Poisson Geometry and Representation Theory of Cluster Algebras

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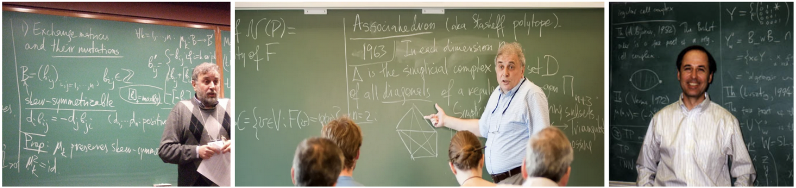
joint work with Shengnan Huang (Northeastern Univ), Thang Lê (Georgia Tech), Greg Muller (Univ Oklahoma), Bach Nguyen (Xavier Univ) and Kurt Trampel (NSA)

Workshop on Integrability, Q-systems and Cluster Algebras
Varna, August 12-17, 2024

Fomin and Zelevinsky



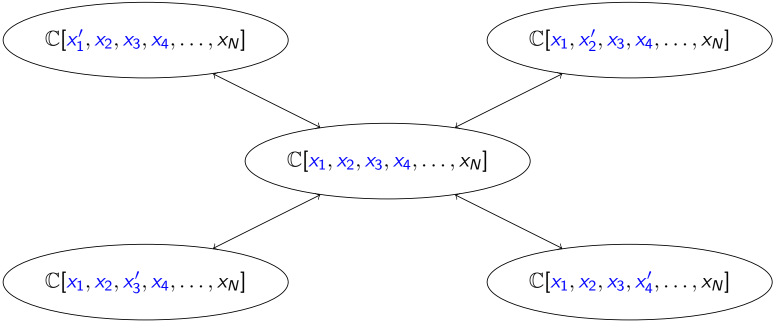
Cluster Algebras were conceived by Sergey Fomin and Andrey Zelevinsky at MIT and Northeastern about 25 years ago.

The above are three photos of Zelevinsky and Fomin giving the first lectures on Cluster Algebras.

They were awarded the 2018 Leroy P. Steele Prize of the American Mathematical Society, for seminar contribution to research.

Cluster Algebras: general picture

[Fomin-Zelevinsky 2000] A cluster algebra is a commutative algebra A, generated by (in general ∞ many) polynomial subalgebras of the same dimension, related by mutation:



Mutation:

$$x_k' = \frac{\text{monomial}_1 + \text{monomial}_2}{x_k}$$

Frozen variables: $x_{m+1}, \ldots, x_N$.

Example

Example. The space of 2×2 matrices. Its polynomial algebra $\mathbb{C}[x_{11}, x_{12}, x_{21}, x_{22}]$ has a cluster structure with two clusters:

$$\mathbb{C}[x_{11}, x_{12}, x_{21}, \Delta] \longleftrightarrow \mathbb{C}[x_{22}, x_{12}, x_{21}, \Delta]$$

where $\Delta = x_{11}x_{22} - x_{12}x_{21}$. Mutation:

$$x_{22} = \frac{x_{12}x_{21} + \Delta}{x_{11}}$$

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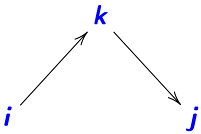
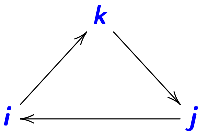


to triangles



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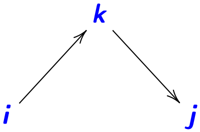

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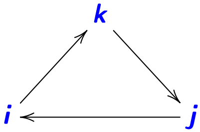
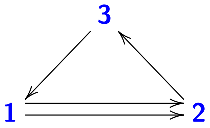


Step III: Cancel out pairs of opposite arrows.

An example: $\mu_3(Q)$

$Q :=$



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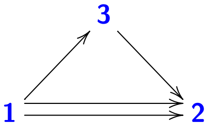

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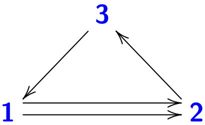

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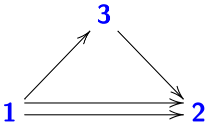
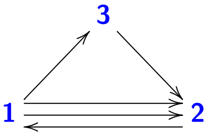
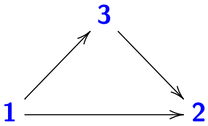


II (2-paths through 3):



