Bootstrapping Form Factors via Integrability and Cluster Algebras

Alexander Tumanov, ENS Paris

Based on:

[Basso, Dixon, AT to appear later this month] [Sever, AT, Wilhelm ’20 - ’21] [Basso, AT ’23]

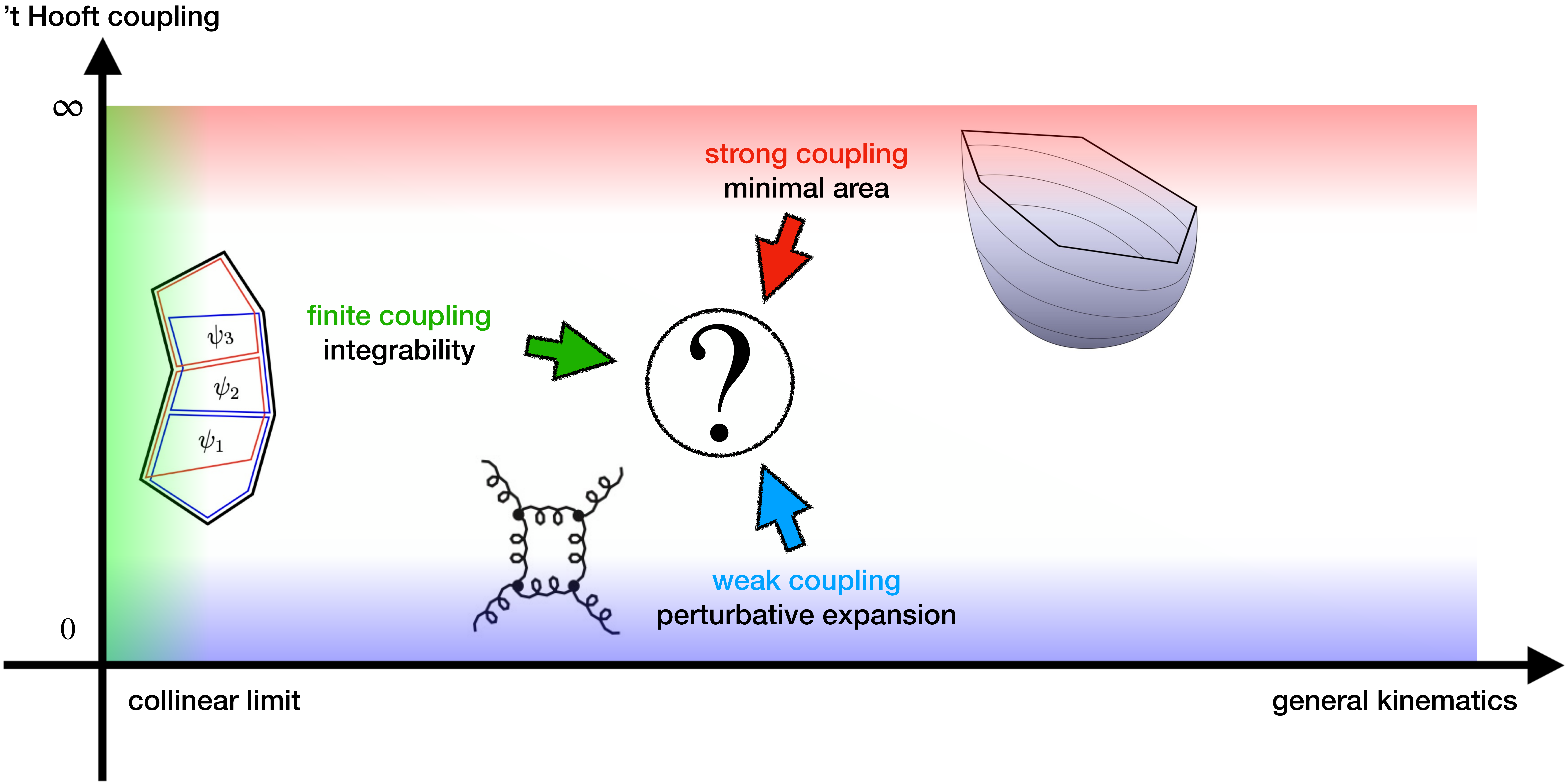
Cluster algebra part is based on:

[Golden, Goncharov, Spradlin, Vergu, Volovich ’13]

[Caron-Huot, Dixon, Drummond, Dulat, Foster, Gürdogan, von Hippel, McLeod, Papathanasiou ’20]

Varna, 14.08.24

Can we “solve” scattering in $\mathcal{N} = 4$ SYM?



Cluster bootstrap

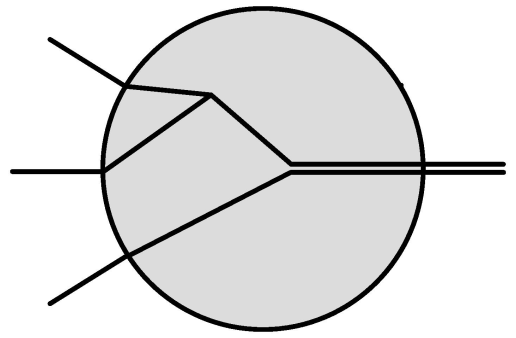
Recent results (2011-2024)

Loop oder \ Observable	Amp 6pt MHV	Amp 6pt NMHV	Amp 7pt MHV	Amp 7pt NMHV	FF 3pt $\mathrm{Tr}\,\phi^2$	FF 3pt $\mathrm{Tr}\,\phi^3$	FF 4pt $\mathrm{Tr}\,\phi^2$	FF 4pt $\mathrm{Tr}\,\phi^3$	FF 4pt $\mathrm{Tr}\,\phi^4$
1	✓	✓	✓	✓	✓	✓	✓	✓	✓
2	✓	✓	✓	✓	✓	✓		✓	✓
3									
4									
5									
6									
7									
8									

✓ = results obtained by traditional methods

Amp = Amplitude, FF = Form Factor
MHV = Maximally Helicity Violating, NMHV = Next to MHV

$$F_{\mathcal{O}}(k_1,\ldots,k_n) = \langle k_1,\ldots,k_n|\mathcal{O}(q)|0\rangle$$



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1	✓	✓	✓	✓	✓	✓	✓	✓	✓
2	✓	✓	✓	✓	✓	✓	✓ (blue)	✓	✓
3	✓ (blue)	✓ (blue)	✓ (blue)	✓ (blue)	✓ (blue)	✓ (blue)			
4	✓ (blue)	✓ (blue)	✓ (blue)	✓ (blue)	✓ (blue)	✓ (blue)			
5	✓ (blue)	✓ (blue)			✓ (blue)	✓ (blue)			
6	✓ (blue)	✓ (blue)			✓ (blue)	✓ (blue)			
7	✓ (blue)				✓ (blue)				
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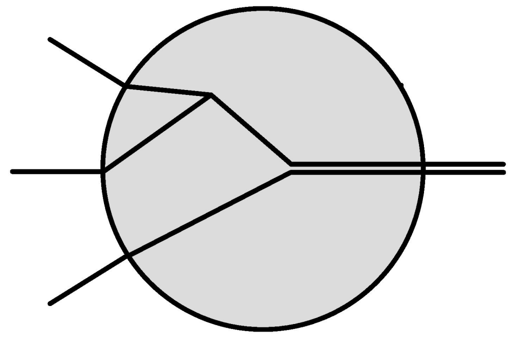

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1	✓	✓	✓	✓	✓	✓	✓	✓	✓
2	✓	✓	✓	✓	✓	✓	✓ (blue)	✓	✓
3	✓ (blue)	✓ (blue)	✓ (blue)	✓ (blue)	✓ (blue) [orange box]	✓ (blue) [red box]		work in progress w/ Basso, Dixon, Drummond & Gurdogan	
4	✓ (blue)	✓ (blue)	✓ (blue)	✓ (blue)	✓ (blue) [orange box]	✓ (blue) [red box]			
5	✓ (blue)	✓ (blue)			✓ (blue) [orange box]	✓ (blue) [red box]			
6	✓ (blue)	✓ (blue)			✓ (blue) [orange box]	✓ (blue) [red box]			
7	✓ (blue)				✓ (blue) [orange box]				
8	✓ (blue)				✓ (blue) [orange box]				

✓ = results obtained by traditional methods

✓ (blue) = perturbative bootstrap results

[red box] = [Basso, Dixon, AT to appear this month]

[orange box] = [Dixon, McLeod, Wilhelm ’20], [Dixon, Gurdogan, McLeod, Wilhelm ’22]

Amp = Amplitude, FF = Form Factor
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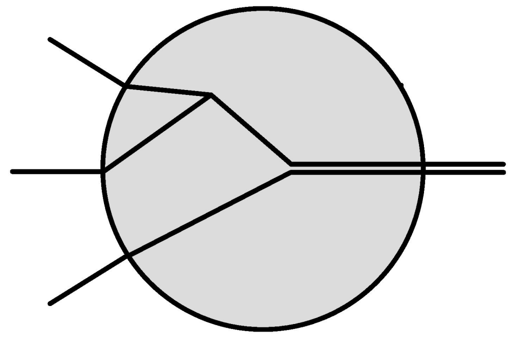
$$F_{\mathcal{O}}(k_1,\,\ldots,\,k_n) = \langle k_1,\,\ldots,\,k_n|\mathcal{O}(q)|0\rangle$$



Symbol

Amplitudes and form factors with small number of external legs live in the space of **generalized polylogarithms**.

This space of functions can be turned into a simple vector space by using the **symbol map**.

Polylogarithmic functions → Maximally iterated co-product (tensor product of "letters")

$$(\log z)^n \quad \rightarrow \quad \underbrace{z \otimes \ldots \otimes z}_{n}$$

$$\mathrm{Li}_n(z) \quad \rightarrow \quad -(1-z) \otimes \underbrace{z \otimes \ldots \otimes z}_{n-1}$$

$$(\text{ansatz at } L \text{ loops}) \quad \sum c_{a_1 \ldots a_{2L}} \underbrace{a_1 \otimes \ldots \otimes a_{2L}}_{(\text{length})=2L}$$

The entries of the symbol a_i are called **letters**. Collection of all the letters is called the **alphabet**.

Constraining the symbol

In this talk, I will talk about two specific three-point form factors, which both have the same symbol alphabet:

$$\mathcal{L}_a = \{a, b, c, d, e, f\} = \{\frac{u}{vw}, \frac{v}{wu}, \frac{w}{uv}, \frac{1-u}{u}, \frac{1-v}{v}, \frac{1-w}{w}\}$$

where $u = \frac{s_{12}}{q^2}, \quad v = \frac{s_{23}}{q^2}, \quad w = \frac{s_{31}}{q^2}, \quad u + v + w = 1,$ q is the momentum carried by the operator

$n = 3$ form factors have two kinematic degrees of freedom. At L loops the ansatz has 6^{2L} terms.

$$a_1 \otimes a_2 \otimes \ldots \qquad \ldots \otimes a_i \otimes a_{i+1} \otimes \ldots \qquad \ldots \otimes a_{2L-1} \otimes a_{2L}$$

First-entry conditions

~~$d \otimes$~~ $\ldots$ ~~$e \otimes$~~ $\ldots$ ~~$f \otimes$~~ $\ldots$

absence of non-physical discontinuities

Steinmann relations

$\ldots$ ~~$a \otimes d$~~ $\ldots$ $\ldots$ ~~$d \otimes a$~~ $\ldots$

$\ldots$ ~~$d \otimes e$~~ $\ldots$

+ dihedral images

related to **cluster algebras**

Final-entry conditions

$\mathrm{Tr}\,\phi^2$: $\ldots$ ~~$\otimes a$~~ $\ldots$ ~~$\otimes b$~~ $\ldots$ ~~$\otimes c$~~

$\mathrm{Tr}\,\phi^3$: $\ldots$ ~~$\otimes abc$~~

loosely related to $\bar{Q}$-equation (supersymmetric Ward identity)

Constraining the symbol

In terms of $x = \dfrac{-q^2}{2\,p_3 \cdot q}$ Bjorken variable, $u = \dfrac{x-1}{x}$

