commute with the e**p**^{*i*}, when with $\sum_{i=1}^{n}$ gether, (16)–(17) generate the full global-symmetry alge- $\mathbf{v} \mathbf{N}$ add motifs *{*1*,* 3*,...,N* ≠4*, N* ≠2*}* switching halfway between

*n <u>x nm n nn n n***_{***n***}** *n n*</u></sub> $\sum_{i=1}^{n}$ *ancetrum.* The spectrum and $\sum_{i=1}^{n}$ and $\sum_{i=1}^{n}$ ne speculum occomes very simple at q 1 and 15 of The spectrum becom \mathbf{r} tivelation relation relation relation relation relation relation relations are \mathbf{r} simple at $q=1$ and N od *Externes symmetry.* The spectrum hecomes very simand specified the degeneration of $\sum_{i=1}^{n}$ \overline{C} and **N** odd $\frac{1}{\sqrt{2}}$ the fermionic representation. The key to define The spectrum becomes **very simple** at q=i and **N odd**

- quasi-momentum $p = -i \log G$ $p = \frac{2\pi}{N}$ *N* • quasi-momentum $p = -i \log G$ $p = \frac{2\pi}{N} \sum_m \mu_m \mod 2\pi$ $\frac{1}{\sqrt{1-\frac{1}{$ $p = \frac{2\pi}{N} \sum \mu_m \mod 2\pi$ \mathbf{r} branches, realising chiral and (up to a shift) \mathbf{r} relates to the Yangian of gl(1*|*1). A detailed study of this • quasi-momentum $p = -i \log G$ $p = \frac{2\pi}{N} \sum_m \mu_m \mod 2\pi$
- changes the creation and annihilation and annihilation \mathbf{r} (*fi*) = *f* ⁺ e*nion* • chiral Hamiltonian tive, (1)–(1)–(1) super-spin chain chained graduate global chain chain chain chain chain chain chain chain. For
1) super-spin chains a long-spin chain. For chains a long-spin chain chain chain chain chain chain chain. For
 odd length the alternation charge breaks periodical charge breaks periodical charge breaks periodical charge b the entirely Hamiltonian

$$
E_{\{\mu_m\}}^{\text{L}} = \sum_{m=1}^{M} \varepsilon_{\mu_m}^{\text{L}} \qquad \qquad \varepsilon_n^{\text{L}} = \begin{cases} n, & n \text{ even}, \\ n - N, & n \text{ odd}, \end{cases} \qquad \qquad 0.
$$

 non -chiral Hamiltonian *N ^m µ^m* mod 2*fi* setting the eigenvalue • non-chiral Hamiltonian *Global symmetry.* The model can be seen as a local can be seen as a local can be seen as a local can be seen as α $\overline{111}$ for $\overline{11}$ tive, the mon-chiral Har

$$
E_{\{\mu_m\}} = \sum_{m=1}^{M} \varepsilon_{\mu_m} \qquad \qquad \varepsilon_n = |\varepsilon_n^{\text{L}}|
$$

 $\overline{}$

N

How to solve the model for N odd **EVALUATE ENEXUAL ENERGY ENERGY POSTAGER POWER** has eigenvalues > 0, with *E* = 0 for the empty motif, and **Example 10 × 2** + 2³ **EXP** + 2³ **EXP** + 2³ + 2 $\mathbf{H}_{\mathbf{Q}\mathbf{W}}$ to solve the model for $\mathbf{N}_{\mathbf{Q}}$ more than the House

We want to solve it in terms of (non-unitary) fermions \longrightarrow use quasi-translations *Extended symmetry.* The parties of the pa els near the maximum are missing. *Explicit diagonalisation.* Let us the vertical ver *Extended symmetry.* The parent model has quantumels near the maximum are missing. **Exploration diagonalisation.** Let us the contract of the term of the contract of the specifical spec which commute with the e^pident commute with $\frac{1}{2}$ n terms of (non-unitary) fermions \longrightarrow use quay spectively. This maximum corresponds to the one or two

eracies (21). As interesting the start with the start with the first site and translate the fermions via lattice and use the stranslation operator: the property of the stranslation operator: • start with the first site and **translate the fermions** via eracies (21). As will see are also are also are already visible in the set of \mathcal{E} long-range hopping. It is not translationally invariant: start with the first site and **translate the fermions v b** start with the f motifs *{*1*,* 3*,...,N* ≠4*, N* ≠2*}* switching halfway between site and **translate the fermions** via

$$
\Phi_i \equiv G^{1-i} f_1 G^{i-1}, \qquad \Phi_i^+ \equiv G^{1-i} f_1^+ G^{i-1}
$$