III: $\mu_3(Q) =$



Cluster and Upper Cluster Algebras

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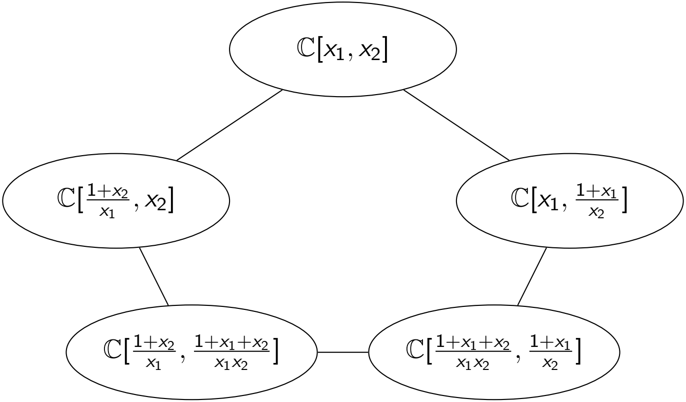
Laurent Phenomenon [Fomin–Zelevinsky]: $\mathbf{A}(Q) \subseteq \mathbf{U}(Q)$.
Important consequences if one can prove equality.

Example: type A_2

The simplest nontrivial cluster algebra is the one of type A_2 without frozen variables. It has 5 clusters:



Example: the Grassmannians $\mathrm{Gr}(2, d)$

The homogeneous coordinate ring $\mathcal{O}(\mathrm{Gr}(2, d))$ is the algebra with generators p_{ij}, $1 \leq i < j \leq d$ subject to the Plücker relations:

$$p_{ik} p_{j\ell} = p_{ij} p_{k\ell} + p_{i\ell} p_{jk} \quad \text{for} \quad 1 \leq i < j < k < \ell \leq d.$$

Cluster algebra, seeds parametrized by the triangulations of the d-gon:

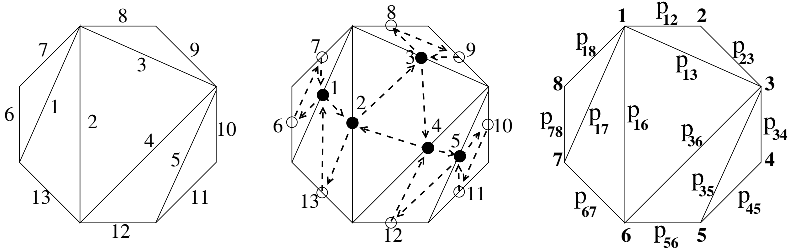


Figure: A triangulation T of an octagon, the quiver $Q(T)$, and labeling of T.

The GSV Poisson structures I

$I(Q)$= incidence matrix of Q of size $N \times m$.

Definition. The quiver Q and a skew-symmetric integer matrix $\Lambda \in M_N(\mathbb{Z})$ are called compatible if

$$I(Q)^\top \Lambda = \begin{bmatrix} D & 0 \end{bmatrix},$$

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Theorem [Gekhtman-Shapiro-Vainshtein]

Each upper cluster algebra $\mathbf{U}(Q)$ with a compatible matrix $\Lambda = (\lambda_{jk}) \in M_N(\mathbb{Z})$ possesses a Poisson algebra structure such that

$$\{x_j, x_k\} = \lambda_{jk} x_j x_k, \quad \forall 1 \le j, k \le N.$$

For every cluster $(x''_1, \ldots, x''_N)$ of $\mathbf{U}(Q)$

$$\{x''_j, x''_k\} = \lambda''_{jk} x''_j x''_k, \quad \forall 1 \le j, k \le N$$

for some $\lambda''_{jk} \in \mathbb{Z}$.

The GSV Poisson structures II

We get a Poisson structure π on the complex affine Poisson variety

$$Y(Q) := \mathrm{MaxSpec}\, U(Q).$$

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Vast classes of important complex affine Poisson varieties arise as special cases

1. Schubert cells in flag varieties with so called standard Poisson structures;
2. Double Bruhat cells, Bott–Samelson varieties for simple Lie groups with the standard Poisson structures;
3. Simple Lie groups with Belavin-Drinfeld Poisson structures and many others.

