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NON-PERTURBATIVE ASPECTS OF QUANTUM GAUGE THEORIES AND SUPERSTRING THEORY

T H E S I S

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The present thesis, submitted for obtaining the scientific degree *"Doctor of Sciences"*, consists of introductory, main and reference parts. The short introductory part outlines the place, the objectives and the significance of the research topics and the main results of the thesis within the framework of contemporary theoretical physics. The main (proper) part provides a systematic and detailed description of the contents and principal results in the included full-text copies of author's publications, which are an inseparable integral part of the thesis. The latter represent *31 selected papers* (out of the 92 works in the full list of author's scientific papers), with *599 independent citations* (out of the 854 independent citations of all author's works) and with *impact-factor 75.400* (out of the total impact-factor 165.730 of all author's works).

The unifying theme of all included scientific papers are various non-perturbative aspects of modern quantum gauge theories and of the related quantum theory of supersymmetric strings under the following topical subdivision:

(a) Three-dimensional gauge theories – $1/N$ expansion, dynamical spontaneous breakdown of internal and discrete space-time symmetries, non-perturbative particle spectra, phase transitions and critical behaviour. These topics include 9 papers [A1–A9] with 77 independent citations and impact-factor 8.944.

(b) Quantum anomalies – anomalous breaking of discrete space-time symmetries; stochastic quantization – symmetries and their anomalous breakdown. These topics include 9 works [B1–B9] with 103 independent citations and impact-factor 21.384.

(c) Super-Poincare covariant quantization of supersymmetric strings; covariant off-shell superspace formulation of super-gauge theories with extended supersymmetry. Here 13 papers [C1–C13] are included with 419 independent citations and impact-factor 45.072.

The concluding reference part contains the list of the principal scientific contributions in the thesis, the list of the selected included author's papers, citation and impact-factor indices.

Acknowledgements

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Contents

1 Objectives and Significance of the Main Results

Throughout the last few decays the concept about grand unification of all fundamental interactions at ultra-high energies occupies a central place in modern theory of elementary particles [1] (for the current string theory perspective, *a.k.a.* "string phenomenology" or "string-inspired model building", see *e.g.* [2]). The principal idea of the grand unification is to describe via minimal number of fundamental parameters the whole multitude of physical laws of the microscopic world:

(i) All the initial large *gauge* symmetries of the fundamental particle interactions at (ultra-)high energies and the patterns of their dynamical breakings at lower energy scales down to the observable gauge symmetry of the standard model at energy scale $\Lambda_0 \simeq 100$ GeV.

(ii) Dependence of the structure and strength of the particle interactions on the energy scales - effective coupling constants.

(iii) Non-trivial and varying structure of the pertinent physical ground states (the vacuums), i.e., the different phases of the quantum field systems and the relevant phase transitions among them conceptually similar to the phases and the phase transition phenomena in condensed matter systems.

(iv) Understanding of the mechanisms for formation of the particle spectra – particle masses, charges and their (possible) composite structure, "confinement" and "deconfinement" phenomena, coherent excitations of the fundamental fields - solitons (magnentic monopoles, vortices, strings *etc*).

At low (with respect to the string theory scale) energies the realistic models of elementary particles are described by $D = 4$ quantum gauge field theories which do not allow for exact solutions (henceforth dimensionality of space-time will be denoted by D). The standard method of their treatment is the "naive" perturbation theory with respect to the coupling constants g_i entering the interacting part of the corresponding Langrangian:

$$L = -\frac{1}{2}\Phi^* K(\partial_x)\Phi + \sum_i g_i L_i(\Phi) \tag{1}$$

Here Φ is a collective notation for the set of fields with different spins and internal degrees of freedom; $K(\partial_x)$ is the kinetic differential operator (of second or first order); $L_i(\Phi)$ contain various interaction terms of degrees higher than 2. The naive perturbation theory approach in quantum field theory has several serious drawbacks:

(a) The requirement for renormalizability of the model (1) imposes strong restrictions on the allowed form of $L_i(\Phi)$, in particular, the coupling constants g_i must be either dimensionless or with a positive mass dimension.

(b) Because of the renormalization of the pertinent ultraviolet divergences, the renormalized coupling constants generically depend on the renormalization mass scale Λ – $g_i^{ren} = g_i^{ren}(\Lambda)$. This fact in many cases constitutes a significant obstacle, *e.g.*, in quantum chromodynamics where the effective coupling grows in the infrared regimes and renders ordinary perturbation theory meaningless.