The spectrum parent model and model and the control of the set of the T_{max} is newled in due to αN *The spectrum.* The spectrum and degeneracies of the • the transformation is **periodic** due to $G^N = 1$

$$
\Phi_{i+N} = \Phi_i \,, \qquad \Phi_{i+N}^+ = \Phi_i^+
$$

µm+1 *> µ^m* + 1 *,* 1 6 *m<M.* (18) $\frac{1}{2}$ the commutation relations are non-local (but) *i*_{**lo pay is that the commutation relations are non-local** (but the pay is that the commutation relations are **non-local**} *P* \cdot 111 **c** a linear combination of anticommutators of the nested • the price to pay is that the commutation relations are non-local (but translationally invariant) *{µm}*, consisting of integers 1 6 *µ^m < N* increasing as

$$
\left\{\Phi_i, \Phi_j^+\right\} = -(1+t_{j-i})\,,\quad \{\Phi_i, \Phi_j\} = \left\{\Phi_i^+, \Phi_j^+\right\} = 0
$$

How to solve the model for N odd *The spectrum.* The spectrum and degeneracies of the **paramely in the model for N odd** trum is real, cf. \mathbf{P} interactions is that the spectrum is that the spectrum is extremely simple as \mathcal{L}

• next we use the Fourier modes of the quasi-translated fermions T_{c} the Fourier modes of the quest trends to for and the *r* ourier modes of the quasi-transfaced te. *modes of the quasi-translated fermions*

$$
\tilde{\Psi}_n \equiv \frac{a_n}{N} \sum_{j=1}^N e^{-2i\pi nj/N} \Phi_j, \quad \tilde{\Psi}_n^+ \equiv \frac{a_n}{N} \sum_{j=1}^N e^{2i\pi nj/N} \Phi_j^+ \qquad a_0 \equiv \text{i and } a_n \equiv \text{i}^{n+1/2} \text{ else}
$$

I and the properties *lations* we rescaled the *M* μ *n* μ • to get canonical commutation relations we rescaled the Fourier modes • By global symmetry, the descendant *[|]*0*, n*Í Ã ^F⁺ cies for the motifs to account for the full Hilbert space. ω modes are caused by ω

$$
\{\tilde{\Psi}_n, \tilde{\Psi}_m^+\} = \delta_{nm} \,, \quad \{\tilde{\Psi}_n, \tilde{\Psi}_m\} = \{\tilde{\Psi}_n^+, \tilde{\Psi}_m^+\} = 0
$$

des are **generators of the gl(1|1) algebra** • the zero modes are **generators of the gl(1|1) algebra**

$$
\frac{1}{a_0}\,\tilde{\Psi}_0 = \sum_{i=1}^N f_i = F_1\,,\quad \frac{1}{a_0}\,\tilde{\Psi}_0^+ = \sum_{i=1}^N f_i^+ = F_1^+
$$

^des are linear combinations of the two-site onerators *k* \mathbf{r} *s* \mathbf{r} **C** \mathbf{S} \mathbf{S} $\ddot{}$ $\mathop{\mathsf{ex}}\nolimits$ are linear combinations of the two-site one • the other modes are linear combinations of the two-site operators $\frac{1}{2}$

$$
\frac{1}{a_n}\,\tilde{\Psi}_n = \sum_{i=1}^{N-1} M_{ni}\,g_i\,,\quad \frac{1}{a_n}\,\tilde{\Psi}_n^+ = \sum_{i=1}^{N-1} \bar{M}_{ni}\,g_i^+\qquad \qquad g_i \equiv f_i + f_{i+1}\,,\quad g_i^+ = f_i^+ + f_{i+1}^+
$$

How to solve the model for N odd The other modes are explicit linear combinations of the other modes are explicit linear combinations of the other combinations of the α et for N odd $2²$ and $2²$ In terms of these fermionic modes, H^l is diagonal: *N* ÿ≠1 **How to solve the model for N odd** $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ ˜ *ⁿ* = ;
; *i*=1 \mathbf{A} *an* ˜ ⁺ *n* = ÿ≠1 *i*=1 *i d*el for N odd *N* ÿ≠1

• in these variables **the chiral Hamiltonian** becomes purely quadratic variables the chiral Hamiltonian becomes purely \mathcal{O} on and matching the result with the e variables the chiral Hamiltonian becomes purely quadratic is the fermionic vacuum, then by \mathcal{L}_1 then by (26) the Fock states the Fock states the Fock states the Fock states that \mathcal{L}_2 parent-model eigenstates at q=i, this gives the full twooni: **n** becomes purely quadratic

$$
\mathrm{H}^{\text{\tiny L}}=\sum_{n=1}^{N-1}\varepsilon_{n}^{\text{\tiny L}}\,\tilde{\Psi}_{n}^{+}\,\tilde{\Psi}_{n}
$$