In Minkowski kinematics, $u < 0, v > 0, w > 0$

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$$a_1 \otimes a_2 \otimes \ldots \qquad \ldots \otimes a_i \otimes a_{i+1} \otimes \ldots \qquad \ldots \otimes a_{2L-1} \otimes a_{2L}$$

First-entry conditions

$\cancel{d \otimes} \ldots \quad \cancel{e \otimes} \ldots \quad \cancel{f \otimes} \ldots$

absence of non-physical discontinuities

Steinmann relations

$\ldots \cancel{a \otimes d} \ldots \quad \ldots \cancel{d \otimes a} \ldots$

$\ldots \cancel{d \otimes e} \ldots$

+ dihedral images

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Final-entry conditions

$\mathrm{Tr}\,\phi^2$: $\ldots \cancel{\otimes a} \quad \ldots \cancel{\otimes b} \quad \ldots \cancel{\otimes c}$

$\mathrm{Tr}\,\phi^3$: $\ldots \cancel{\otimes abc}$

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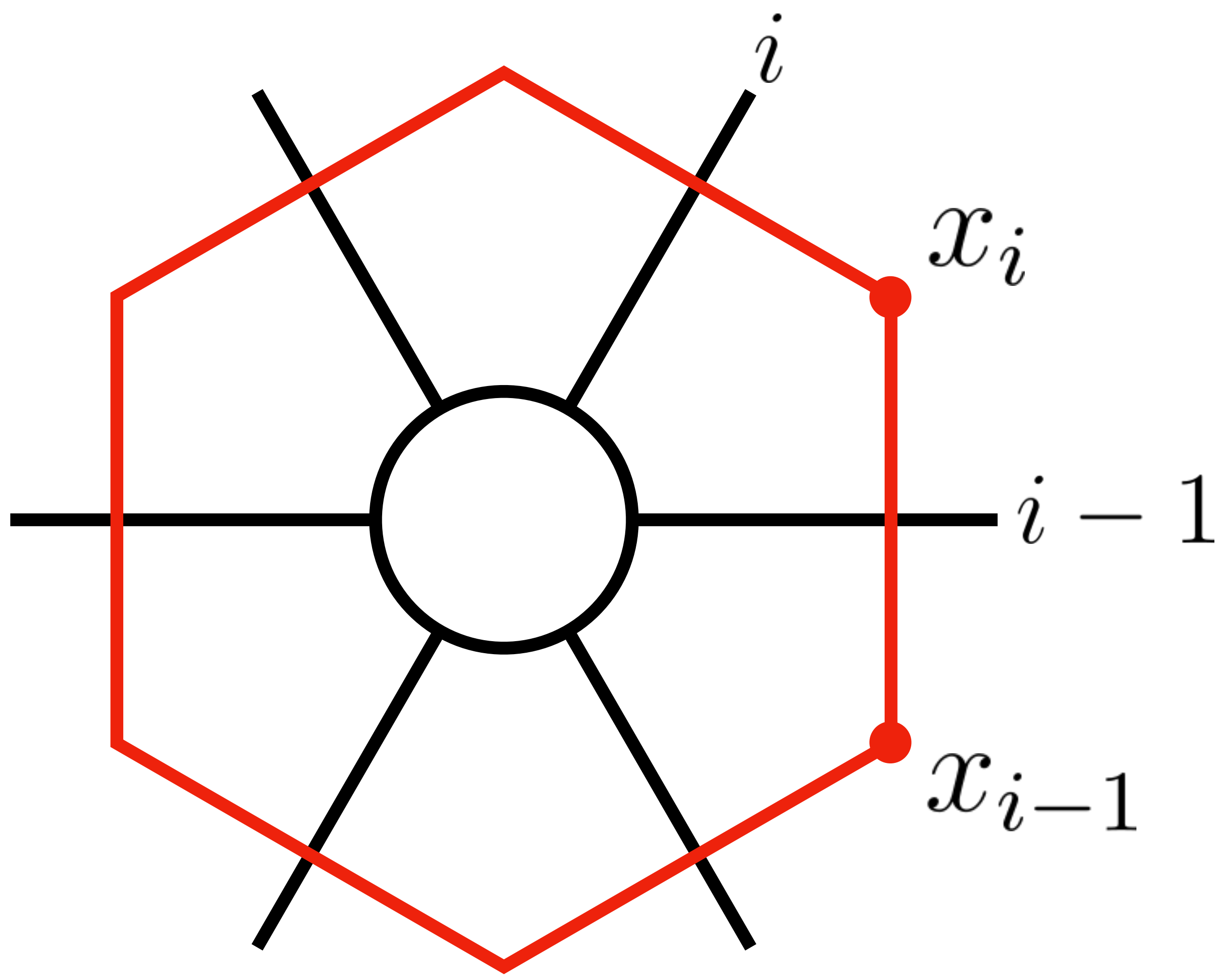
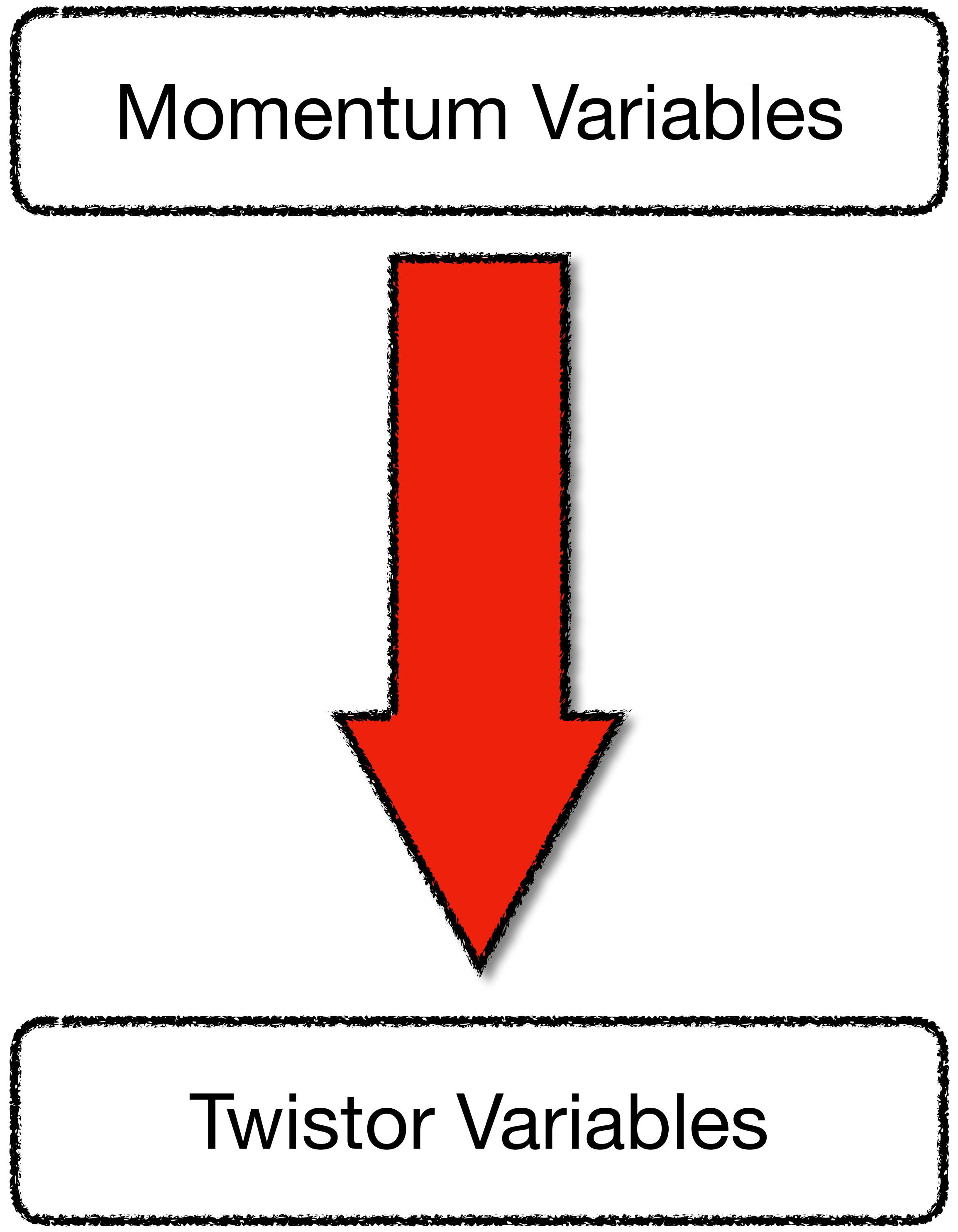
Relation to cluster algebras

n-point amplitude is described by the $Gr(4,n)$ cluster algebra.

Twistor space $Gr(4,n)$





$$p_i^2 = 0 \qquad \sum_{i=1}^{n} p_i = 0$$

$$x_i^{\alpha\dot\alpha} - x_{i-1}^{\alpha\dot\alpha} = p_i^{\alpha\dot\alpha} = \lambda_i^{\alpha}\tilde\lambda_i^{\dot\alpha}$$

$$Z_i = \begin{pmatrix} \lambda_i^{\alpha} \\ x_i^{\alpha\dot\alpha}\lambda_{i\alpha} \end{pmatrix}$$

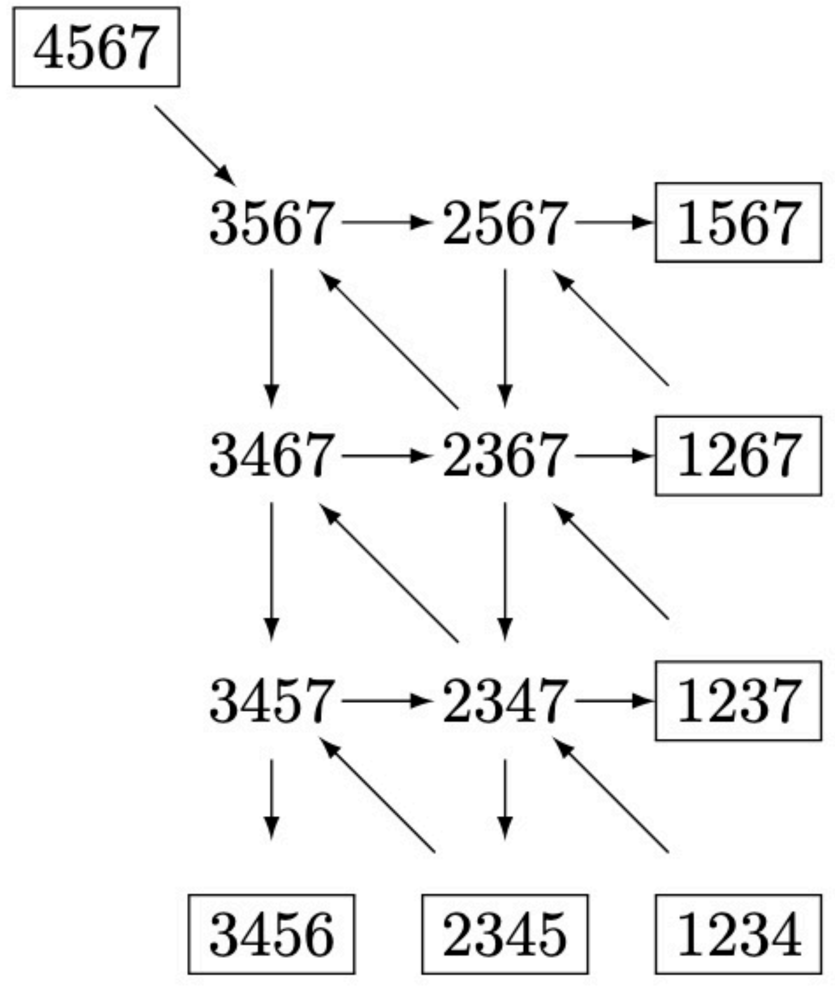
Plücker coordinates on $Gr(4,n)$: $\langle i\, j\, k\, l\rangle = \mathrm{Det}\{Z_i, Z_j, Z_k, Z_l\}$ (dual conformal invariant invariants)

Relation to cluster algebras

$Gr(4,n)$ cluster algebras

Plücker coordinates = cluster variables

Typical cluster for $Gr(4,7)$:



Mutation rule: $a_i \to \dfrac{\prod_{i\to j} a_j + \prod_{j \to i} a_j}{a_i}$

6pt two-loop MHV amplitude:

$$R_6^{(2)} = \sum_{\text{cyclic}} \text{Li}_4\left(-\frac{\langle 1234\rangle\langle 2356\rangle}{\langle 1236\rangle\langle 2345\rangle}\right) - \frac{1}{4}\,\text{Li}_4\left(-\frac{\langle 1246\rangle\langle 1345\rangle}{\langle 1234\rangle\langle 1456\rangle}\right)$$
$$+\,(\text{products of lower-transcendentality polylogs with the same arguments})$$

x-variables = cross ratios of a-variables

Definition: $x_i = \dfrac{\prod_{j\to i} a_j}{\prod_{i \to j} a_j}$

Mutation rule: $x_i \to \begin{cases} \dfrac{1}{x_i} & \text{if mutated on } i \\ x_i\left(1 + x_k^{-\text{sign}\{b_{ik}\}}\right)^{-b_{ik}} & \text{if mutated on } k \neq i \end{cases}$

Relation to cluster algebras

Amplitude function spaces are determined by the $Gr(4, n)$ cluster algebras

Cluster x-variables = Arguments of polylogarithms

Basis of cluster x-variables = Symbol alphabet

Two cluster variables are non-adjacent if they never occur in the same cluster

Cluster non-adjacency = Steinmann relations

For the form factors, we don't have the cluster algebra yet, but we do have the Steinmann relations!