(c) Obviously, naive perturbation theory is incapable to describe most quantum field theory phenomena inherent to the grand unification picture outlined above ((i)–(iv)) which are intrinsically non-perturbative, *e.g.* confinement/deconfinement, phenomena involving coherent excitations (solitons), whose masses are of the order of $1/g_i$ [3] *etc.*

A contribution toward the non-perturbative treatment of quantum gauge theories is contained in authors papers [A1–A9] which constitute the first part of the thesis. Here we study in great detail the properties of $D = 3$ quantum gauge theories with fermions, including supersymmetric gauge theories, within the framework of the non-perturbative $1/N$ expansion (N being the number of "flavors" of the corresponding matter fields). Although $D = 3$ gauge theories are simpler than the realistic $D = 4$ gauge theories, the former retain most of the crucial qualitative features of the latter. Thus, in our papers [A1–A9] (see next section) we find for the first time in the literature explicit realizations of the non-perturbative mechanisms modelling the principal physical properties of the realistic gauge theories ((i)–(iv)) above: dynamical mass generation, including dynamical generation of gauge-invariant masses for the gluons; multiple phases defined via *more than one* order parameters, which are related to dynamical spontaneous breakdown (and restoration) not only of continuous internal symmetries, but also discrete space-time reflection symmetries; non-perturbative particle spectra, qualitatively different in the various phases, including particle "confinement" in some of the phases and their "deconfinement" in other phases.

Here for the first time in the literature we have proved explicitly the renormalizability of naively non-renormalizable (within the standard perturbation theory) quantum field theory models, including those containing four-fermion interactions. Furthermore, we have developed a systematic quantum field theoretic approach for description of the pertinent phase transitions and critical behaviour of the $D = 3$ gauge theories with fermions, including supersymmetric gauge theories, within the non-perturbative $1/N$ expansion. In particular, we provide explicit construction of the critical theories at the second order phase transition points, which turn out to be three-dimensional supersymmetric nonlinear sigma-models. The latter are nontrivial examples of $D = 3$ quantum *conformal gauge theories with fermions* whose anomalous operator dimensions are explicitly calculable with our $1/N$ expansion techniques.

$D = 3$ gauge theories with fermions provide an arena for systematic non-perturbative study of quantum *anomalous* (not spontaneous) symmetry breaking of discrete space-time symmetries (P- and T-reflections). The primary importance of these *parity-violating* anomalies lies in the fact that discrete space-time reflection symmetries in $D = 3$ are close qualitative analogs and, correspondingly, allow for a deeper understanding of the dynamical anomalous breaking of chiral symmetry in $D = 4$ quantum chromodynamics. Parity-violating anomalies are also relevant in various other important topics, *e.g.*, in the problem of fermion number fractionization [4], the Hall effect [5], anomaly cancellations in higher-dimensional (Kaluza-Klein) field theories [6], *etc.* In our papers [B2,B3,B4] included in the thesis we have proposed an adequate systematic non-perturbative approach for description of the anomalous breaking of discrete space-time symmetries based on the notion of Atiyah-Patodi-Singer topological "eta"-invariant. The gauge-invariant "eta"-function regularization of odd-dimensional fermionic determinants (effective actions of quantized fermions) in the presence of non-trivial non-vanishing background gauge fields at infinity is shown to give both anomalous and spontaneous violation of parity. We have explicitly calculated the competitive contributions of both types of anomalies to the magnitude of induced currents and charges in a static or constant uniform backgrounds.

Stochastic quantization is a relatively new method of quantization of field theory models first proposed by Parisi and Wu [7] (for a review, see Ref.[8]). The main idea of stochastic quantization is to consider D-dimensional Euclidean quantum field theory as the equilibrium limit of an $(D + 1)$-dimensional statistical system coupled to a thermal reservoir. This statistical system evolves in a new fictitious time direction τ where the coupling to a heat reservoir is given by means of a stochastic ("white") noise field. In the equilibrium limit $\tau \to \infty$ stochastic averages become identical to ordinary Euclidean correlation functions (vacuum expectation values).

One of the main advantages of stochastic quantization of gauge field theories lies in the fact that it avoids the use of gauge-fixing and Faddeev-Popov ghost field terms customary in the customary quantization of gauge theories within the standard (naive) perturbation theory. Also, the invariant stochastic regularization of ultraviolet divergencies explicitly preserves all the underlying symmetries. The latter fact underscores the relevance of the results in our papers [B1,B5,B6,B7,B8,B9] included in the thesis. Here for the first time in the literature we have found and thoroughly studied the explicit mechanisms for the appearance of dynamical anomalous (and spontaneous) breakdown of the symmetries in the limiting equilibrium ordinary quantum field theory in spite of their manifest preservation by the stochastic regularization for any finite stochastic time. In particular, we have found in our paper [B8] the stochastic quantization's analog of the famous Noether theorem about symmetries versus conserved currents. Also, we have found the mechanism for topological quantization of physical parameters in theories with topologically nontrivial phase spaces within the stochastic quantization framework. Therefore, all results in [B1,B5,B6,B7,B8,B9] are of primary importance for the self-consistency of the stochastic quantization scheme.