Root of Unity Cluster Algebras I

Fix an integer $\ell \geq 1$ and a primitive ℓ-th root of unity

$$\varepsilon^{1/2} \in \mathbb{C}.$$

Choose a skew-symmetric matrix

$$\Lambda = (\lambda_{ij}) \in M_N(\mathbb{Z}/\ell),$$

and consider the root of unity quantum torus

$$\mathcal{T}_{\varepsilon,\Lambda} := \frac{\mathbb{C}\langle y_1^{\pm 1}, \ldots, y_N^{\pm 1}\rangle}{(y_j y_k - \varepsilon^{\lambda_{ij}} y_k y_j)}$$

Root of Unity Cluster Algebras II

We define the mutation $\mu_k(\Sigma) := (y_1, \dots y_{k-1}, y'_k, y_{k+1}, \dots, y_N; \mu_k(Q))$ where

$$y'_k := \varepsilon^{n'/2} y_k^{-1} \prod_{j \to k} y_j + \varepsilon^{n''/2} y_k^{-1} \prod_{i \to k} y_i$$

same integers n' and n''. Compatibility between Λ and $I(Q)$.

Definition of root of unity quantum cluster algebras

Define

- The root of unity quantum cluster algebra $\mathbf{A}_\varepsilon(Q)$:= the $\mathbb{C}$-subalgebra of $\mathrm{Frac}(\mathcal{T}_{\varepsilon,\Lambda})$ generated by all cluster variables.
- The root of unity upper quantum cluster algebra $\mathbf{U}_\varepsilon(Q)$:= the intersection in $\mathcal{T}_{\varepsilon,\Lambda}$ of all mutated mixed quantum tori

$$\mathbb{C}\langle (y''_1)^{\pm 1}, \dots, (y''_m)^{\pm 1}, y_{m+1}, \dots, y_N \rangle.$$

Root of Unity Cluster Algebras III

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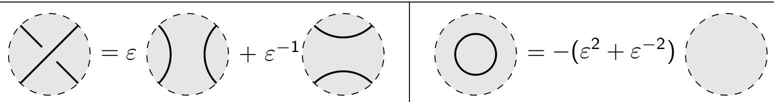
This is a vast class including many important subfamilies:

1. Lie theory: De Concini-Kac-Procesi quantum groups of various nature $\mathcal{O}_\varepsilon(G)$, $\mathcal{O}_\varepsilon(B_+ w B_+/B_+)$, ...
2. Topology: Kauffman bracket skein algebras of oriented surfaces.



The Kauffman skein relation | The value of the unknot

Canonical central subalgebras

Define the subalgebra of $\mathbf{U}_\varepsilon(Q)$

$$\mathbf{C}_\varepsilon(Q) := \bigcap_{\text{all seeds}} \mathbb{C}[(y_1'')^{\pm\ell}, \ldots, (y_m'')^{\pm\ell}, y_{m+1}^\ell, \ldots, y_N^\ell].$$

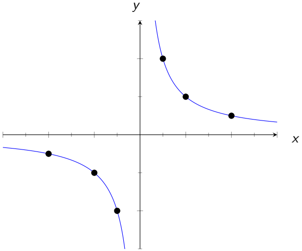
Theorem [Trampel-Nguyen-Y]

For all root of unity quantum cluster algebras $\mathbf{U}_\varepsilon(Q)$ and ℓ-th primitive roots of unity $\varepsilon^{1/2}$ such that ℓ is odd,

1. $\mathbf{C}_\varepsilon(Q)$ is a central subalgebra of $\mathbf{U}_\varepsilon(Q)$ and
2. $\mathbf{C}_\varepsilon(Q)$ is isomorphic to the classical upper cluster algebra $\mathbf{U}(Q)$.

T-orbits of Symplectic Leaves

Example: $\left(\mathbb{C}^2, \pi = (xy-1)\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}\right)$ with $\mathbb{C}^\times$-action $x \mapsto tx$, $y \mapsto t^{-1}y$:



Torus Actions

Given

$$\nu := (\nu_1, \ldots, \nu_N) \in \mathrm{Ker}\, I(Q)^\top,$$

there is a $\mathbb{C}^\times$-action on the upper cluster algebra $\mathbf{U}(Q)$ by Poisson automorphisms, such that

$$\psi_\nu(t)(x_j) = t^{\nu_j} x_j, \quad \forall t \in \mathbb{C}^\times, 1 \le j \le N.$$

One packages the $\mathbb{C}^\times$-actions into and action of the torus

$$T(Q) := (\mathbb{C}^\times)^{\dim \mathrm{Ker}\, I(Q)^\top},$$

on $\mathbf{U}(Q)$ by Poisson automorphisms.

Main Theorem on the Poisson structures of CAs

Geometry of the GSV Poisson structures:

$$T(Q) \curvearrowright (Y(Q), \pi).$$

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Theorem [Muller-Nguyen-Trampel-Y]

Let $\mathbf{U}(Q)$ be any finitely generated upper cluster algebra admitting a compatible skew-symmetric matrix.