Intermediate results

$\mathrm{Tr}\,\phi^2$

L	2	3	4	5	6	7	8
symbols in $\mathcal{C}$	48	249	1290	6654	34219	????	????
dihedral symmetry	11	51	247	1219	????	????	????
$(L-1)$ final entries	5	9	20	44	86	191	191
L^{th} discontinuity	2	5	17	38	75	171	164

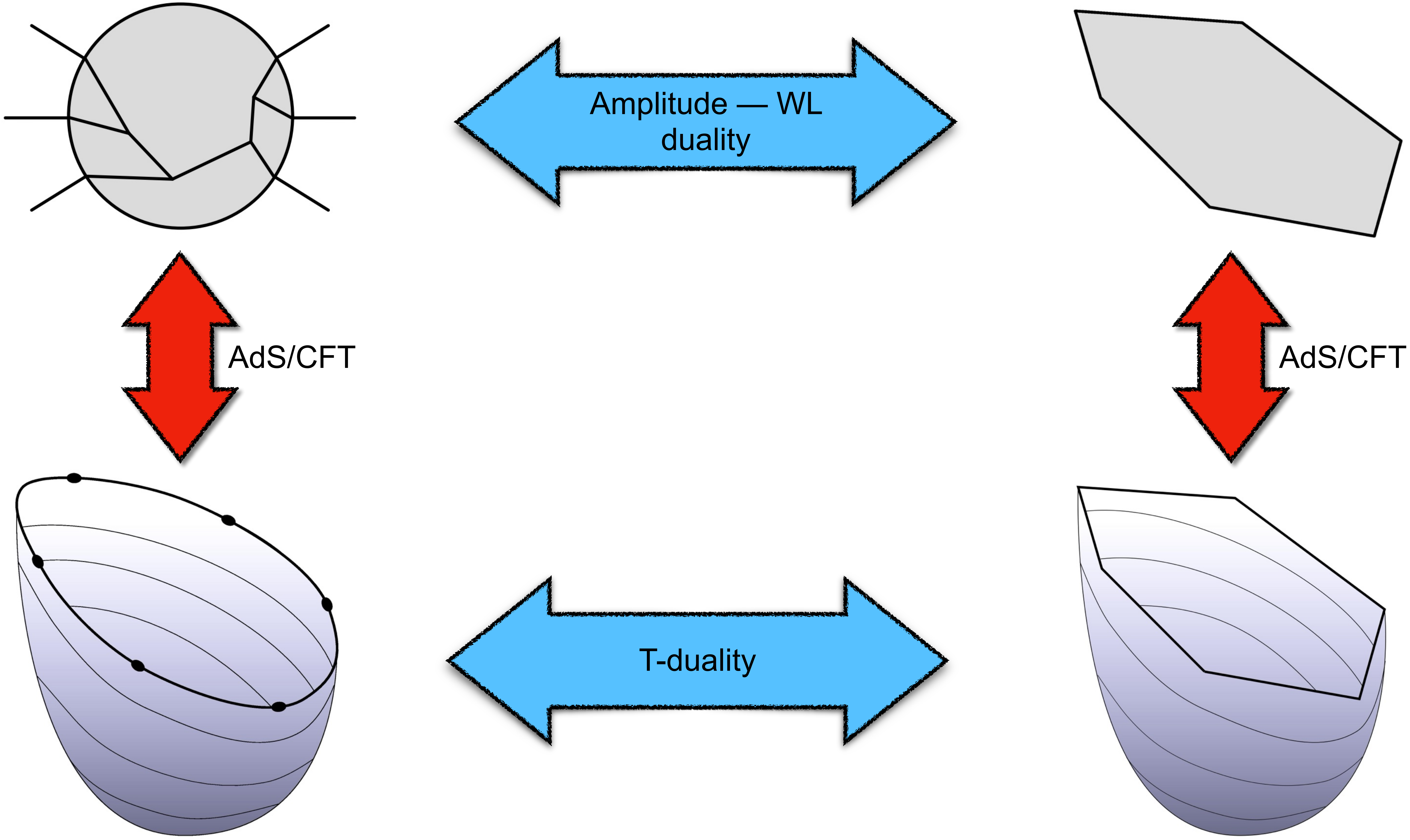
$\mathrm{Tr}\,\phi^3$

L	2	3	4	5	6
functions in $\mathcal{C}$	52	284	1495	$\sim$8000	?????
dihedral symmetry	13	63	302	$\sim$1400	????
$(L-1)$ final entries	4	15	47	190	407
$(L+1)^{\mathrm{st}}$ discontinuity	3	13	43	182	394

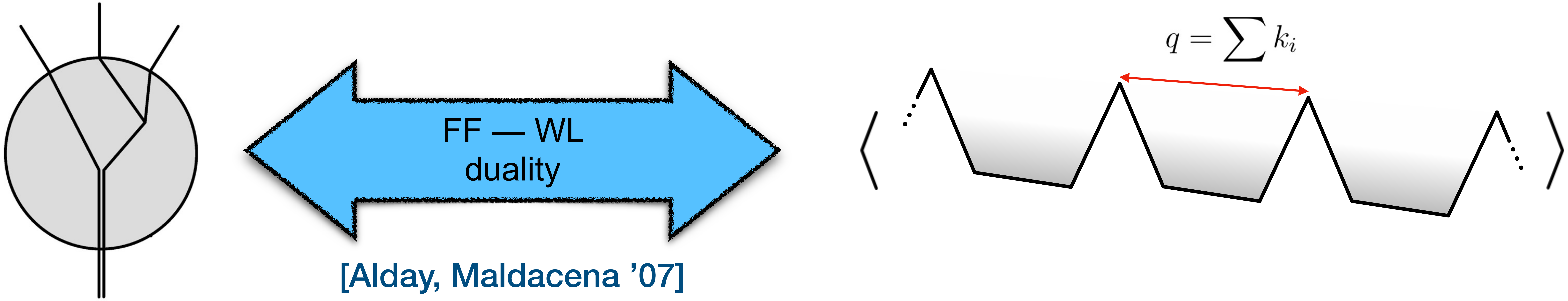
Integrability

Amplitude — Wilson Loop duality

[Alday, Maldacena '07]



Form Factor — Wilson loop duality for $\mathrm{Tr}\,\phi^2$

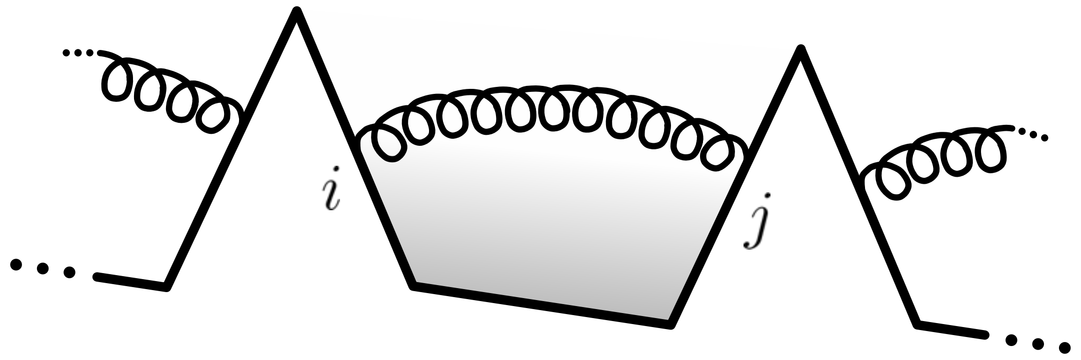


$\mathrm{Tr}\,\phi^2$

[Alday, Maldacena '07]

[Brandhuber, Spence, Travaglini, Yang '10]

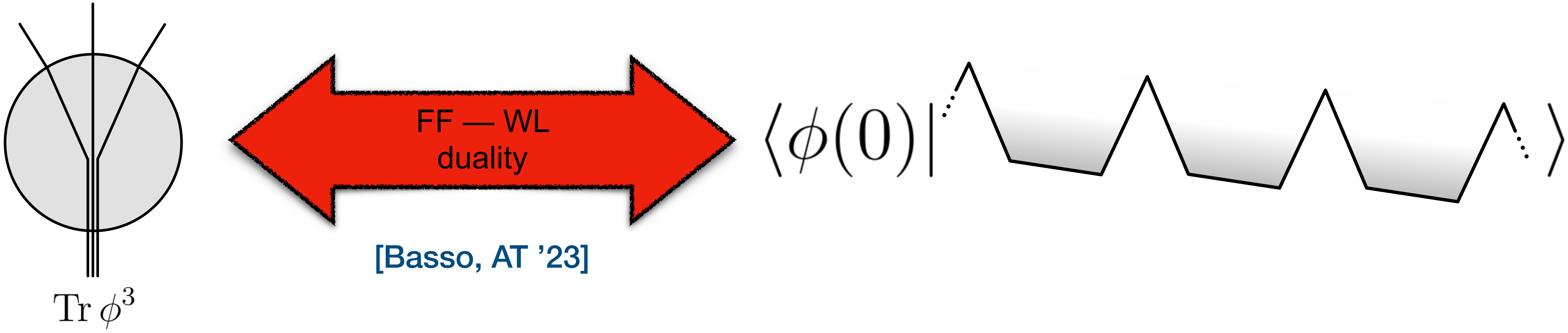
At one loop, corrections arise from dressing the Wilson loop with gluon exchanges between edges:

$$W_{2,n} = 1 + g^2 \sum_{i<j}$$  $$+ \mathcal{O}\left(g^4\right)$$

$\mathrm{Tr}\,\phi^k$

number of particles

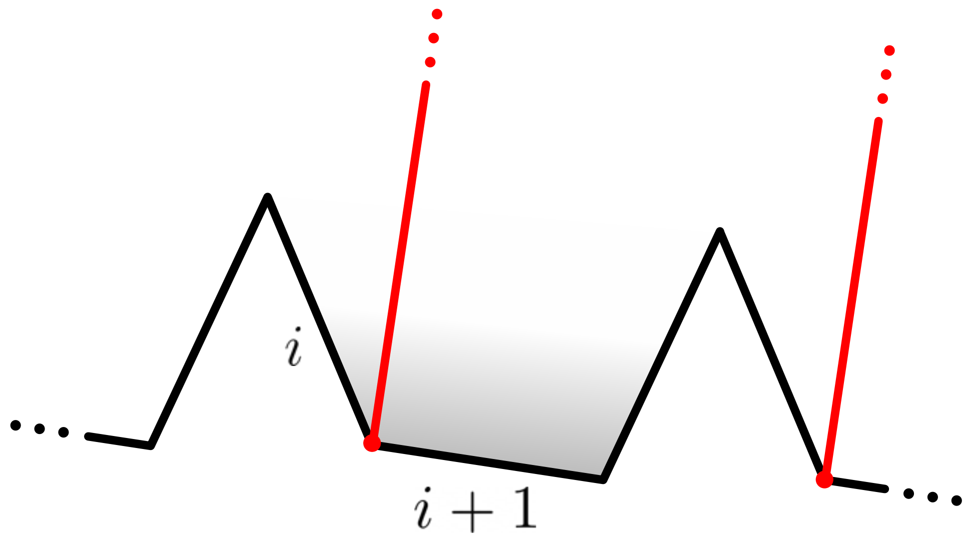
Form Factor — Wilson loop duality for $\mathrm{Tr}\,\phi^3$



