Chiral symmetry restoration in Gross – Neveu model

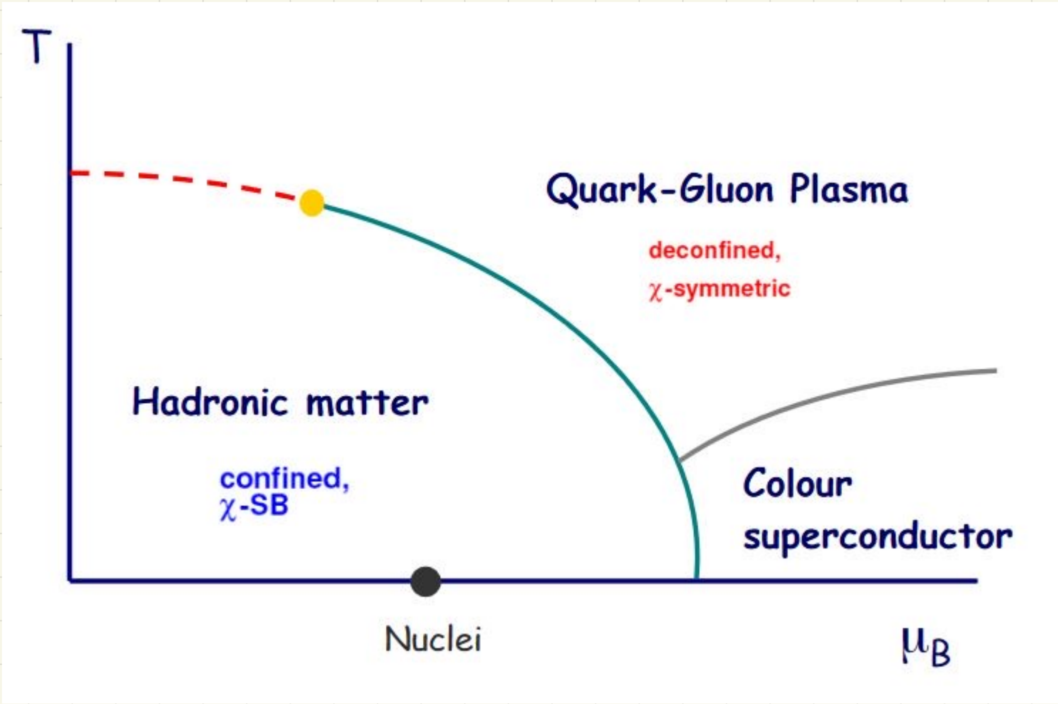
K. Zarembo (Nordita & NBI)

V. Melin, Y. Sekiguchi, P. Wiegmann and K. Z., 2404.07307

Integrability, Q - systems and Cluster Algebras

Varna, 12.08.24

QCD phase diagram



Hadronic phase: $\langle \bar{\psi} \psi \rangle \neq 0$

Quark-gluon plasma: $\langle \bar{\psi} \psi \rangle = 0$

Gross - Neveu model

(1+1) d

$$\mathcal{L} = i\bar{\Psi}_i \not{\partial} \Psi_i + \frac{g^2}{2}\left(\bar{\Psi}_i \Psi_i\right)^2 \qquad i = 1 \dots N$$

- Asymptotically free — Anselm '59

- Dimensional transmutation: — Gross, Neveu '74

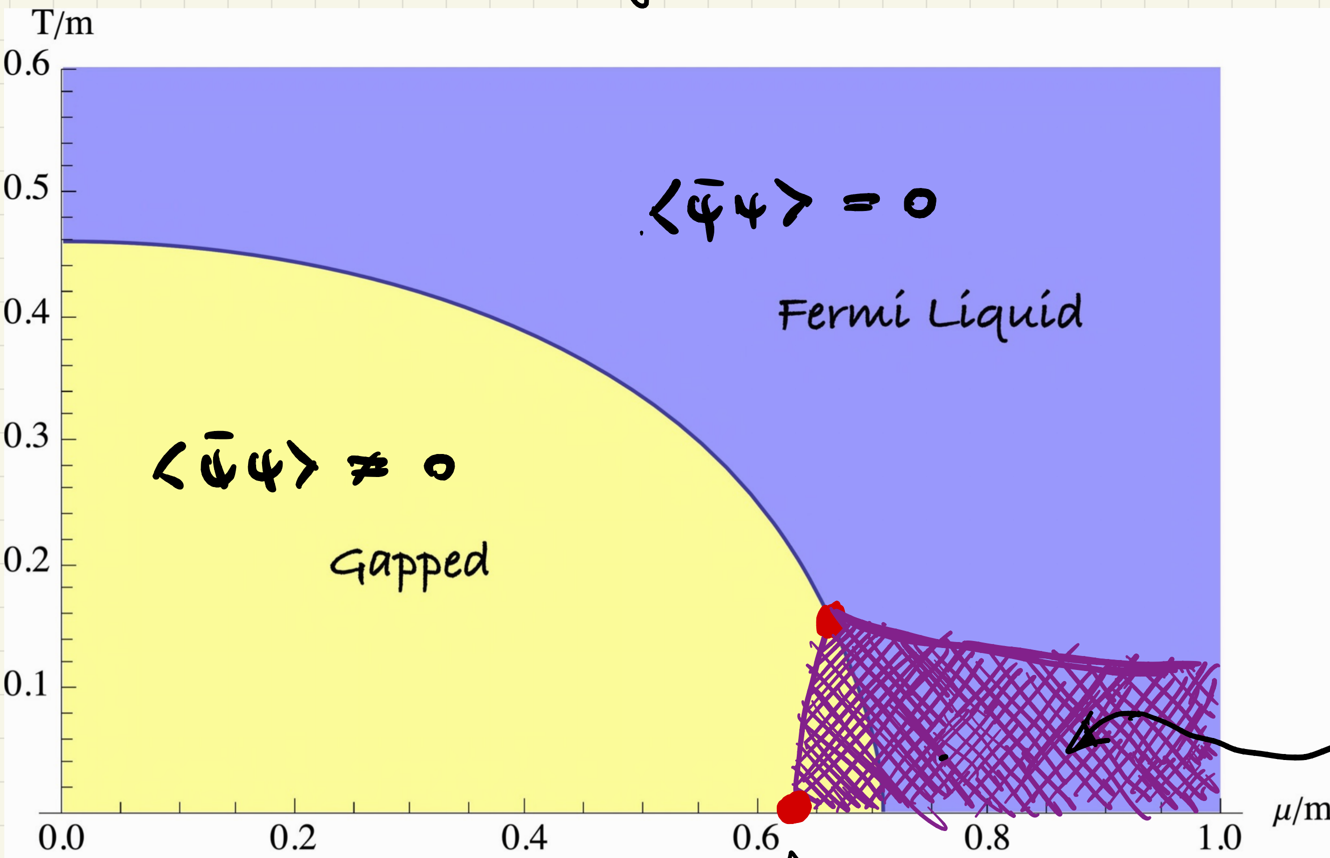
$$m \simeq \Lambda\, e^{-\frac{\pi}{\lambda}} \qquad \lambda = g^2 N$$