The GSV Poisson structure on $Y(Q)$ always has a Zariski open (dense) $T(Q)$-orbit of symplectic laves and it is explicitly given by

$$Y(Q)^{\mathrm{reg}} \backslash \mathcal{V}(x_{m+1} \dots x_N).$$

Proof: Arguments with the anticanonical bundle of $Y(Q)^{\mathrm{reg}}$ + normality.

Analysis of Theorem

Tempting to conjecture:

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But this is incorrect even if we require $Y(Q)$ to be smooth:

- The symplectic foliations of the Belavin-Drinfeld Poisson structures were classified ([Y] 2001 PhD thesis under N. Reshetikhin) and this property does not hold for them.
- Gekhtman-Shapiro-Vainshtein constructed cluster algebra structures on some of them.

Affine $A_1^{(1)}$ Example I

Consider the $A_1^{(1)}$ cluster algebra for

$$I(Q) := \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}, \qquad \Lambda := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

without frozen variables. $\mathrm{Ker}\, I(Q)^T = \{0\}$, so $T(Q)$ is trivial.

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Berenstein-Fomin-Zelevinsky gave the presentation

$$\mathbf{U}(Q) = \mathbb{C}[x_1, x_2, x_1', x_2']/(x_1 x_1' - x_2^2 - 1, x_2 x_2' - x_1^2 - 1).$$

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There is a nicer presentation in terms of the following element

$$z := x_1'x_2' - x_1x_2 \in \mathbf{U}(Q).$$

We have

$$Y(Q) = \mathcal{V}(f) \subset \mathbb{C}^3, \quad \text{where} \quad f(x_1, x_2, z) := x_1x_2z - (x_1^2 + x_2^2 + 1)$$

and one checks that $Y(Q)$ is smooth.

Affine $A_1^{(1)}$ Example II

The GSV Poisson structure is a Poisson structure with potential:

$$\{x_1, x_2\} := f_z = x_1 x_2,$$
$$\{x_2, z\} := f_{x_1} = x_2 z - 2x_1,$$
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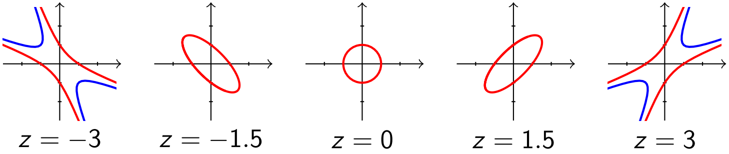
Our theorem gives that

$$(Y(Q), \pi) \text{ is a symplectic manifold},$$

i.e., the whole $Y(Q) = \mathrm{MaxSpec}\,\mathbf{U}(Q)$ is a single symplectic leaf.

Affine $A_1^{(1)}$ Example III

The cross sections of $Y(Q)$ with $z = \mathrm{const}$ give all conics through $(0, \pm i)$ and $(\pm i, 0)$ except for the singular one $x_1 x_2 = 0$, classically known as pencil of conics:



The red curves represent the imaginary parts of the cross sections (conics) and the blue curves their real parts.

Affine $A_1^{(1)}$ Example IV

The clusters of $\mathbf{U}(Q)$ are the pairs

$$(x_n, x_{n+1}), \quad n \in \mathbb{Z}$$

for x_n recursively defined by

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The union of the cluster tori is

$$Y(Q) \backslash \{(0, \pm i, 0), (\pm i, 0, 0)\}.$$

This is the "local part" that was previously understood.

Cayley–Hamilton algebras [Procesi]

Setting: A is an algebra with trace $\mathrm{tr} : A \to C$ for a central subalgebra C: $\mathrm{tr}(za) = z\,\mathrm{tr}(a)$ and $tr(ab) = \mathrm{tr}(ba)$ for all $a, b \in A$, $z \in C$.

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In the setting of an algebra with trace, use the same expressions to define the n-th characteristic polynomial of $a \in A$

$$\chi_{a,n}(t) \in C[t] \quad \text{using} \quad \mathrm{tr}(a), \ldots, \mathrm{tr}(a^n).$$

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Definition [Procesi]

A Cayley-Hamilton algebra of degree n is an algebra A with trace $\mathrm{tr} : A \to C$ such that

$$\chi_{a,n}(a) = 0, \quad \forall a \in A \quad \text{and} \quad \mathrm{tr}(1) = n.$$

Cayley-Hamilton Structures on Cluster Algebras

Theorem [Huang-Lê-Y]

Assume that the order ℓ of the root of unity $\varepsilon^{1/2}$ is odd. Then:

1. The pair $(\mathbf{U}_\varepsilon(Q), \mathbf{C}_\varepsilon(Q))$ has a canonical structure of Cayley-Hamilton algebra of degree ℓ^N.
2. The following are equivalent:
   - $\mathbf{U}_\varepsilon(Q)$ is a finitely generated $\mathbb{C}$-algebra.
   - $\mathbf{U}_\varepsilon(Q)$ is a fin generated module over $\mathbf{C}_\varepsilon(Q)$ and $\mathbf{C}_\varepsilon(Q) \cong U(Q)$ is a finitely generated commutative $\mathbb{C}$-algebra.