[Basso, AT ’23]

To compensate for the charge at the infinity, the Wilson loop needs to be “charged” accordingly. [Caron-Huot ’10]

$$W_{3,n} = \sum_i$$



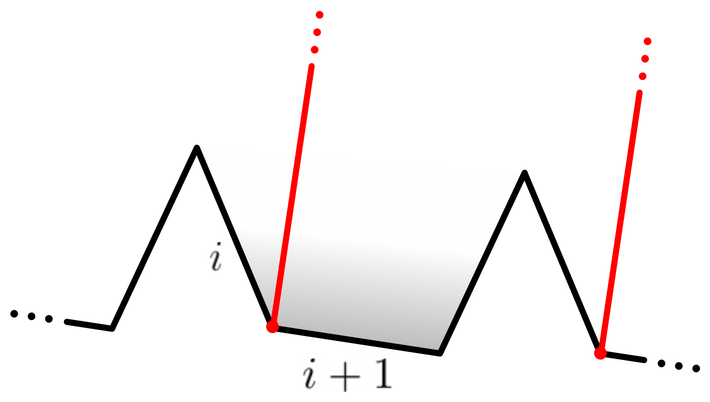
$$+\,\mathcal{O}\left(g^2\right)$$

In the general case of $\mathrm{Tr}\,\phi^k$, the asymptotic state consists of $k-2$ zero-momentum scalars.

$m = 2$ amplituhedron

Similar to [Caron-Huot, Coronado, Muhlmann '23]

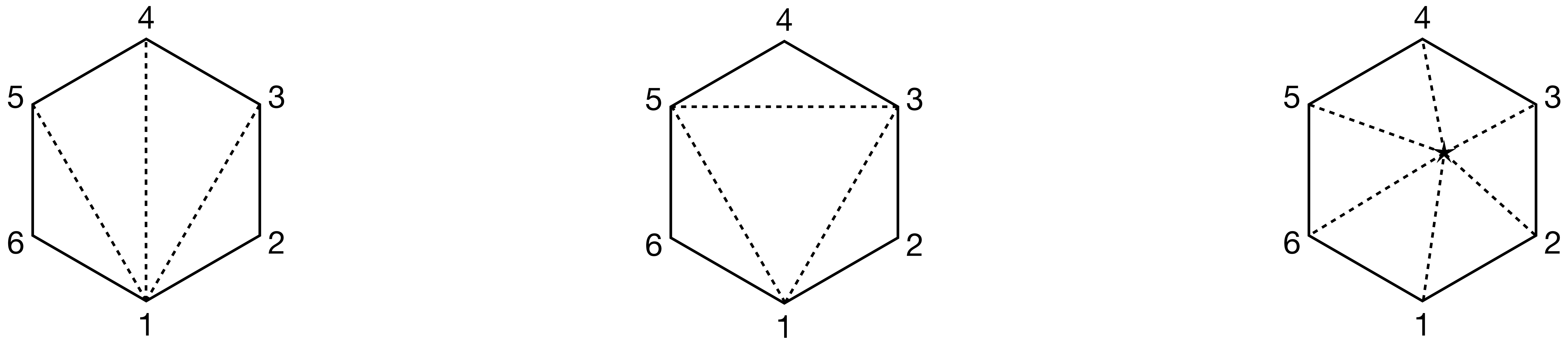
$$W_{3,n}^{\text{tree}} = \sum_{i=1}^{n}$$



$$= -\sum_{i=2}^{n-1} (1ii+1) \quad \text{where} \quad (ijk) = \frac{\delta^{0|2}(\langle ij\rangle \eta_k^- + \langle jk\rangle \eta_i^- + \langle ki\rangle \eta_j^-)}{\langle ij\rangle\langle jk\rangle\langle ki\rangle}$$

These 3-brackets are $2D$ versions of the standard R-invariant $[ijklm] = \dfrac{\delta^{0|4}\left(\langle [ijkl\rangle \eta_{m]}\right)}{\langle ijkl\rangle\langle jklm\rangle\langle klmi\rangle\langle lmij\rangle\langle mijk\rangle}$

This result is nothing but a triangulation of a polygon:



$$W_{3,n}^{\text{tree}} = -\sum_{i=2}^{n-1}(1ii+1) \qquad W_{3,n}^{\text{tree}} = -\sum_{T\in\mathcal{T}_n}(T_1T_2T_3) \qquad W_{3,n}^{\text{tree}} = -\sum_{i=1}^{n}(*ii+1)$$

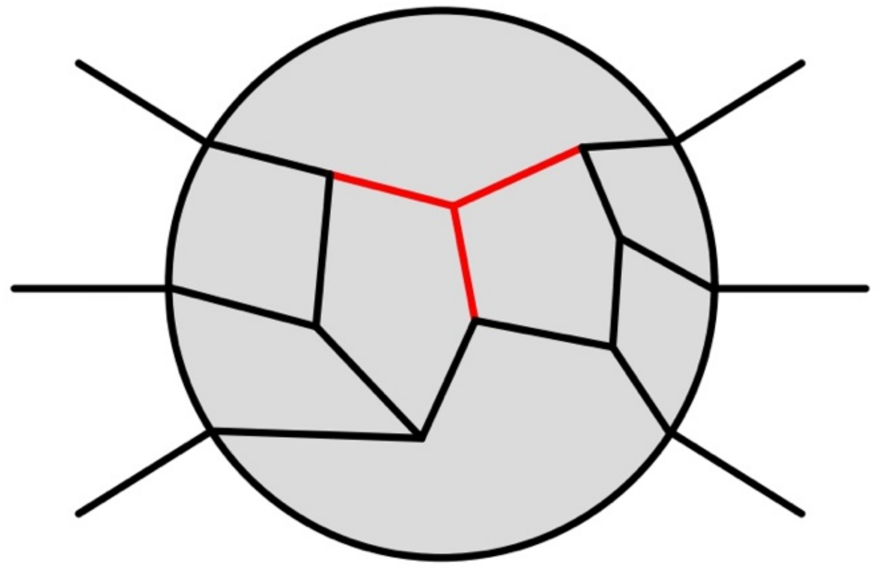
For general k we find $W_{k,n}^{\text{tree}} = \dfrac{1}{(k-2)!}\left(W_{3,n}^{\text{tree}}\right)^{k-2}$. This forms an amplituhedron $A_{m,n,k'}$ with $m = 2$ and $k' = k-2$.

Wilson Loop OPE & Form Factor OPE

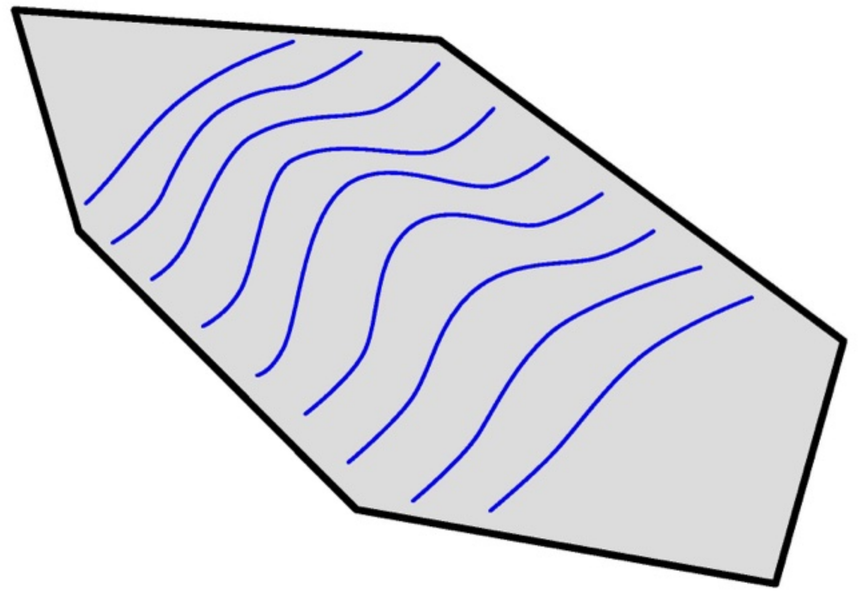
[Alday, Gaiotto, Maldacena, Sever, Vieira ’11]

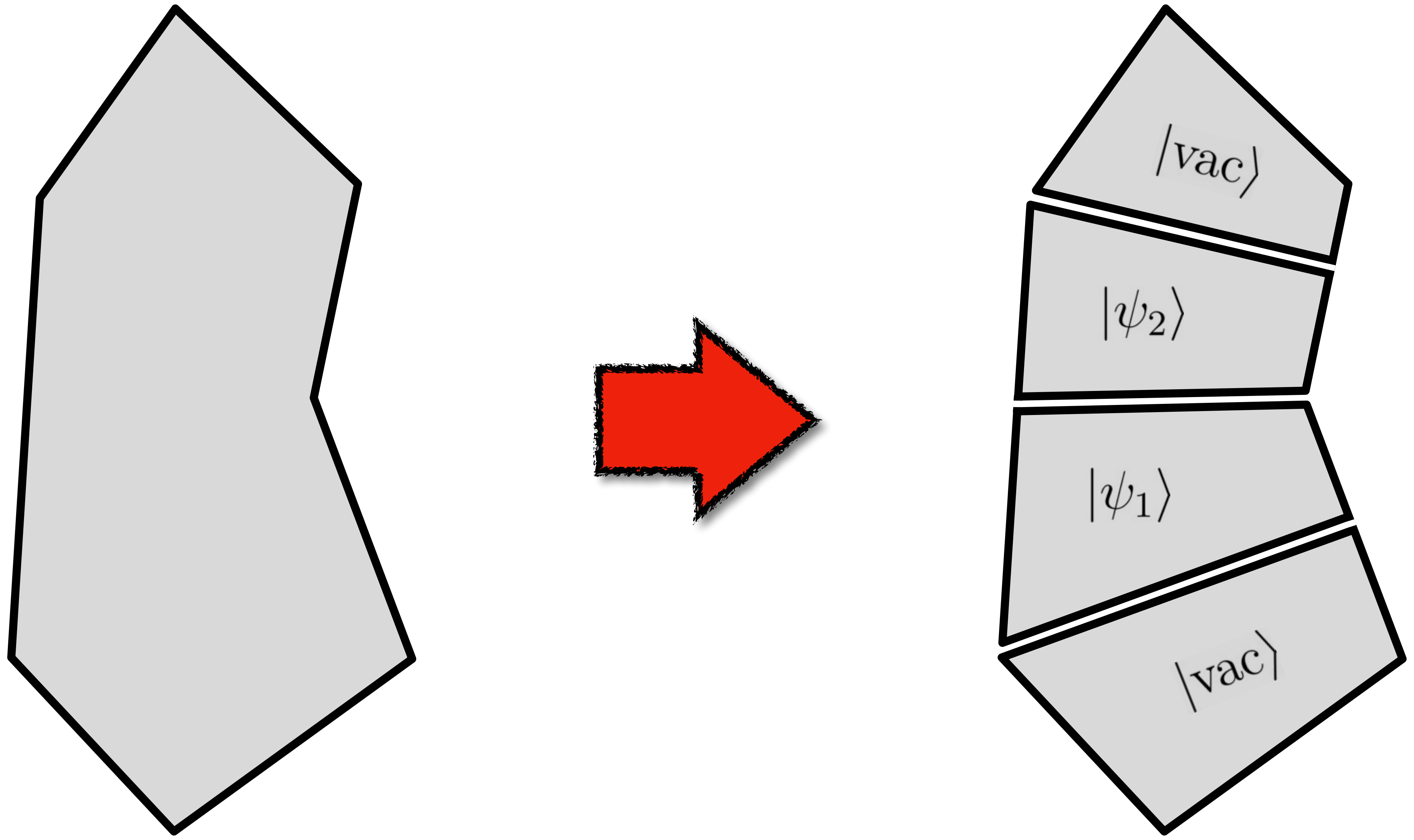
[Sever, AT, Wilhelm ’20]