- Chiral symmetry breaking:

$$\langle \bar{\Psi} \Psi \rangle = \text{const}\ m$$

Finite temperature and density

Large-N phase diagram:



Thies, Urlichs '03
Schnetz, Thies, Urlichs '04
Thies '06
Başar, Dunne, Thies '09

Chiral crystal:

$$\langle \bar\psi \psi \rangle \sim \mathrm{sn}(x; k)$$

cnoidal wave

$$\boxed{\mu_c = \frac{2}{\pi} m}$$

Thies, Urlichs '03

Brazovskii, Kirova '81

Başar, Dunne, Thies '09

Large-N solution

Gross, Neveu '74

$$\mathcal{L} = i\bar{\psi}_i \not{\partial} \psi_i - \sigma \bar{\psi}_i \psi_i - \frac{N}{2\lambda}\sigma^2$$

$$\sigma = -\frac{\lambda}{N}\bar{\psi}\psi \qquad \text{on-shell}$$

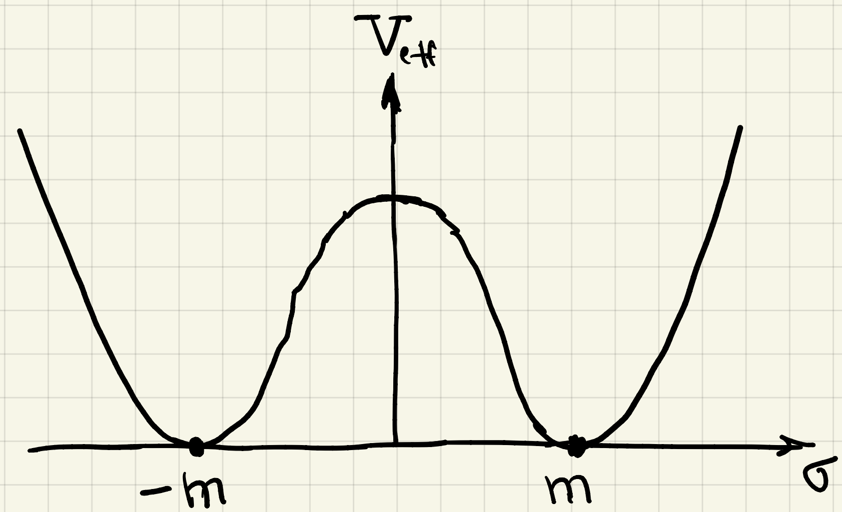
- integrate out ψ_i

$$S_{\text{eff}} = -N\left[\frac{1}{2\lambda}\int d^2x\, \sigma^2 + i \ln\det(i\not{\partial} - \sigma)\right]$$

$N \to \infty \implies$ Mean-field exact

Effective potential:

$$V_{eff}(\sigma) = \frac{\sigma^2}{2\lambda} - \frac{1}{2}\int \frac{d^2p}{(2\pi)^2} \ln(p^2+\sigma^2)$$



- Chiral $\mathbb{Z}_2$ $\sigma \to -\sigma$ is spontaneously broken

$\langle \sigma \rangle \equiv m$ - gives mass to fermions:

$$i\bar{\psi}_i \partial\!\!\!/ \psi_i - \sigma \bar{\psi}_i \psi_i \quad \text{becomes} \quad i\bar{\psi}_i \partial\!\!\!/ \psi_i - m \bar{\psi}_i \psi_i$$

Gap equation:

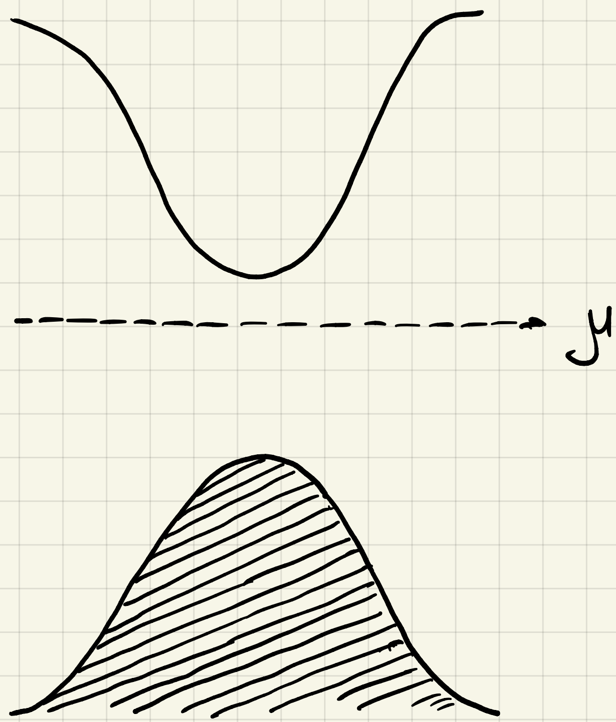
$$\frac{1}{\lambda} = \int \frac{d^2p}{(2\pi)^2} \frac{1}{p^2 + m^2}$$