- it can be diagonalised on the Fourier Fock space spanned by diagonalised on the Fourier Fock space spanned by $\tilde{\tau}$ spectrum requires $\tilde{\tau}$ and $\tilde{\tau}$ and $\tilde{\tau}$ $\ket{n_1,\ldots,n_M}\equiv \tilde{\Psi}^+_{n_1}\ldots\tilde{\Psi}^+_{n_M}\ket{\varnothing}$ $0\leqslant n_1<\cdots$ $\int_{\mathbb{R}^2} |x|^2 dx$ $\int_{\mathbb{R}^2} |x|^2 dx$ $\int_{\mathbb{R}^2} |x|^2 dx$ $f(x)$ $f(x)$ $f(x)$ $f(x)$ $f(x)$ $\langle {\bf r} \rangle = {\bf \tilde{v}}_{n_1}^+ \dots {\bf \tilde{v}}_{n_M}^+$ (*n*₁, ..., $n_M \rangle \equiv {\bf \tilde{v}}_{n_1}^+ \dots {\bf \tilde{v}}_{n_M}^+$ $\ket{\mathbf{by}} \qquad \ket{n_1,\ldots,n_M} \equiv \Psi^+_{n_1}\ldots \Psi^+_{n_M}\ket{\varnothing}$ $\frac{1}{2}$ **Fourier Fock space spanned by** *Outlook.* We obtained and analysed a long-range fermi- $\ket{n_1,\ldots,n_M}\equiv \Psi^+_{n_1}\ldots \Psi^+_{n_M}\ket{\varnothing}$ \ldots, r $\langle M \rangle \equiv \Psi_{n_1}^{\perp} \ldots \Psi_{n_M}^{\perp} \ket{\varnothing}$ e spanned by $\overline{p}_1, \ldots, n_M \rangle \equiv \tilde{\Psi}^+_{n_M}$. *Á*l $0 \leq n_i \leq \ldots \leq n_i \leq N$
- compatibility with the motif rule thanks to $\varepsilon_n^L + \varepsilon_{n+1}^L = \varepsilon_2^L$ mode numbers *{nm}* with 0 6 *n*¹ *< ··· < n^M < N*. The mode numbers *{nm}* with 0 6 *n*¹ *< ··· < n^M < N*. The nks to ε_n^L -*E*
E
E
F
E
E
F
E
E $\frac{1}{2}$ $\frac{1}{2}$ with the motif rule thanks to

$$
\varepsilon_{n}^{\textrm{L}}+\varepsilon_{n+1}^{\textrm{L}}=\varepsilon_{2n+1\,\textrm{mod}\,N}^{\textrm{L}}
$$

 $O \leqslant \log_1 1$

E
E
E
F
E
F
E
F
E
E
F
E
E Unit al Mannivollian is quartic • the **non-chiral Hamiltonian** is quartic *Á*l *ⁿ* + *Á*^l *ⁿ*+1 = *Á*^l ²*n*+1 mod *^N .* (32) *Á*l *ⁿ* + *Á*^l *ⁿ*+1 = *Á*^l $\frac{1}{2}$

$$
H = \sum_{n=1}^{N-1} \varepsilon_n \tilde{\Psi}_n^+ \tilde{\Psi}_n + \sum_{\substack{1 \le m < n < N \\ 1 \le r < s < N}} \tilde{V}_{mn;rs} \tilde{\Psi}_m^+ \tilde{\Psi}_r \tilde{\Psi}_s \qquad \qquad \tilde{V}_{mn; m+k, n-k} = (-1)^{k+1} 4 \delta_{m \text{ odd}}
$$
\n
$$
0 \le 2k < n-m
$$
\neigenvalues

\nstation rule:

\n
$$
\varepsilon_m^L + \varepsilon_n^L = \varepsilon_r^L + \varepsilon_s^L
$$

N even *e*[*l,m*] ⌘ [*el,*[*el*+1*,...*[*em*1*, em*]*...*]] = [[*...*[*el, el*+1]*,...em*1]*, em*]*, i* 6 *l < m < j.* with \mathbf{e}^{k} , and the experimental \mathbf{e}^{k} , and the experimental \mathbf{e}^{k}