$$\sum_{L=0}^{\infty} g^{2L}$$

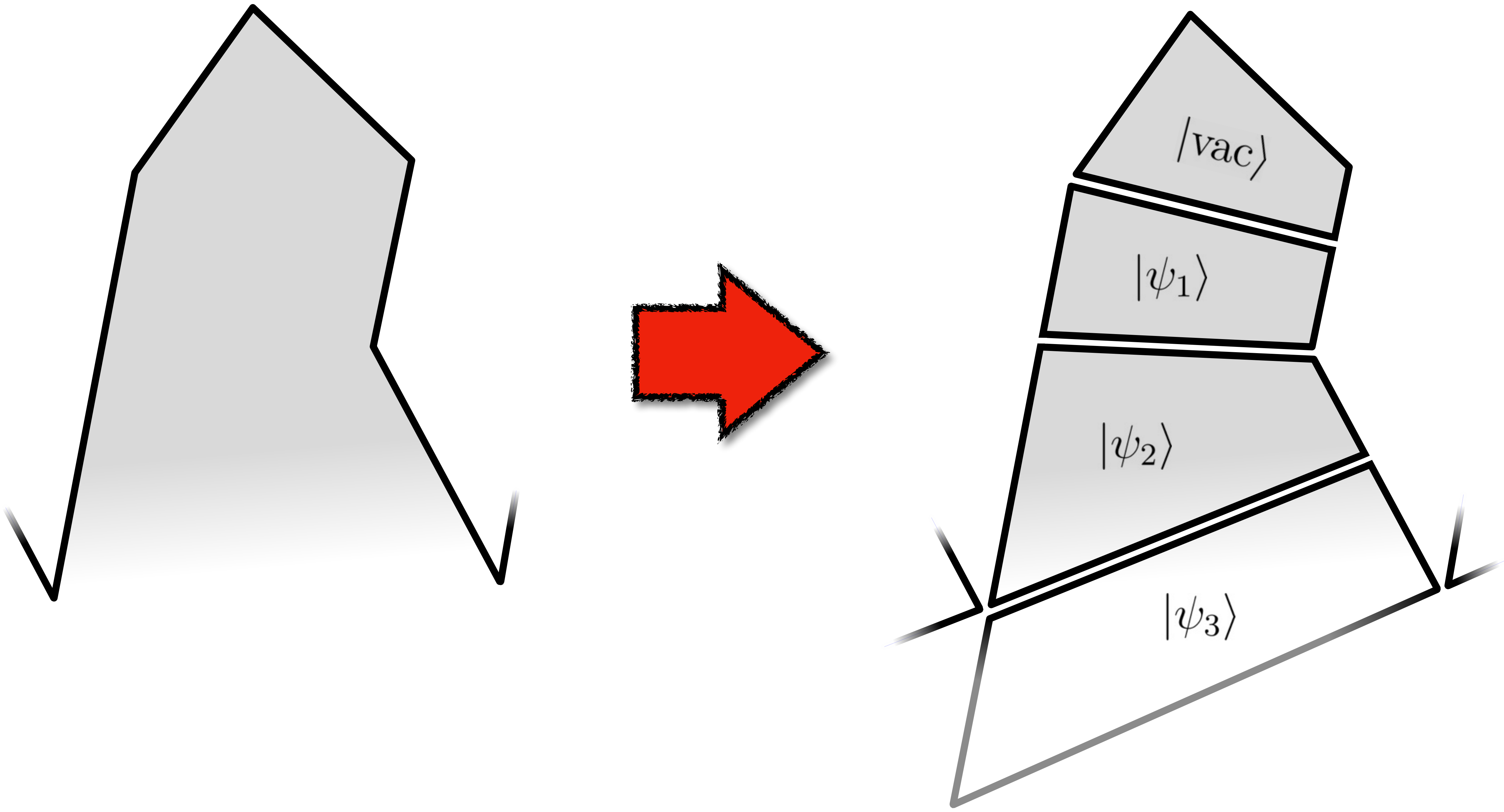


$$= \sum_{\psi} e^{-E(\psi)\tau}$$





$$\mathcal{W}_7 \sim \sum_{\psi_i} P(0|\psi_1)\, P(\psi_1|\psi_2)\, P(\psi_2|0)$$



$$\mathcal{W}_{5,\mathcal{O}} \sim \sum_{\psi_i} P(0|\psi_1)\, P(\psi_1|\psi_2)\, P(\psi_2|\psi_3)\, F_{\mathcal{O}}(\psi_3)$$

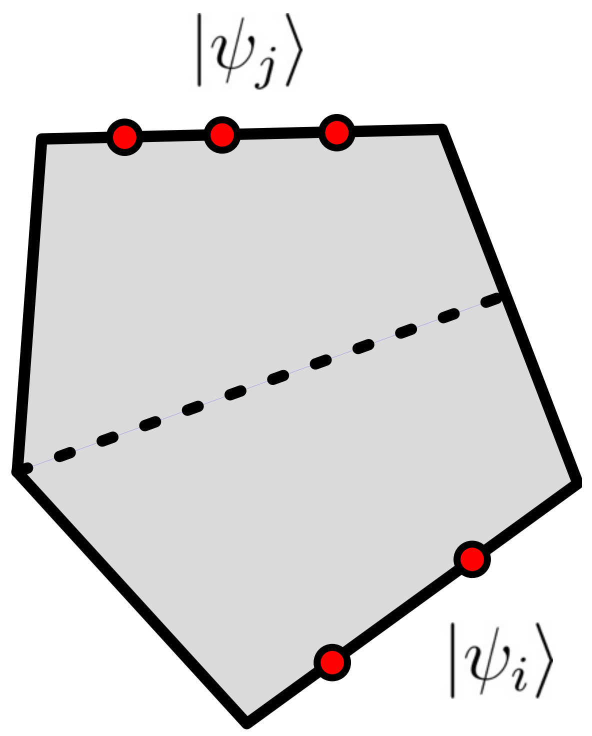
Pentagon & form factor transitions

States can be thought of as field insertions on the edges of the Wilson loop. They are characterized by the number of particles, their species, and rapidity (momentum):

$$|\psi\rangle = |\{a_1, u_1\}, \ldots, \{a_n, u_n\}\rangle$$

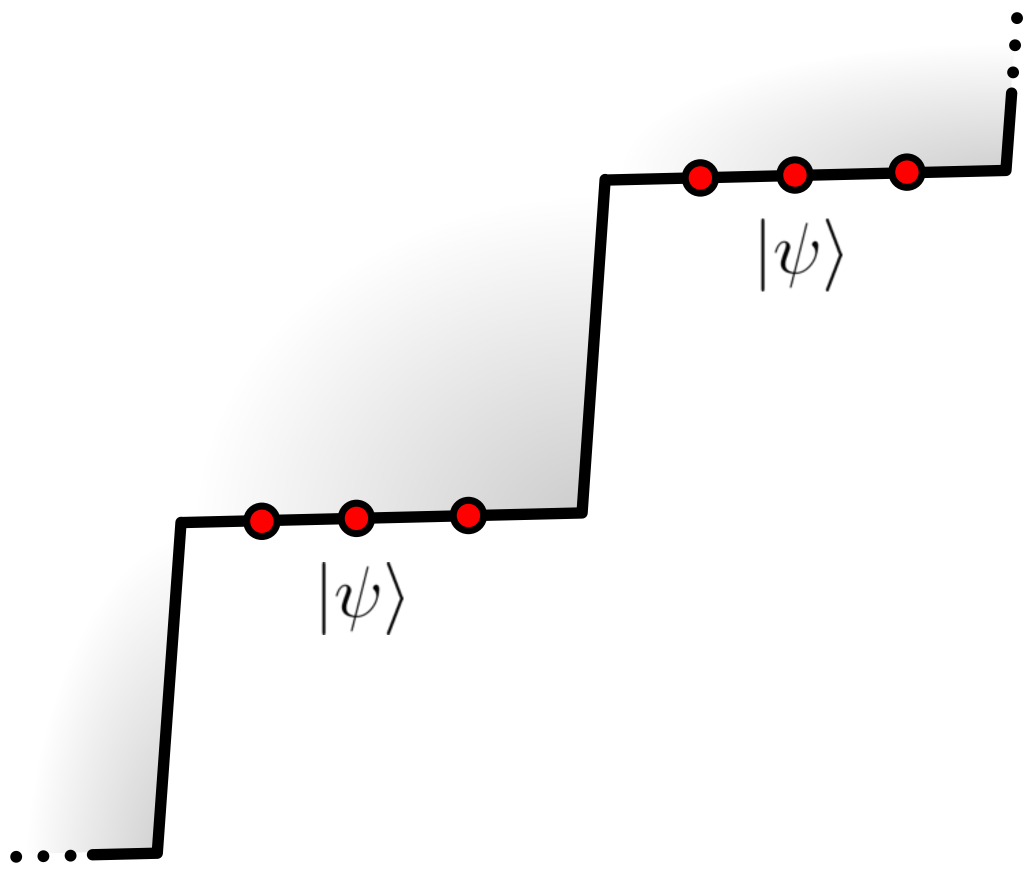
where $a_i = \underbrace{\phi,\ \psi,\ \bar{\psi},\ F,\ \bar{F}}_{\text{twist 1}},\ \underbrace{DF,\ D\bar{F},\ \bar{D}F,\ \bar{D}\bar{F}}_{\text{twist 2}},\ D^2F,\ \ldots$

Pentagon transition $P(\psi_i|\psi_j)$



[Basso, Sever, Vieira '13 - '14]