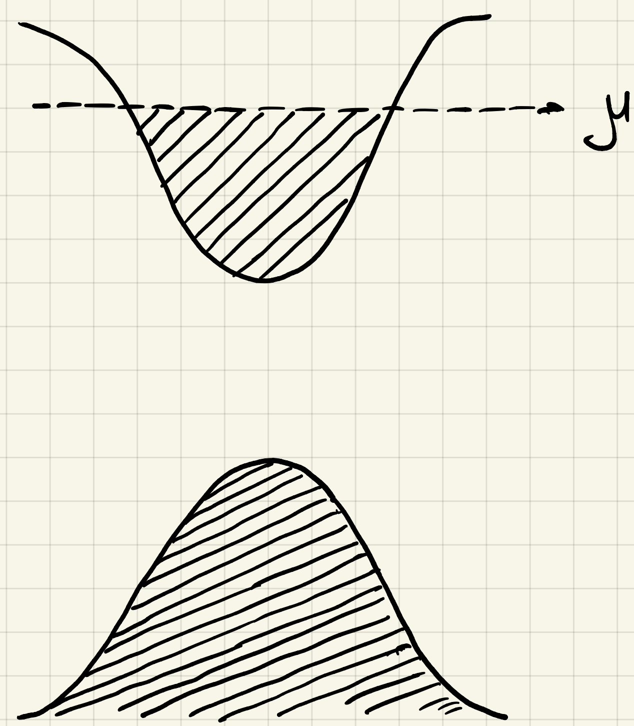
$$\Downarrow$$

$$m = \Lambda e^{-\frac{\pi}{\lambda}}$$

Peierls instability

Peierls '30, 55 Frölich '54

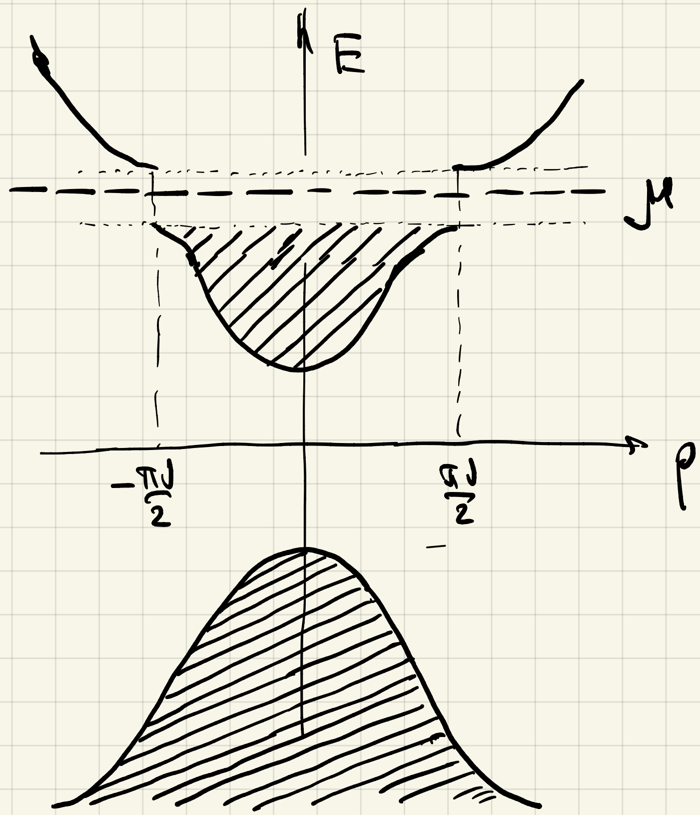




Very stable

Unstable

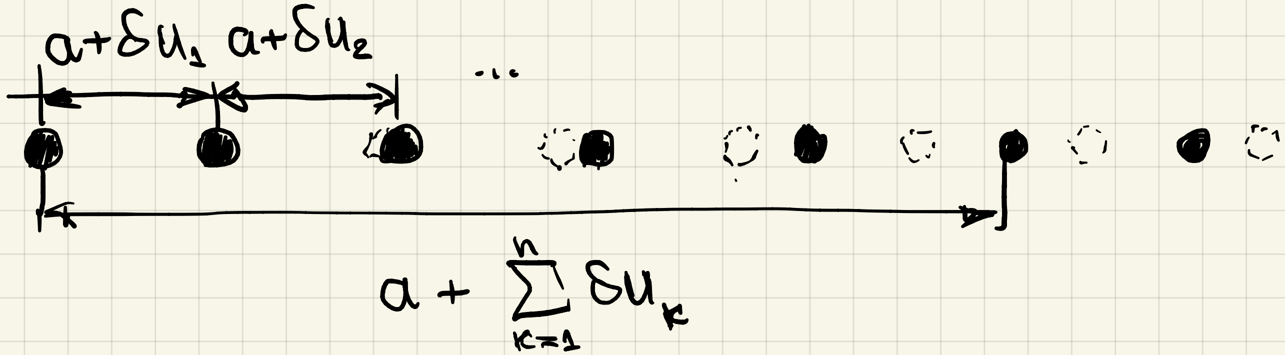
- emergent crystalline structure opens a bandgap around Fermi level:



$$\langle \sigma \rangle \simeq A \sin \pi \nu x$$

- 1d crystals do not exist

Peierls '34 Landau '38



$|\delta u| \ll a$

$$\left|\sum_{k=1}^{n}\delta u_k\right| \sim \sqrt{n}\,\delta u \sim a \quad \text{for} \quad n=\frac{a^2}{\delta u^2}$$

$\hookrightarrow$ no long-range order

Peierls '34

Coleman – Mermin – Wagner theorem

- $\mathbb{R} \times \mathbb{Z}_2 \to \mathbb{Z}$ symmetry breaking is forbidden in 1d

Mermin, Wagner '66 Coleman '73

Goldstone boson (phonon)

- either gets gapped :

$$\langle u(x)\, u(0) \rangle \propto e^{-\frac{|x|}{\xi}}$$

- or disorders the order parameter:

$$\langle u(x)\, u(0) \rangle \propto \frac{1}{|x|^{\beta}}$$

Berezinskii '71

Kosterlitz, Thouless '73

Quasi-long-range order

Large-N and infinite-volume limits do not commute

Witten '78

Quasi-long-range order:

$$\langle \bar{\Psi}\psi(x)\, \bar{\Psi}\psi(0) \rangle \simeq \frac{\sin \pi \nu x}{|x|^{\alpha/N}}$$

Cicconi, Di Pietro, Serone '22, 23

Beyond mean-field

- Does phase transition exist at finite N or becomes a crossover?

- Transition may only happen for $N > N_c$. If so, what is N_c?

- How accurate is large-N approximation?

- Is spectrum gapless or system develops non-perturbative (e^{-N}) gap?

Integrability

- GN model is integrable
  - exact spectrum
  - exact S-matrix

Zamolodchikov, Zamolodchikov '79

Karowski, Thun '81

$$\Downarrow$$

TBA

Symmetries

$$\mathcal{L} = i\bar{\Psi}_i \not{\partial} \psi_i - \sigma \bar{\Psi}_i \psi_i - \frac{N}{2\lambda}\sigma^2$$

Global $O(2N)$:

- rotates real and imaginary components of ψ_i $i = 1 \dots N$

Chiral $\mathbb{Z}_2$:

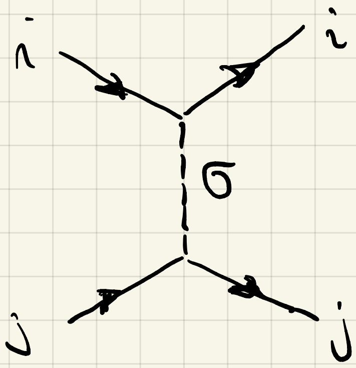
$$\psi_i \to \gamma_3 \psi_i$$

$$\sigma \longrightarrow -\sigma$$

$N \geq 2$:

$\beta = -\frac{N-1}{\pi N}\lambda^2$ ($O(2)$ model is CFT - equiv. to Thirring)

Bound states



leads to attractive interaction in the anti-symmetric $[ij]$ channel.