• for N even, $N = 2L$ there are singularities (poles) in the coefficients of the conserved quantities, which can be regularised as \mathcal{L}_{max} elements have poles in \mathcal{L}_{max} **en**, $N = 2L$ there are singularities (poles) in the coefficients of α is a viological as ''
''' '' '' *n*_{*i*} *n*_{*i*}*n***_{***n***}^{***n***}** *n_{<i>i*}</sub>^{*n*} *n_i*^{*n*} *n_i*^{*n*} *n_i*^{*n*} *n_i*^{*n*} *n_i^{<i>n*} *n_i*^{*n*} *n_i*^{*n*} *n_i*^{*n*} *n_i^{<i>n*} *n_i*^{*n*} *n_i*^{*n*} *n_i*^{*n*} *n_i^{<i>n*} *n*_{*i*} *n_i*^{*n*}

$$
t_k \equiv \tan \frac{\pi k}{N}
$$
, $k \neq L$ and $t_L = t_L(\alpha) \equiv \tan \frac{\pi (L + \alpha)}{N} = \cot \frac{\pi \alpha}{N}$

• the **residues of these poles** give a collection of conserved quantities esidues of these poles give a collection of conserved qu *ei e ^j ei* = *ei ,* if *|i j|* = 1*, n* of conse *n*=*j*+1 *t r*ved quantities *ni,n Nl,Nk*;*Nj,Ni* . The residues of these poles give a conection of conserved quantities

$$
\big[\mathscr{G},\mathscr{H}^C\big]=\big[\mathscr{G},\mathscr{H}\big]=\big[\mathscr{H}^C,\mathscr{H}\big]=0
$$

$$
\mathscr{G} \equiv (1+t_{N-1}e_{N-1})\cdots(1+t_{L+1}e_{L+1})e_L(1+t_{L-1}e_{L-1})\cdots(1+t_1e_1),
$$

\n
$$
\mathscr{G} \equiv (1-t_1e_1)\cdots(1-t_{L-1}e_{L-1})e_L(1-t_{L+1}e_{L+1})\cdots(1-t_{N-1}e_{N-1}).
$$

\n
$$
\mathscr{G}\mathscr{G} = \mathscr{G}\mathscr{G} = 0
$$

$$
\mathcal{H} = \sum_{1 \leq i \leq j < N} h_{ij} e_{[i,j]},
$$
\n
$$
h_{ij} = \lim_{\alpha \to 0} \frac{\pi^2 \alpha^2}{2N^2} \left(h_{ij}^{\text{L}}(\alpha) + h_{ij}^{\text{R}}(\alpha) \right)
$$
\n
$$
\mathcal{H}^{\text{C}} = \sum_{1 \leq i \leq j < k \leq l < N} h_{ij;kl} \left\{ e_{[i,j]}, e_{[k,l]} \right\}
$$
\n
$$
h_{ij;kl} = \lim_{\alpha \to 0} \frac{\pi^2 \alpha^2}{2N^2} \left(h_{ij;kl}^{\text{L}}(\alpha) - h_{ij;kl}^{\text{R}}(\alpha) \right)
$$

 h **e** chiral and non-ch the roles the chiral and non-chiral hamiltonians is interchanged Universite Paris–Saclay, CNRS, CEA, Institut de Physique Th ´ orique, 91191 Gif-sur-Yvette, France ´ *increasement and non-emitary fermionic is interchanged* $\frac{1}{2}$ the reles the chiral and non-chiral hamiltonians is interehanced the roles the chiral and non-chiral hamiltonians is interchanged

> $T = \begin{bmatrix} 1 & 0.1 & 1 \end{bmatrix}$ in a that we remove by taking the poles in a that we remove by taking the most double poles in a that we remove by taking the most double poles in a that we remove by taking the most double the • the eigenvalues of these conserved quantities are all zero (with **Jordan blocks** of size up to $L+1$) $\frac{1}{2}$ are conserved conserved charges are in $\frac{1}{2}$ and $\frac{1}{2}$ and $\frac{1}{2}$ and $\frac{1}{2}$ and $\frac{1}{2}$ and $\frac{1}{2}$

To do list

- Study the system for **even length:** spectrum identically zero; Jordan blocks
- Identify the **extended symmetry:** $gl(1|1)$ Yangian?
- Interpret the **staggering** & the **linear dispersion relations** in the odd case
- **CFT limit:** gl(1|1) Kac-Moody algebra?
- **Free field** realisation and **vertex operators algebra**
- **Wave functions** in the fermionic representation & **Macdonald polynomials**
- **Other roots of unity**: $q^3=1$ and $gl(2|1)$ symmetry