Form factor transition $F_{\mathcal{O}}(\psi)$



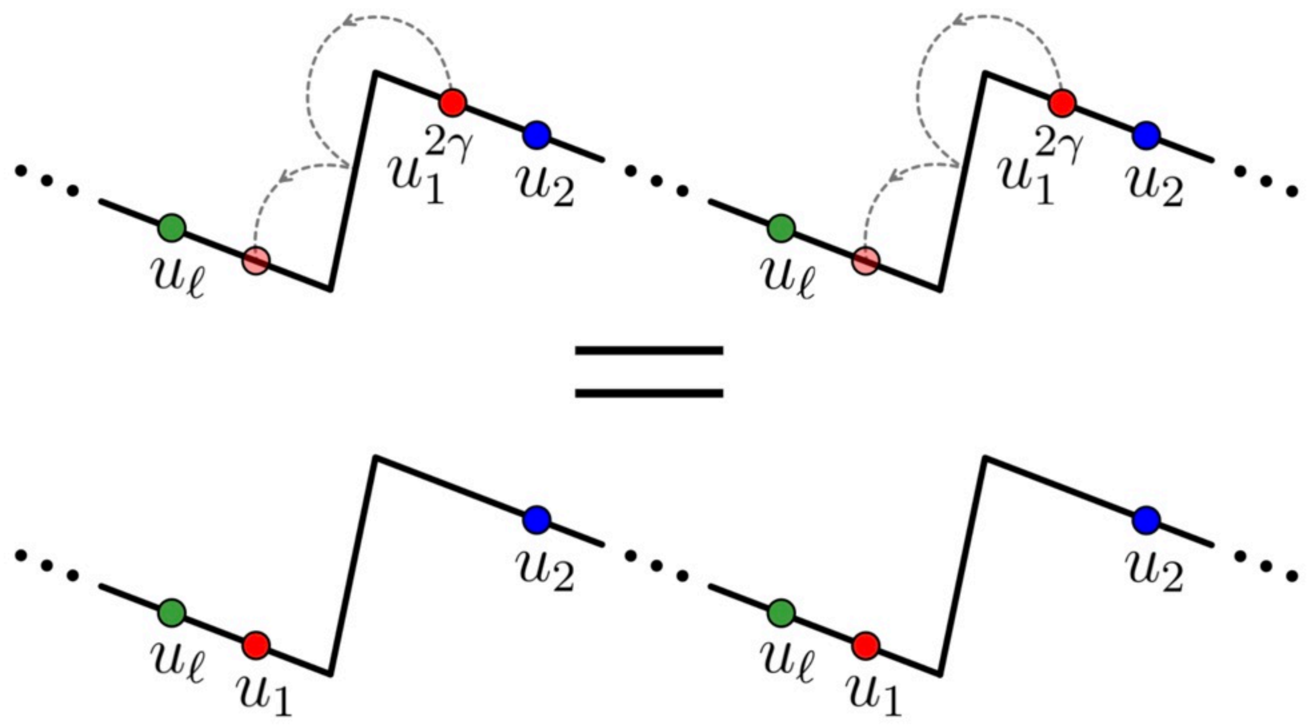
Other types of constraints

1 Watson relation $F(\ldots, u_i, u_{i+1}, \ldots) = S(u_i, u_{i+1})\, F(\ldots, u_{i+1}, u_i, \ldots)$

2 Crossing symmetry

Mirror transformation ($u \to u^{\gamma} = u + i$) shifts an excitation to the neighbouring edge. If after a certain number of these the object goes back to itself, it's crossing symmetric.

$$F\left(u_1^{2\gamma}, \ldots, u_\ell\right) = F\left(u_2, \ldots, u_\ell, u_1\right)$$



3 Reflection symmetry, square limit, etc…

Main building block [for scalar excitations]

$$\mathbb{K}_{ij} = 2j(-1)^{ij+j} \int_0^\infty \frac{dt}{t} \frac{J_i(2gt) J_j(2gt)}{e^t - 1}$$

[Beisert, Eden, Staudacher ’07]

$\alpha-$tilt

$$\mathbb{K}(\alpha) = 2\cos(\alpha) \begin{pmatrix} \cos(\alpha)\mathbb{K}_{\circ\circ} & \sin(\alpha)\mathbb{K}_{\circ\bullet} \\ \sin(\alpha)\mathbb{K}_{\bullet\circ} & \cos(\alpha)\mathbb{K}_{\bullet\bullet} \end{pmatrix}$$

[Basso, Dixon, Papathansiou ’20]

The transitions are built by defining the phase factors:

$$f_a^{[\alpha]}(u,v) = \frac{1}{\cos^2\alpha} \left[\kappa_\alpha^u \, \mathbb{Q} \frac{1}{1+\mathbb{K}(\alpha)} \tilde{\kappa}_\alpha^v - \tilde{\kappa}_\alpha^u \, \mathbb{Q} \frac{1}{1+\mathbb{K}(\alpha)} \kappa_\alpha^v \right]$$

$$f_s^{[\alpha]}(u,v) = \frac{1}{\cos^2\alpha} \left[\kappa_\alpha^u \, \mathbb{Q} \frac{1}{1+\mathbb{K}(\alpha)} \kappa_\alpha^v - \tilde{\kappa}_\alpha^u \, \mathbb{Q} \frac{1}{1+\mathbb{K}(\alpha)} \tilde{\kappa}_\alpha^v \right]$$

fixed functions

Tilted transition:

$$P_{\phi\phi}^{[\alpha]}(u|v) = \frac{\Gamma(iu - iv)}{g^2\, \Gamma\left(\frac{1}{2} + iu\right) \Gamma\left(\frac{1}{2} - iv\right)} \exp\left[J_\phi(u) + J_\phi(-v) + i f_a^{[\pi/4]}(u,v) + f_s^{[\alpha]}(u,v) \right]$$

pentagon transition

$\alpha = \pi/4$

$\alpha = 0$

form factor transition

$$P_{\phi\phi}(u|v) = P_{\phi\phi}^{[\pi/4]}(u|v) \qquad Q_{\phi\phi}(u|v) = P_{\phi\phi}^{[0]}(u|v)$$

Back to bootstrapping: imposing the constraints

$\mathrm{Tr}\,\phi^2$

L	2	3	4	5	6	7	8
symbols in $\mathcal{C}$	48	249	1290	6654	34219	????	????
dihedral symmetry	11	51	247	1219	????	????	????
$(L-1)$ final entries	5	9	20	44	86	191	191
L^{th} discontinuity	2	5	17	38	75	171	164
collinear limit	0	1	2	8	19	70	6
OPE $T^2 \ln^{L-1} T$	0	0	0	4	12	56	0
OPE $T^2 \ln^{L-2} T$	0	0	0	0	0	36	0
OPE $T^2 \ln^{L-3} T$	0	0	0	0	0	0	0
OPE $T^2 \ln^{L-4} T$	0	0	0	0	0	0	0
OPE $T^2 \ln^{L-5} T$	0	0	0	0	0	0	0

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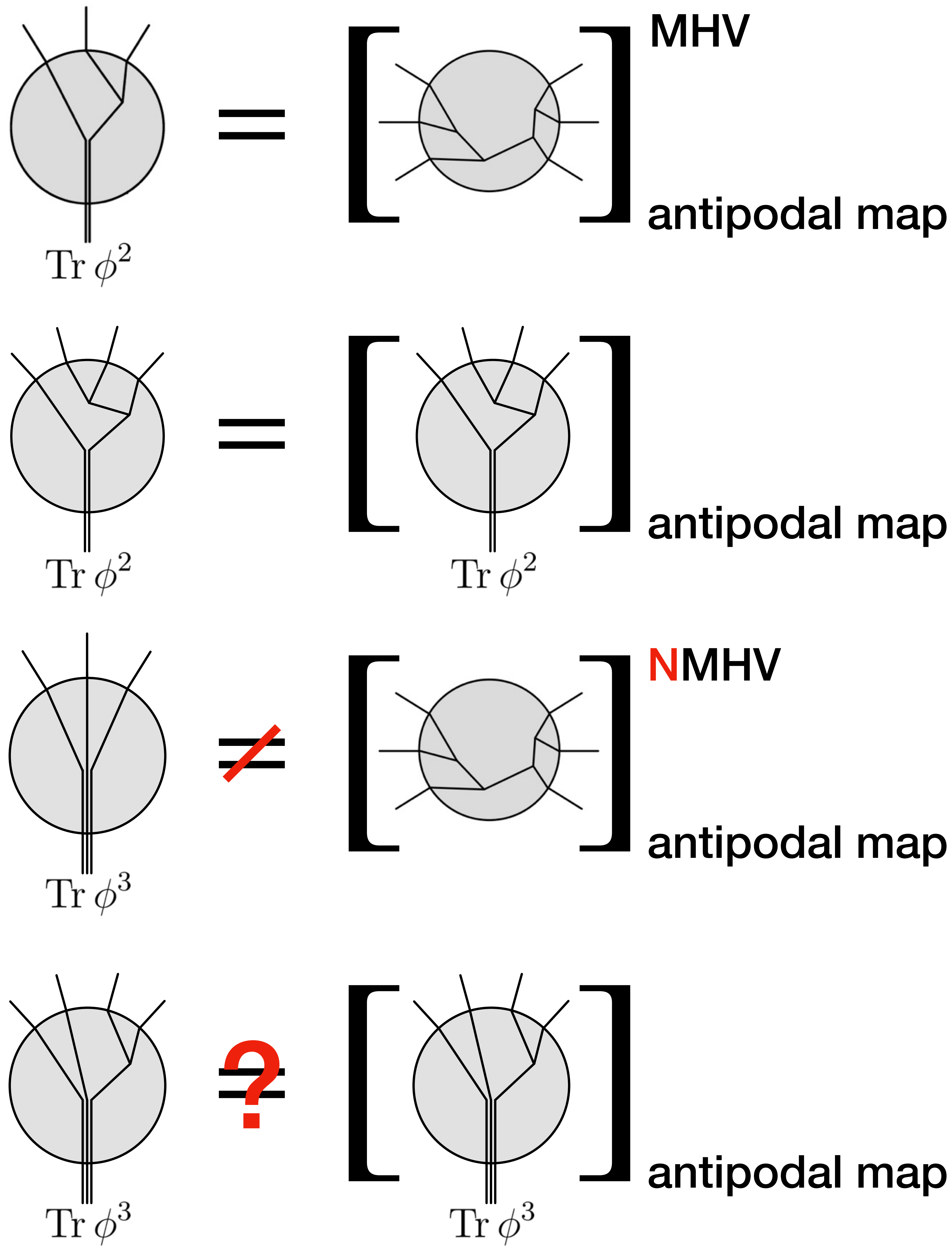
OPE $T^1 \ln^{L-1} T$	1	6	31	158	???
OPE $T^1 \ln^{L-2} T$	0	2	20	137	322
OPE $T^1 \ln^{L-3} T$	0	0	4	103	272
OPE $T^1 \ln^{L-4} T$	0	0	0	50	190
OPE $T^1 \ln^{L-5} T$	0	0	0	0	64
OPE $T^1 \ln^{L-6} T$	0	0	0	0	0

Antipodal duality



[Dixon, Gurdogan, McLeod, Wilhelm ’21]

[Dixon, Gurdogan, Liu, McLeod, Wilhelm ’22]

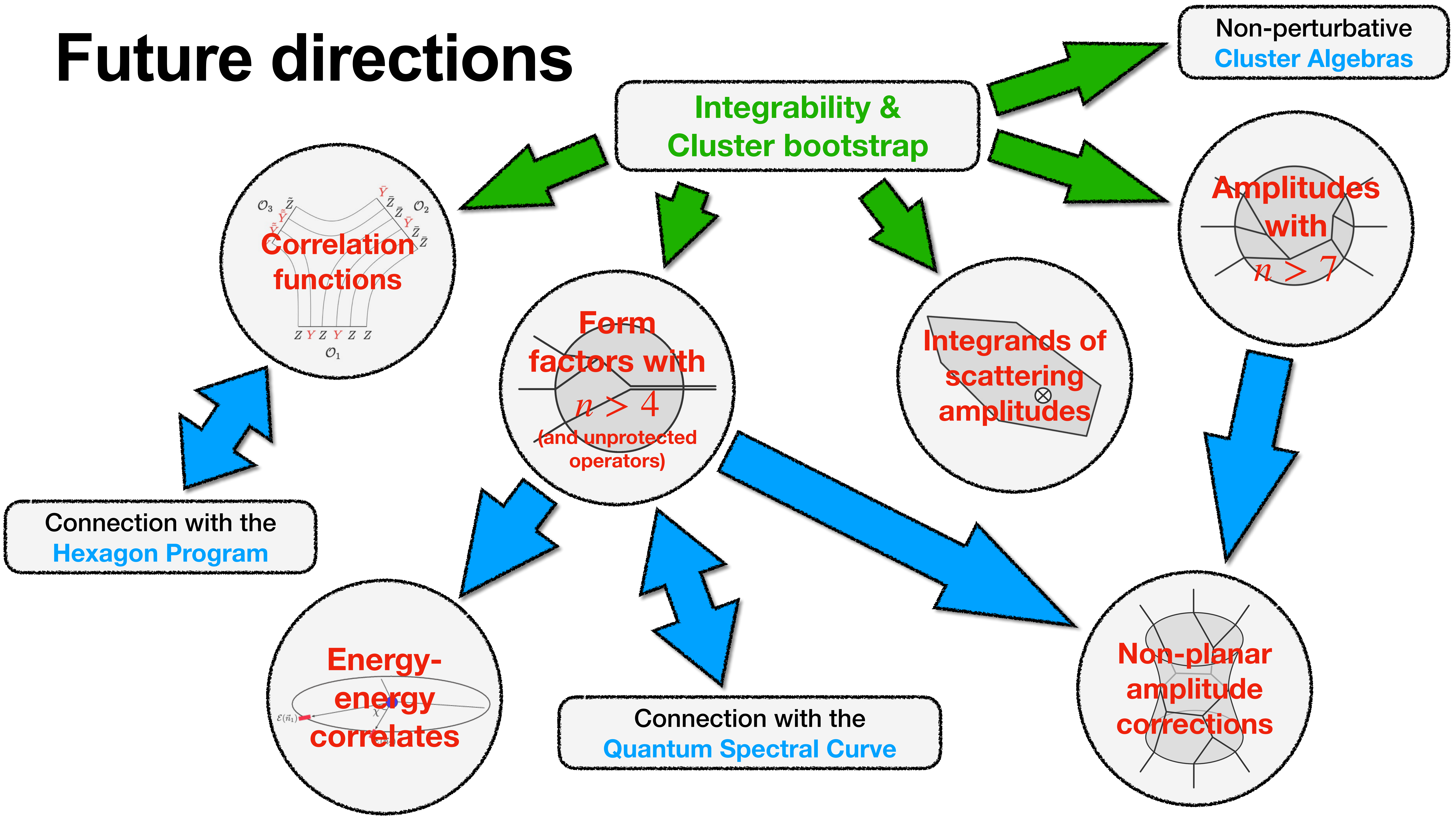
Conclusions

1. Form factors of all BPS operators can now be computed non-perturbatively.

2. Three- and four-loop form factors of $\mathrm{Tr}\,\phi^2$ and $\mathrm{Tr}\,\phi^3$ can now be bootstrapped through high-loop orders.