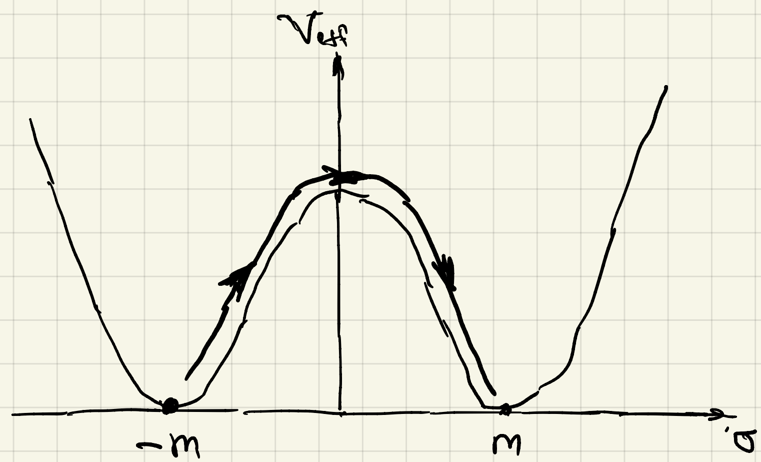
bound states:

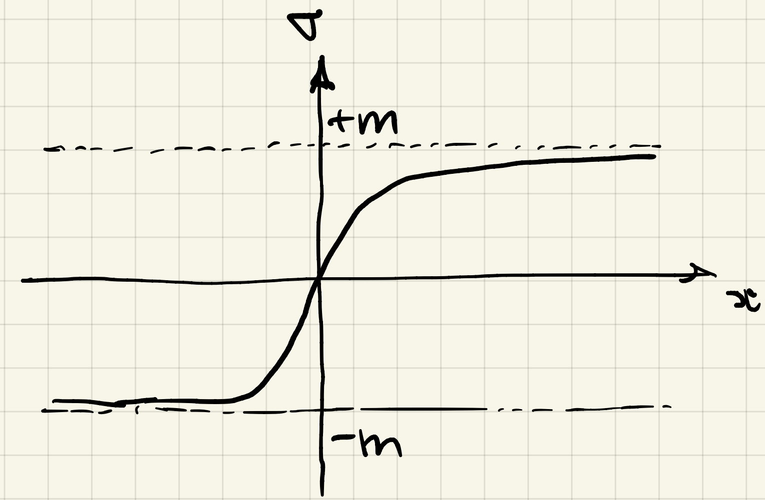
$$\psi_{[i_1} \cdots \psi_{i_a]}$$

$$a = 1, \dots, N-2$$

$$\boxed{m_a = \frac{m \sin \frac{\pi a}{2N-2}}{\sin \frac{\pi}{2N-2}}}$$

Solitons





Gross, Dashen, Hasslacher, Neveu '75

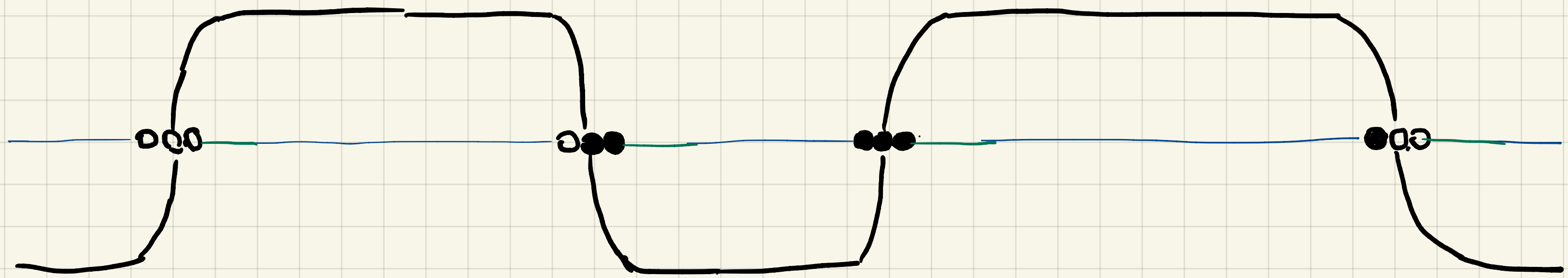
$$m_s \propto N m$$

Exact formula:

$$\boxed{m_s = \frac{m}{2\sin\frac{\pi}{2N-2}}}$$

Fermion zero-modes

Ex: O(6) model , $N = 3$



$B = -\frac{3}{2}$ $B = +\frac{1}{2}$ $B = +\frac{3}{2}$ $B = -\frac{1}{2}$

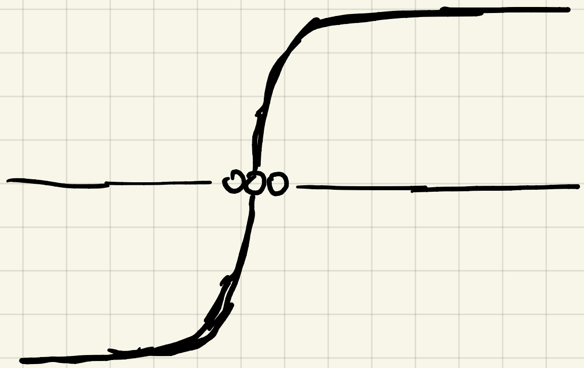
Internal structure of a soliton:

$|s_1 \dots s_N\rangle \qquad s_i = \pm \qquad 2^N$ states

- Solitons transform in spinor rep. of $O(2N)$

Witten '78

Fractional fermion charge



$|\circ\circ\circ\rangle$ $\quad B = -\frac{3}{2}$ $\quad \bar{s}$

$|\circ\circ\bullet\rangle$ $\quad B = -\frac{1}{2}$ $\quad s$

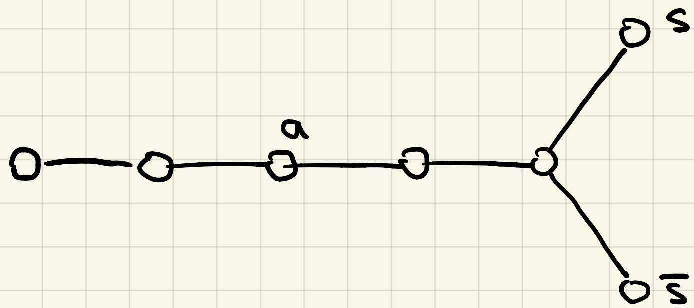
$|\circ\bullet\bullet\rangle$ $\quad B = +\frac{1}{2}$ $\quad \bar{s}$

$|\bullet\bullet\bullet\rangle$ $\quad B = +\frac{3}{2}$ $\quad s$

$$\boxed{\text{Max. baryon charge : } \frac{N}{2}}$$

Exact spectrum

Zamolodchikov, Zamolodchikov '79



	Mass	Baryon charge
Fermion	m	1
Bound states	$m\dfrac{\sin\frac{\pi a}{2N-2}}{\sin\frac{\pi}{2N-2}}$	a
Solitons	$\dfrac{m}{2\sin\frac{\pi}{2N-2}}$	$\dfrac{N}{2}$