Modern string theory is considered as the most plausible candidate for a unifying theory of all fundamental forces in Nature at ultra-high energies of the order of the Planck mass scale. For the first time in history of physics string theory provides an adequate consistent quantization of gravity, including the understanding of such fundamental cosmological objects like quantum black holes and wormholes (for a review, see Refs.[9]), it underscores the relevance of extra space-time dimensions, the relevance of supersymmetry, it lays new ways to build grand unified theories in particle physics phenomenology [2], it inspires new types of cosmological scenarios such as the so called "brane-worlds", which indicate that our own Universe could be just one copy of many other parallel universes embedded in higher-dimensional space-time [10].

One of the main building blocks in string theory are the fundamental strings with space-time supersymmetry – the so called Green-Schwarz superstrings [11] (see also [9]). The latter are special kinds of gauge theories on the two-dimensional string world sheet. The characteristic feature of Green-Schwarz superstrings from the point of view of their canonical Hamiltonian description is that they represent infinite-dimensional dynamical systems with constraints of both first-class as well as second-class according to Dirac classification [12], which are mixed in a *Lorentz non-covariant* way. To this end let us recall that upon quantization first-class constraints (generating gauge symmetries) and second-class constraints (implementing reduction of the number of physical degrees of freedom) are treated in essentially different ways. Because of the latter fact the naive separation of the first- and second-class constraints in the Green-Schwarz superstring models leads to loss of relativistic invariance, thus necessitating a qualitatively new approach to the problem of manifestly Lorentz-covariant superstring quantization.

In the series of our works [C1–C13] included in the thesis we have provided for the first time in the literature a consistent systematic solution to the above mention problem of covariant Green-Schwarz superstring quantization. This is achieved with the help of the introduction of a special set of auxiliary (pure gauge) bosonic spinorial variables on the string world-sheet, which implement the separation of the pertinent Hamiltonian constraints into first-class and second-class in a manifestly Lorentz-covariant manner. Our original approach has been employed, further developed and applied by various other authors for a manifestly relativistically covariant treatment of super p-branes and super Dirichlet p-branes [13], which subsequently were realized to constitute important building blocks of non-perturbative string theory [9]. As a non-trivial application of our approach we have found a new manifestly covariant off-shell unconstrained superfield formulation of supersymmetric gauge field theories with extended supersymmetry.

The above sketchy review convincingly demonstrates that the theme and objectives of the present thesis fit entirely within the framework of some of the most actively developing research areas of modern theoretical and mathematical physics.

For a complete list of the all scientific contributions in the thesis, please, consult the last Section 7.

Remark about the cited literature. Due to the obvious unfeasibility to supply a complete exhaustive list of all relevant references pertaining to the material in the present thesis, we have restricted ourselves in providing only citations of ground-laying, key or review publications. References to the selected author's papers included as part of the thesis are given according to the numbering in the attached list in Section 6.

2 Non-Perturbative Aspects of Three-Dimensional Gauge Theories With Fermions

2.1 $1/N$ Expansion of Vector QFT Models - General Remarks

$1/N$ expansion of quantum field theory (QFT) models with internal $U(N)$ (or $O(N)$) "flavor" symmetry, where the fundamental matter fields belong to the vector representation (*vector QFT models* for short), is one of the principal, most well understood and systematically developed non-perturbative methods. Here the term "non-perturbative" means that $1/N$ expansion is qualitatively different from standard ("naive") perturbation theory w.r.t. coupling constant(s), as its diagrams involve new types of internal propagator lines corresponding to auxiliary "flavor"-singlet composite fields which are given by infinite resummation of (subsets of) ordinary Feynman diagrams. The latter results in improved ultraviolet behavior of $1/N$ diagrams which coupled with the fact that $1/N$ is dimensionless expansion parameter makes renormalizable those vector QFT models which are non-renormalizable w.r.t. the ordinary perturbation theory. Another important general property of $1/N$ expansion is that it exhibits *linear* realization of $U(N)$ (or $O(N)$) "flavor" symmetry in QFT models with a *nonlinearly* realized "flavor" symmetry, *i.e.*, the nonlinear sigma-models.