3. We have constructed new types of OPE building blocks that corresponding to tilted BES kernels.

Future directions



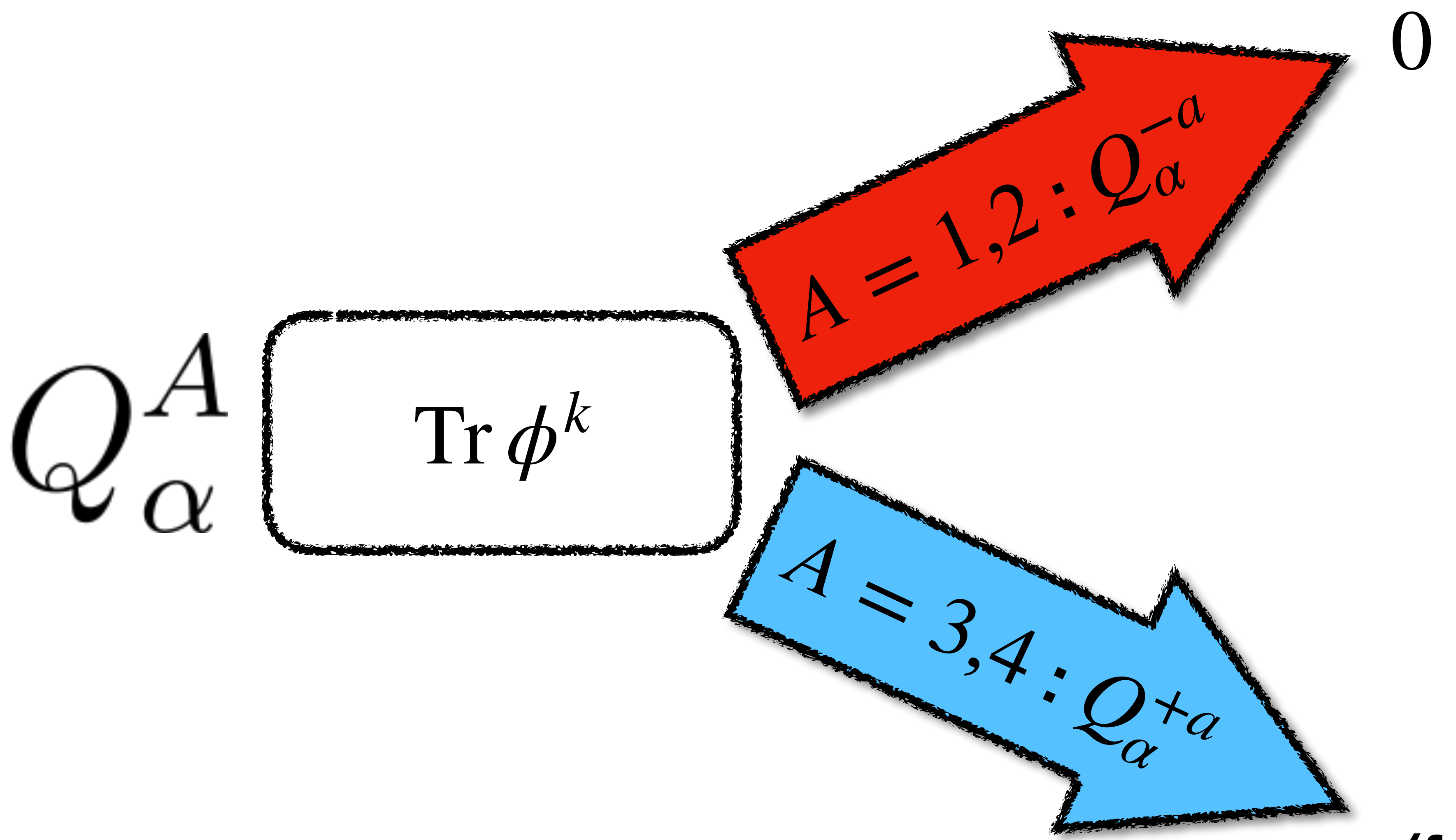
Thank you!

Form factors of $1/2-$BPS operators

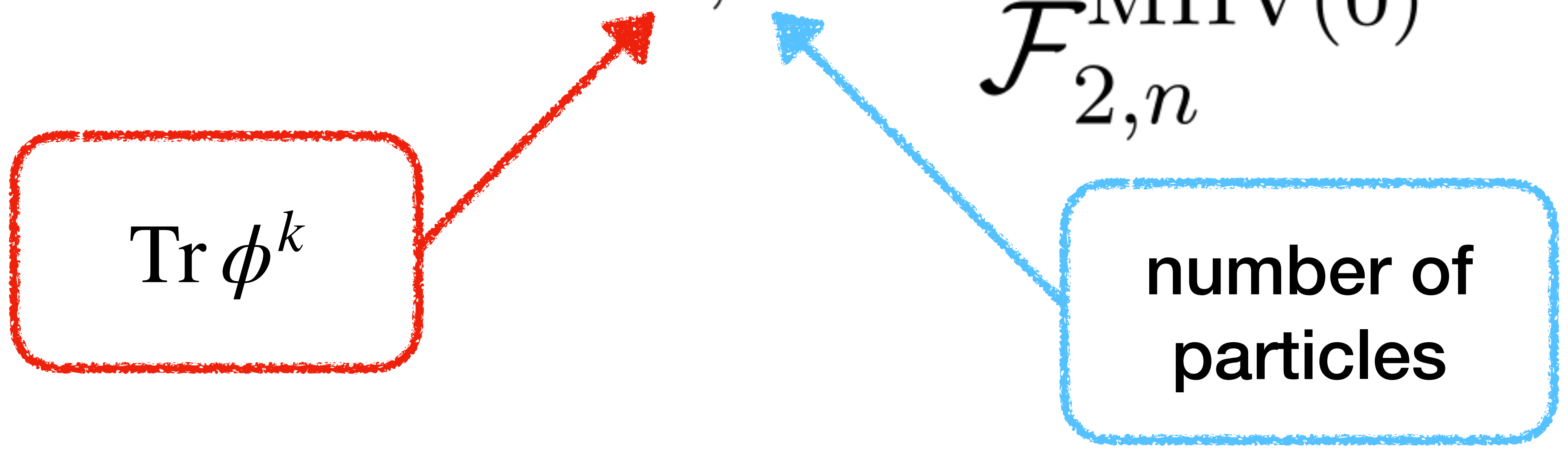
The simplest class of local operators $\mathcal{N}=4$ SYM are the [protected] $1/2$-BPS operators.

$$\mathrm{Tr}\,\phi_{AB}^2 \qquad \mathrm{Tr}\,\phi_{AB}^3 \qquad \ldots \qquad \mathrm{Tr}\,\phi_{AB}^k \qquad \ldots$$

Without any loss of generality we may simply fix A, B. From now on these indexes will be omitted.



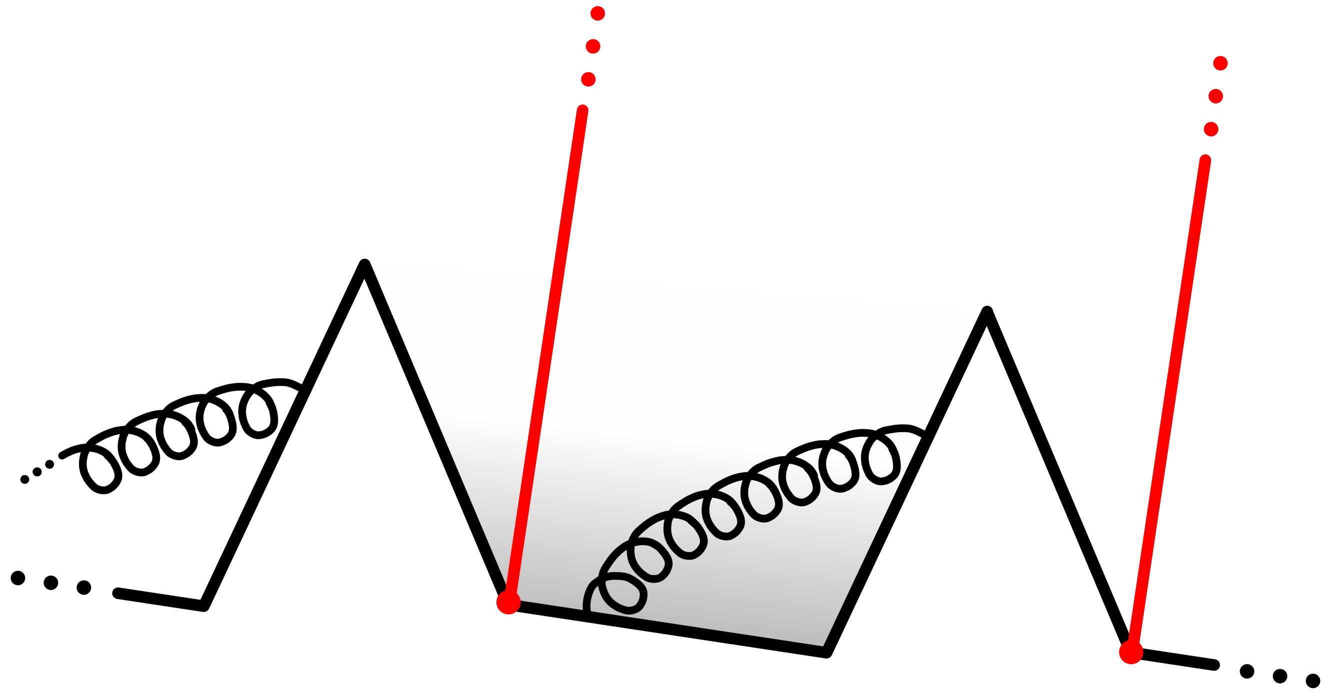
$$W_{k,n} = \frac{\mathcal{F}_{k,n}}{\mathcal{F}_{2,n}^{\mathrm{MHV}(0)}}$$