Finite baryon density

n_i – # of particles of type i

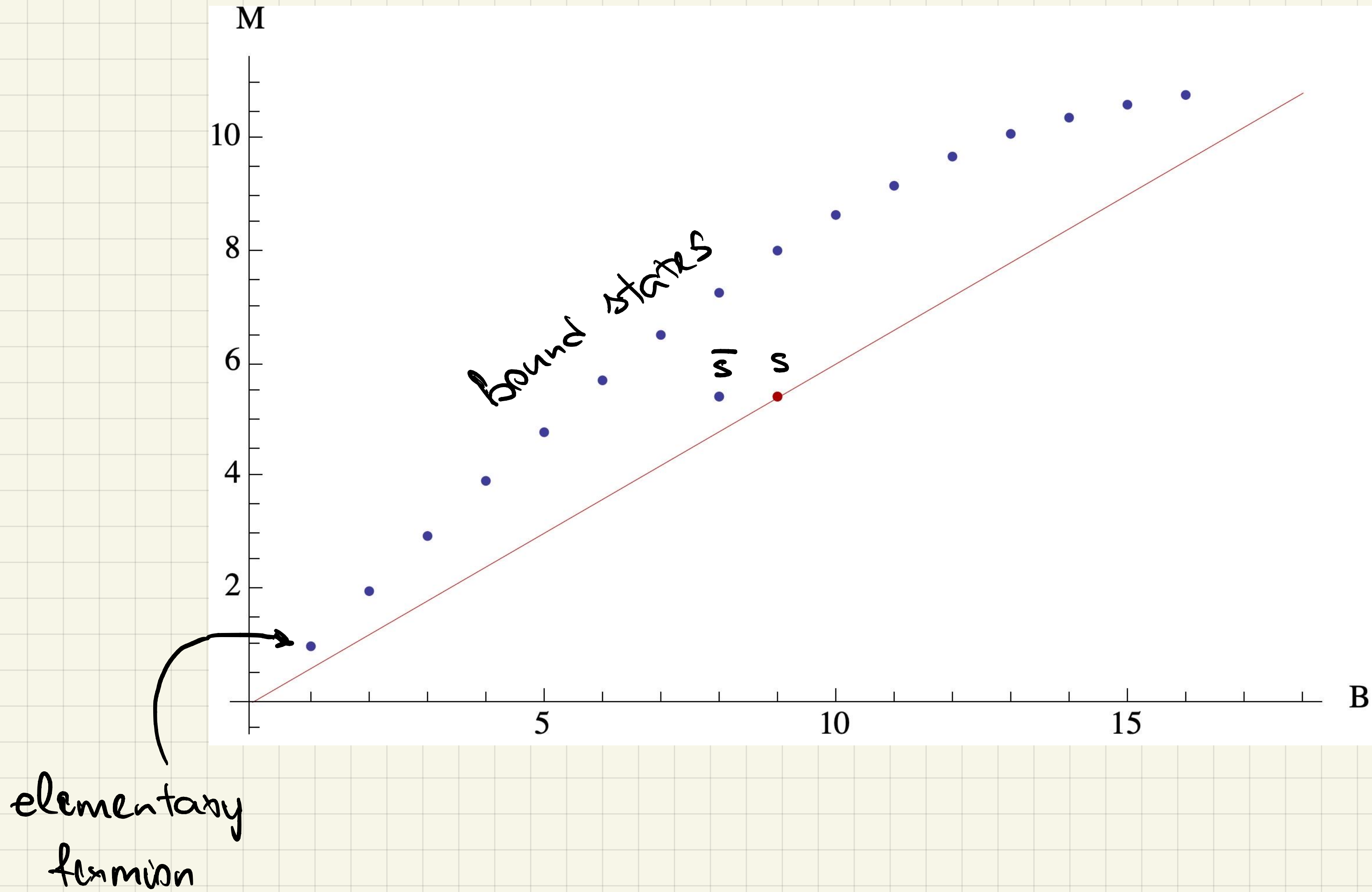
Total baryon charge:

$$B = n_i B_i$$

Total energy:

$$E = n_i m_i$$

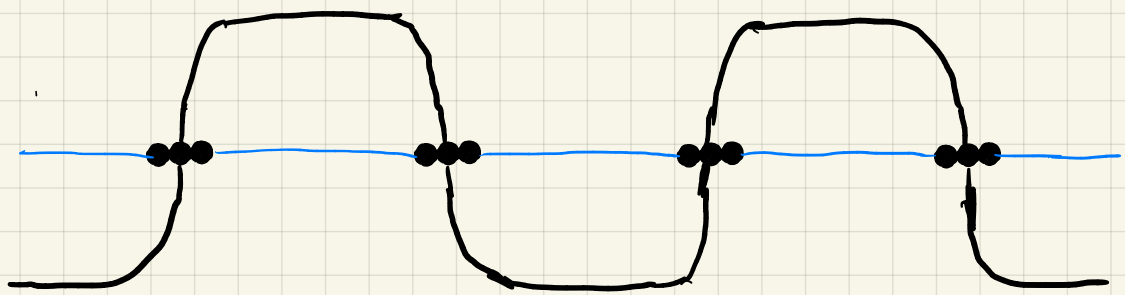
- Smallest energy for given baryon charge is achieved by smallest m_i / B_i



• solitons are most energy efficient !

Soliton crystal

Ground state at finite baryon density.

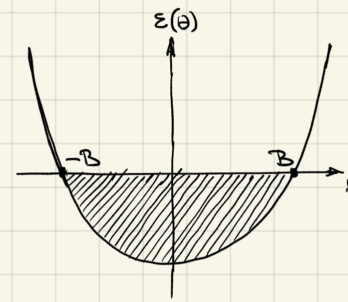


- All baryon charge resides in soliton zero modes

TBA equations

$$\varepsilon(\theta) - \int_{-B}^{B} d\theta' \, K(\theta-\theta')\, \varepsilon(\theta') = M\cosh\theta - \mu$$

$$\varepsilon(\pm B) = 0$$



$$K(\theta) = \frac{1}{2\pi i} \frac{d \ln S(\theta)}{d\theta}$$

Free energy:

$$F = \frac{m}{2\pi} \int_{-B}^{B} d\theta \cosh\theta \; \varepsilon(\theta)$$

Phase transition

- Solution exists only for $\mu > M$

$$\mu_s = \mu B_s = \frac{\mu N}{2}$$

$$m_s = \frac{m}{2 \sin \frac{\pi}{2N-2}}$$

- Soliton crystal exists for $\mu_s > m_s$

$$\boxed{\mu_c = \frac{m}{N \sin \frac{\pi}{2N-2}}}$$

$$\mu_c \overset{N \to \infty}{\simeq} \frac{2m}{\pi}$$ agrees w. mean field

TBA equations for solitons

$$\varepsilon(\theta) - \int_{-B}^{B} d\theta' \, K(\theta-\theta')\, \varepsilon(\theta') = m_s \cosh\theta - \mu_s$$

$$m_s = \frac{m}{2\sin\frac{\pi}{2N-2}} \qquad \mu_s = \frac{\mu N}{2}$$

$$K(\theta) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi}\, e^{-i\omega\theta} \left[1 - \frac{e^{\frac{\pi|\omega|}{2N-2}} \left(\tanh\frac{\pi\omega}{2} + \tanh\frac{\pi\omega}{2N-2} \right)}{4\sinh\frac{\pi\omega}{2N-2}} \right]$$