Corollary [Finite Generation Transfer]. $\mathbf{U}_\varepsilon(Q)$ is a finitely generated $\mathbb{C}$-algebra $\Rightarrow$ $U(Q)$ is a finitely generated commutative $\mathbb{C}$-algebra.

Irreps of Noether Algebras

General picture [Artin, Kaplansky, Posner, Procesi,...]:
A is a (noncommutative) finitely generated $\mathbb{C}$-algebra w/o zero divisors, which is module-finite over a central subalgebra $C \subseteq Z(A)$.

$\mathrm{Irr}(A)$=irreps of A. Kaplansky Thm $\Longrightarrow$ all are fin dimensional. By Schur's lemma, we have the character map:

$$\psi : \mathrm{Irr}(A) \to \mathrm{MaxSpec}(C), \quad V \mapsto \mathrm{Ann}_C(V).$$

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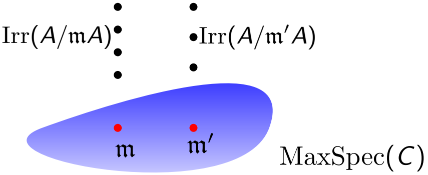
Problem. Describe he fibers of ψ, which are $\psi^{-1}(\mathfrak{m}) = \mathrm{Irr}(A/\mathfrak{m}A)$.



Irreps of Noether Algebras

General picture [Artin, Kaplansky, Posner, Procesi,...]:
A is a (noncommutative) finitely generated $\mathbb{C}$-algebra w/o zero divisors, which is module-finite over a central subalgebra $C \subseteq Z(A)$.

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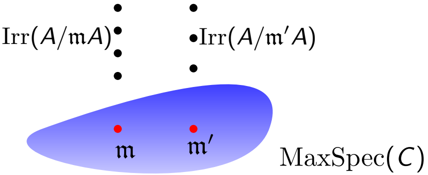

[Brown-Gordon] Fully Azumaya locus of A with respect to C:=the open subset of $\mathfrak{m} \in \mathrm{MaxSpec}(C)$ such that all irreps in $\psi^{-1}(\mathfrak{m})$ have max dim.

Main Thm on Irreps of CAs

Theorem [Muller-Nguyen-Trampel-Y]

For all root of unity upper quantum cluster algebras $\mathbf{U}_\varepsilon(Q)$ and ℓ-th primitive roots of unity $\varepsilon^{1/2}$ such that

1. ℓ is odd
2. and $\mathbf{U}_\varepsilon(Q)$ is a finitely generated algebra such that $\mathbf{U}_\varepsilon(Q) = \mathbf{A}_\varepsilon(Q)$,

the fully Azumaya locus $\mathcal{A} \subset Y(Q)$ of $\mathbf{U}_\varepsilon(Q)$ with respect to $\mathbf{C}_\varepsilon(Q) \cong \mathbf{U}(Q)$ satisfies

$$Y(Q)^{\mathrm{reg}} \backslash \mathcal{V}(x_{m+1} \dots x_N) \subseteq \mathcal{A} \subseteq Y(Q) \backslash \mathcal{V}(x_{m+1} \dots x_N).$$

Affine $A_1^{(1)}$ Example

Return to the acyclic cluster algebra for

$$I(Q) := \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}, \qquad \Lambda := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

without frozen variables.
On the quantum level we have the presentation

$$\mathbf{U}_\varepsilon(Q) = \mathbb{C}[x_1, x_2, x_1', x_2']/(x_1x_2 - \varepsilon x_2x_1, x_1x_1' - \varepsilon^{-1}x_2^2 - 1, x_2x_2' - \varepsilon x_1^2 - 1).$$

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Our second main theorem implies that all irreducible representations of $\mathbf{U}_\varepsilon(Q)$ have dimension ℓ and can be obtained in an explicit way from representations of quantum tori.

Thank you!