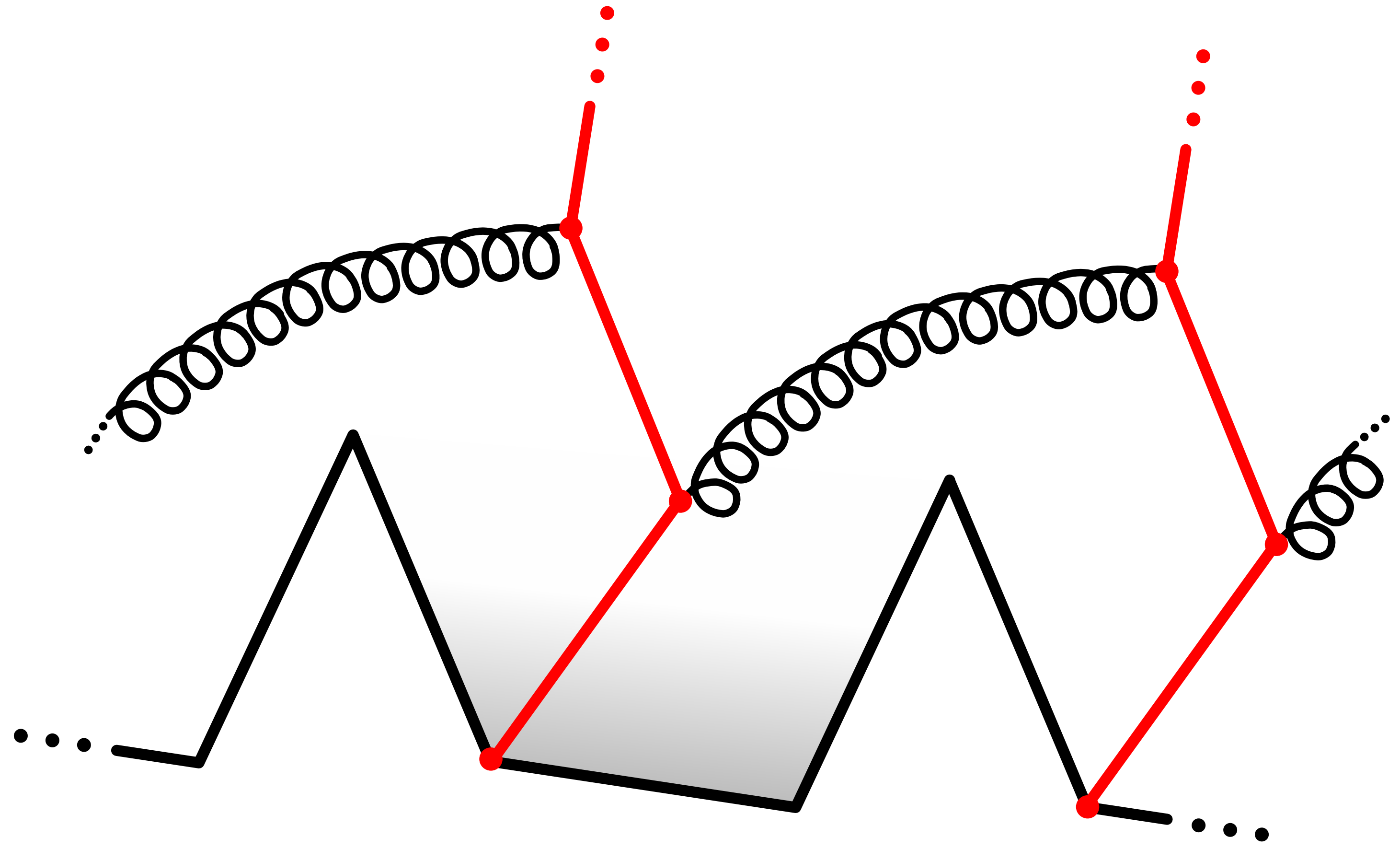
$1/2-$BPS chiral supermultiplet
(for $k=2$ it contains the chiral Lagrangian)

$\mathrm{Tr}\,\phi^3$ one-loop check

We also tested this duality at one loop. Two types of diagrams need to be considered:



Typical 1-loop form factor corrections



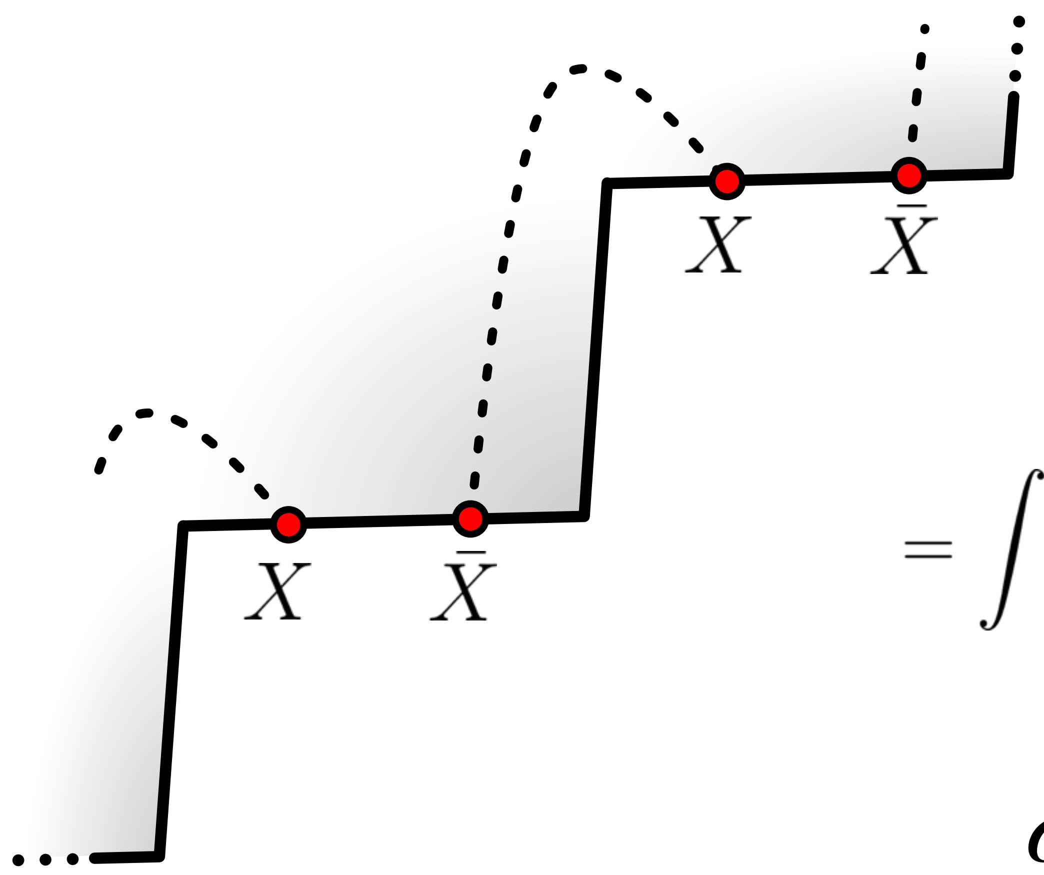
Operator renormalisation diagrams

Because the operator is protected, diagrams of the second kind add up to zero. The diagrams of the first kind add up perfectly to the expected result.

Leading Born-level Form factor transitions

$\mathrm{Tr}\,\phi^2$ The leading contribution comes from the vacuum state, the subleading comes from the singlet states:

$$F^{(0)}_{\mathrm{Tr}\,\phi^2}(\{X,u_1\},\{\bar{X},u_2\}) =$$



$$= \int d\sigma_1 d\sigma_2\, \Psi_{X\bar{X}}(\sigma_1,\sigma_2|u_1,u_2)\,\left(\text{propagator between } x(\sigma_2) \text{ and } x^{[+]}(\sigma_1)\right)$$

σ_1 and σ_2 parametrize the locations of the insertions

The results:

$$F_{F\bar{F}}(u,v) = -1 \times \frac{2}{g^2}\left(u^2+\tfrac{1}{4}\right)\cosh(\pi u)\,\delta(u-v)$$

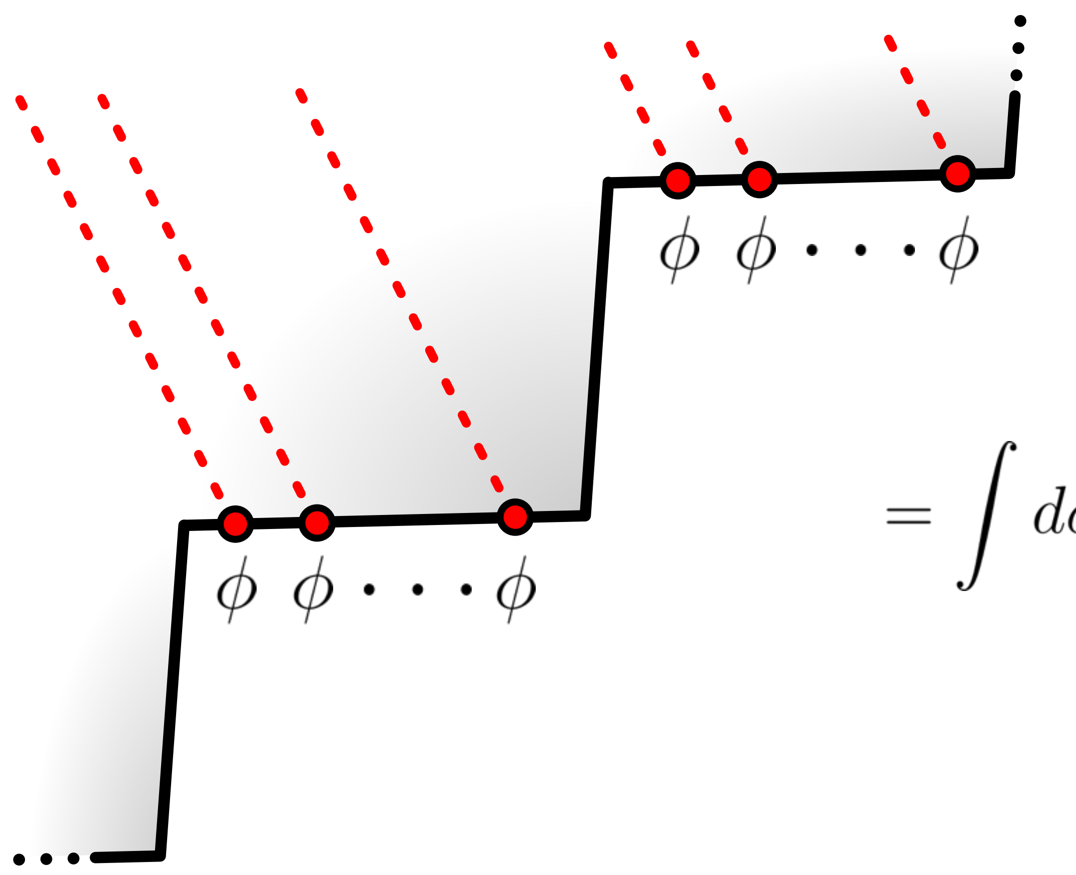
$$F_{\psi\bar{\psi}}(u,v) = +4 \times \frac{2}{g^2}\, u\sinh(\pi u)\,\delta(u-v)$$

$$F_{\phi\bar{\phi}}(u,v) = -6 \times \frac{4}{g^2\,(u-v-2i)\,(u-v-i)}\,\frac{\Gamma\,(iu-iv)}{\Gamma\left(\tfrac{1}{2}+iu\right)\Gamma\left(\tfrac{1}{2}-iv\right)}$$

Leading Born-level Form factor transitions

$\mathrm{Tr}\,\phi^k$ The leading contribution comes from $k-2$ identical scalars:

$$F^{(0)}_{\mathrm{Tr}\,\phi^k}(\{\phi,u_1\},\ldots,\{\phi,u_{k-2}\}) =$$



$$= \int d\sigma_1 \ldots d\sigma_{k-2}\, \Psi_{\phi\ldots\phi}(\sigma_1,\ldots,\sigma_{k-2}|u_1,\ldots,u_{k-2})$$

The results:

$$F_\phi(u) = 1$$

$$F_{\phi\phi}(u,v) = g^2\,\frac{\Gamma\left(\frac{1}{2}-iu\right)\Gamma\left(\frac{1}{2}+iv\right)}{\Gamma\left(iv-iu\right)}$$