Karowski, Thun '81

Large − N limit

$$K(\theta) \simeq -N \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \, e^{-i\omega\theta} \, \frac{\tanh \frac{\pi\omega}{2}}{2\pi\omega} = -N \cdot \frac{1}{2\pi^2} \ln \left| \coth \frac{\theta}{2} \right|$$

$$m_s \simeq N \cdot \frac{m}{\pi}$$

$$\mu_s = N \cdot \frac{\mu}{2}$$

TBA eqs. at large − N.

$$\int_{-B}^{B} d\theta' \, \ln \left| \coth \frac{\theta - \theta'}{2} \right| \, \varepsilon(\theta') = \frac{m}{\pi} \cosh \theta - \frac{\mu}{2}$$

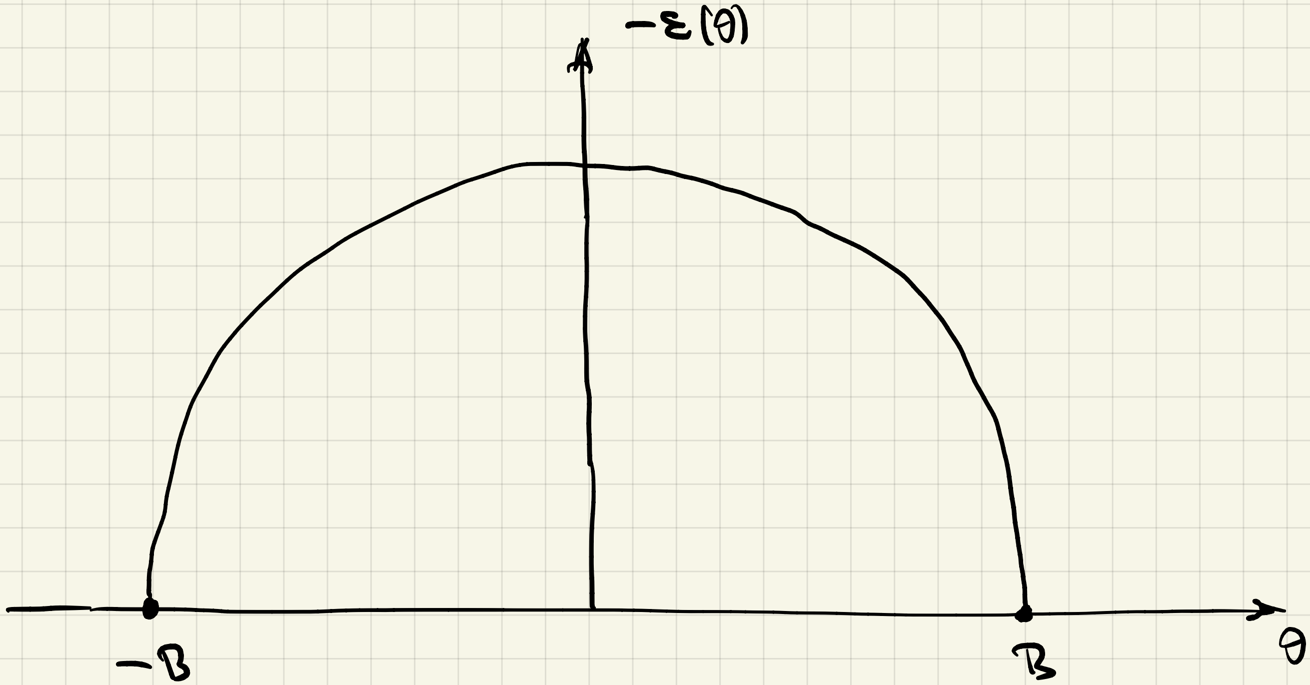
Large-N solution

- differentiate in θ:

$$-\frac{1}{2\pi^2}\int_{-B}^{B}\frac{d\theta'\,\varepsilon(\theta')}{\sinh(\theta-\theta')} = \frac{m}{\pi}\sinh\theta$$

Solution:

$$\boxed{\varepsilon(\theta) = -2m\sqrt{\sinh^2 B - \sinh^2\theta}}$$



complete elliptic int.

$$\frac{\pi \mu}{2m} = \frac{E(k)}{k} \qquad k = \frac{1}{\cosh B}$$

i) $\mu \to \mu_c$

$$\delta \equiv \frac{\pi\mu}{2m} - 1 \simeq \frac{\mu - \mu_c}{\mu_c}$$

$$B \simeq \sqrt{\frac{4\delta}{|\ln\delta|}}$$

ii) $\mu \to \infty$ (perturbative regime)

$$B \simeq \ln\frac{2\mu}{m}$$

Running coupling: $\lambda(\mu) = \dfrac{1}{\ln\frac{\mu}{m} + \text{const}}$

scheme - dependent

$$\boxed{\lambda(\mu) = \frac{\pi}{B(\mu)}}$$ ~ "integrability" scheme

Exact β-function

$$\boxed{\beta = -\frac{\lambda^2}{\pi}\frac{E}{k'K}}$$

$$k' = \sqrt{1-k^2}$$

$$k = \frac{1}{\cosh\frac{\pi}{\lambda}}$$

Weak coupling:

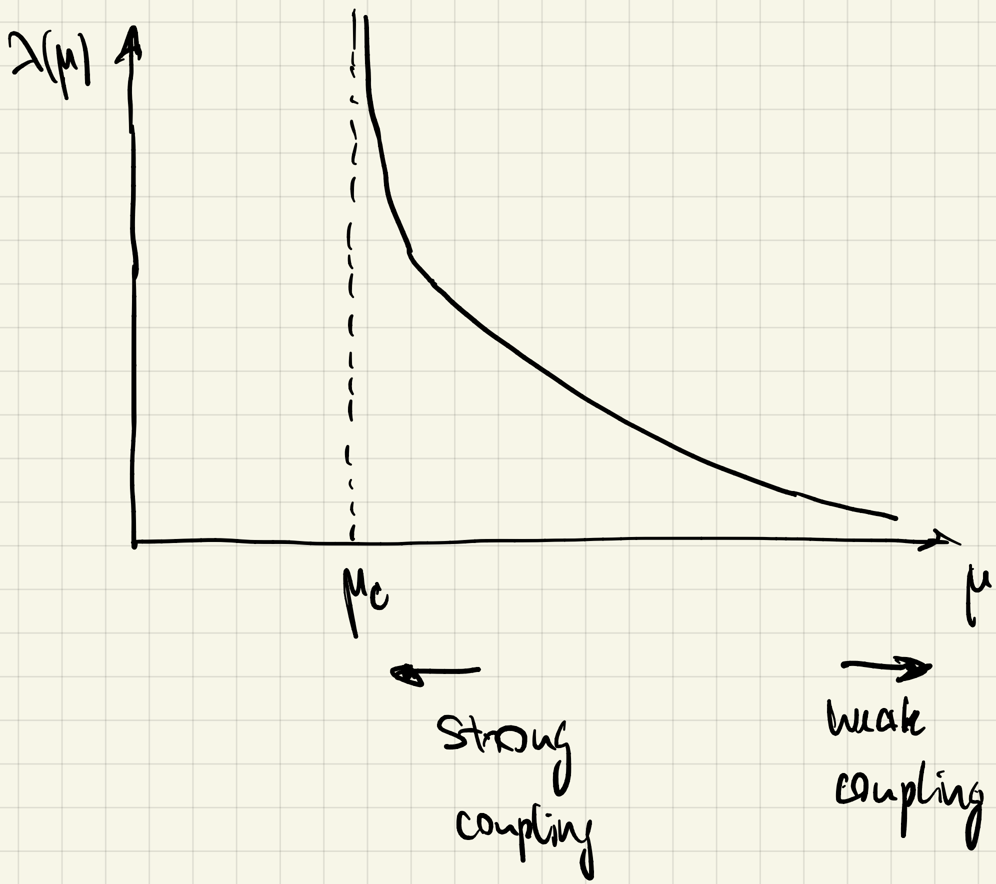
$$\beta = -\frac{\lambda^2}{\pi}\left(1 + e^{-\frac{4\pi}{\lambda}} + \frac{7}{8}e^{-\frac{8\pi}{\lambda}} + \dots\right)$$

OPE? Renormalons?

Strong coupling:

$$\beta \simeq -\frac{\lambda^3}{\pi^2 \ln\frac{4\lambda}{\pi}}$$

$$\lambda(\mu) = \frac{\pi}{B(\mu)}$$



$$\rho(\theta) - \int_{-B}^{B} d\theta' \, K(\theta - \theta') \, \rho(\theta') = M \cosh\theta$$

$$\frac{dp(\theta)}{d\theta} = \rho(\theta)$$

$$\hookrightarrow \begin{cases} \varepsilon = \varepsilon(\theta) \\ p = p(\theta) \end{cases}$$

At large-N:

$$\rho(\theta) \approx \frac{m \sinh 2B}{C_s \sqrt{\sinh^2 B - \sinh^2 \theta}} - 2m \sqrt{\sinh^2 B - \sinh^2 \theta}$$

$\uparrow$ "integration constant"

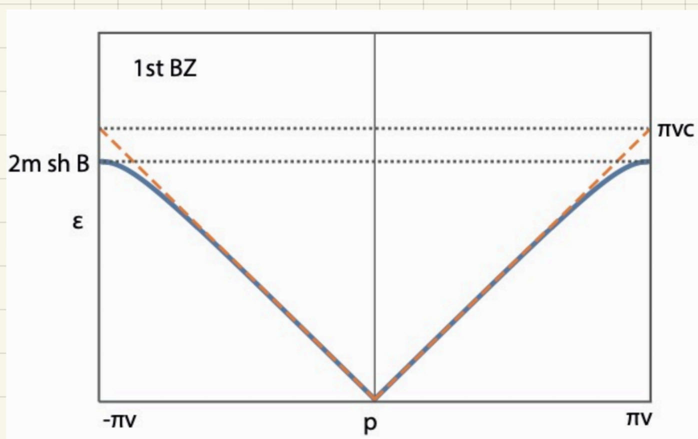
Dispersion relation

$$c_s\, dp = \frac{4\varepsilon_+\varepsilon_- - c_s\,\varepsilon^2}{\sqrt{(4\varepsilon_+^2 - \varepsilon^2)(4\varepsilon_-^2 - \varepsilon^2)}}$$

$$\varepsilon_+ = m \cosh B$$

$$\varepsilon_- = m \sinh B$$

$$c_s = \frac{k' K}{E}$$



$\varepsilon(p) \overset{p\to 0}{\simeq} c_s\,|p| \quad \Rightarrow \quad c_s$ - speed of sound

At $\mu \longrightarrow \mu_c$: $\quad \varepsilon(p) \simeq 2c_s \rho \sin\frac{p}{2\rho}$

$$\rho = \frac{m}{kK} \quad \text{- density of solitons}$$

- dispersion of a phonon in harmonic lattice w. spacing $1/\rho$

Holes in Fermi sea

$\Updownarrow$

Phonons in chiral crystal

Speed of sound

$$c_s = \frac{k' K}{E} \qquad \frac{\pi \mu}{2m} = \frac{E}{k}$$

$\sim$ speed of light

(also conf. $c = \frac{1}{\sqrt{d-1}}$)

Critical slow-down: $c_s \sim \sqrt{\delta |\ln \delta|}$

Elementary fermion

$$\varepsilon_f(\theta) - \int_{-B}^{B} d\theta' \, \mathcal{K}_f(\theta-\theta') \, \varepsilon(\theta') = m\cosh\theta - \mu$$

determined by fermion-soliton phase shift

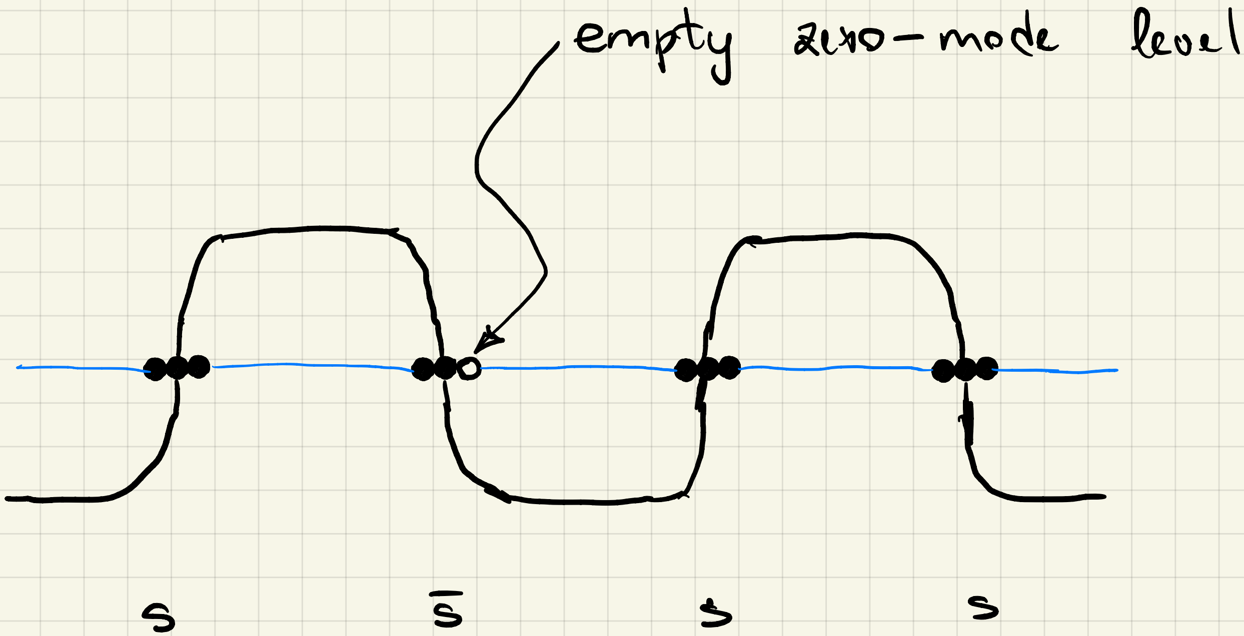
At large-N_c

$$\varepsilon_f(\theta) + \int_{-B}^{B} \frac{d\theta'}{2\pi} \frac{\varepsilon(\theta')}{\cosh(\theta-\theta')} = m\cosh\theta - \mu$$

$$\Downarrow$$

$$\varepsilon_f(\theta) = m\sqrt{\sinh^2 B + \cosh^2\theta} - \mu$$

Hole



changes

O(2N) chirality

Hole dispersion

$$\varepsilon_{\bar{f}}(\theta) - \int_{-B}^{B} d\theta' \, K_{\bar{f}}(\theta-\theta')\, \varepsilon(\theta') = m_s \cosh\theta - \mu_s + \mu$$

$K_{\bar{f}}$: $s\bar{s}$ scattering

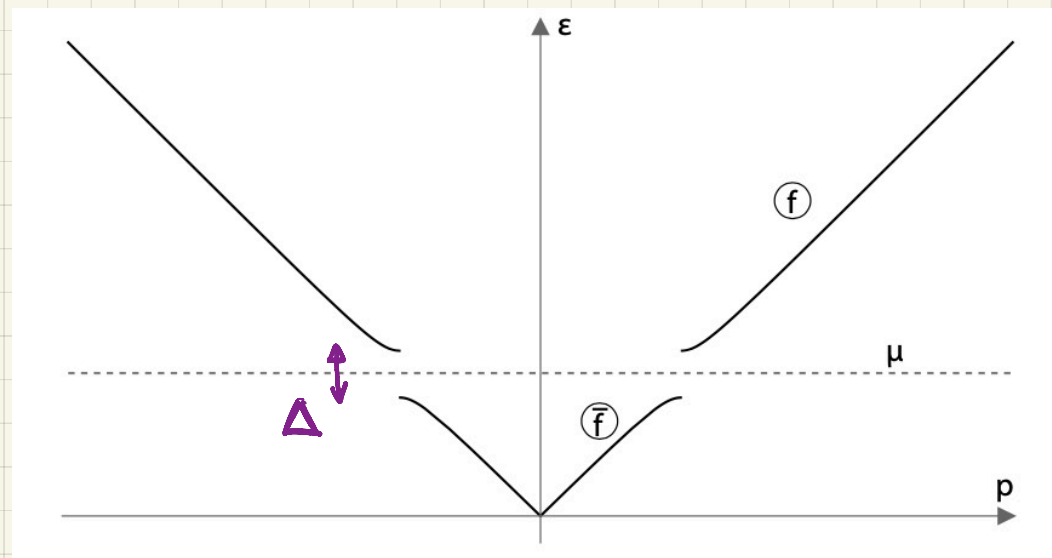
Large-N:

$$K_{\bar{f}}(\theta) \simeq K(\theta) - \frac{\delta(\theta)}{2}$$

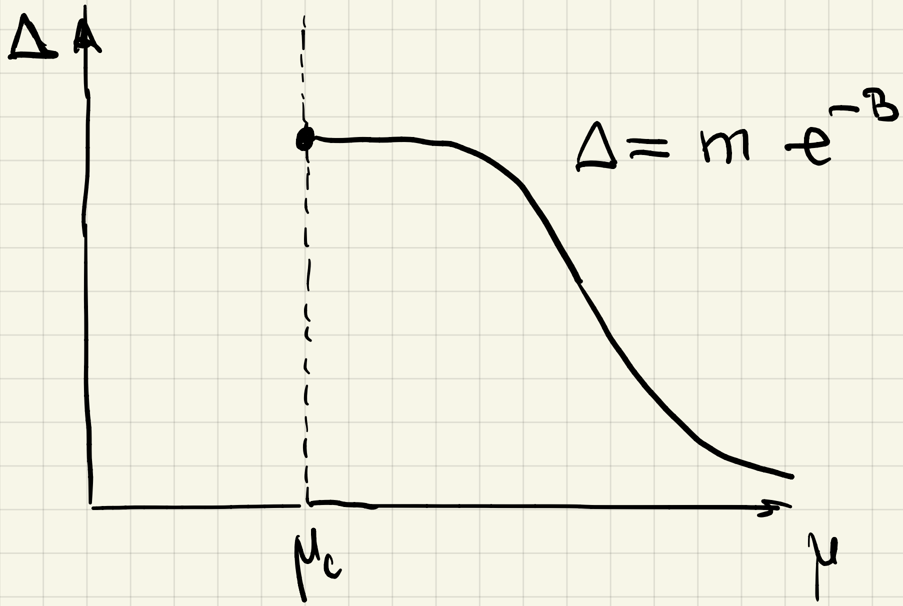
$$\Downarrow$$

$$\varepsilon_{\bar{f}}(\theta) = \mu + \frac{1}{2}\varepsilon(\theta)$$

$$p_{\bar{f}}(\theta) = \frac{1}{2} p(\theta)$$



Spectral gap

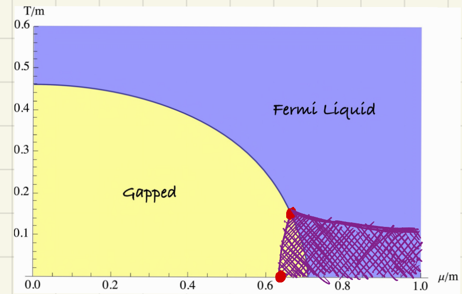


At $\mu \gg m$: $\quad \Delta \sim \frac{m^2}{\mu} \sim \mu\, e^{-\frac{2}{\lambda(\mu)}}$

- the gap is fully non-perturbative, at any coupling

Conclusions

- The quantum critical point in GN model exists for any N



QCP

$$\boxed{\mu_c = \frac{m}{N \sin \frac{\pi}{2N-2}}}$$

- High-density phase has quasi-long-range order (no symmetries broken)

- Phonon remains gapless ($\varepsilon \simeq c_s p$) for any N

- Fermion spectrum is always gapped.