$1/N$ expansion is the main instrument in uncovering and for explicit description of the following important properties of QFT (many of these are among the main topics of the present thesis):

(i) $D = 2$ QFT: dynamical breakdown of classical conformal symmetry via *dimensional transmutation* of coupling constants together with *asymptotic freedom*, as well as construction of *higher local quantum conserved currents* in $D = 2$ integrable models.

(ii) $D \geq 3$ QFT: dynamical breaking of continuous and discrete (space- and time-reflection) symmetries; non-trivial phase structure – several distinct types of phases with *multiple order parameters* and the pertinent *phase transitions*; non-perturbative particle spectra qualitatively different in the various phases; *dynamical mass generation* for the fundamental N-component matter fields; dynamical generation of massive gauge bosons where the standard Higgs mechanism is inoperative; particle confinement in some of the phases, explicit appearance of composite bosons and fermions; renormalization of *non-renormalizable* (w.r.t. ordinary perturbation theory) QFT models.

(iii) Further applications of $1/N$ expansion of vector models in various areas of QFT and statistical mechanics (*i.e.*, Euclidean QFT) include: finite-size effects (finite-size scaling in the nonlinear sigma-models); stochastic Langevin equation in dissipative dynamics; finite-temperature QFT (dimensional reduction crossover at high temperature); Bose-Einstein condensation in weakly interacting Bose gas; multicritical points and double scaling limit; for a comprehensive review, see Refs.[14, 15].

Derivation of $1/N$ expansion via functional integral techniques is based on the following general prescription:

(a) Introduce appropriate set of auxiliary "flavor"-singlet fields and rewrite the original action in a (classically) equivalent form which is quadratic w.r.t. fundamental N-component matter fields;

(b) In the functional integral expression for the generating functional of the quantum correlation functions perform the Gaussian functional integral over the N-component matter fields to obtain an effective action depending only on "flavor"-singlet fields, where the factor N appears as a common factor in front of it in the same way as the Planck constant appears as a common factor $1/\hbar$ in front of the ordinary classical action in the standard functional integral;

(c) Then the $1/N$ expansion becomes "semiclassical" expansion around saddle points of the effective "flavor"-singlet action, which can be viewed as vacuum expectation values of the pertinent "flavor"-singlet fields in the leading order w.r.t. $1/N$ (cf. Eqs.(21) below).

2.2 $1/N$ Expansion of $O(N)$ Non-Linear Sigma-Model

Let us start with the simpler example of $1/N$ expansion in $D = 2,3$ $O(N)$ nonlinear sigma-model whose Lagrangian is given by:

$$\mathcal{L}_{NLSM} = -\frac{1}{2}\partial^\nu \varphi^a \partial_\nu \varphi_a \quad , \quad \vec{\varphi}^{\,2} = \mu^{D-2}\frac{N}{T} \quad , \quad \vec{\varphi} = \left(\varphi^1, \ldots \varphi^N\right) \tag{2}$$

Here and below μ denotes a common mass scale exhibiting the dimensionfull nature of the coupling constants (so that T in (2) and T, g, e^2 in (9) below are dimensionless parameters). In $D = 2$ the $O(N)$ nonlinear sigma-model is the physically most interesting example of a $D = 2$ *completely integrable* field theory model due to its deep analogies with the realistic $D = 4$ gauge theories (asymptotic freedom, dimensional transmutation, instantons). Apart from this (2) describes quantum spin chains, effective low energy degrees of freedom of the high-T_c superconductors, *etc.* In higher dimensions $D \geq 3$ (2) and its generalizations (with more complicated target spaces for the fundamental fields $\varphi^a(x)$, such as complex projective and Grassmannian manifolds [16], see below) play fundamental role in the general field theoretic description of phase transitions and critical phenomena (see Refs.[A1–A9] and subsection 2.4 below).

The $1/N$ expansion for the model (2) is obtained from the generating functional of the time-ordered correlation functions:

$$\begin{aligned} Z[J] &= \int \mathcal{D}\vec{\varphi} \prod_x \delta\left(\vec{\varphi}^{\,2} - \mu^{D-2}N/T\right) \exp\left\{ i\int d^2x \left[-\frac{1}{2}\partial^\nu\vec{\varphi}\,\partial_\nu\vec{\varphi} + \left(\vec{J},\vec{\varphi}\right)\right]\right\} \\ &= \int \mathcal{D}\vec{\varphi}\,\mathcal{D}\alpha \exp\left\{ i\int d^2x \left[-\frac{1}{2}\partial^\nu\vec{\varphi}\,\partial_\nu\vec{\varphi} - \frac{1}{2}\alpha\left(\vec{\varphi}^{\,2} - \mu^{D-2}N/T\right) + \left(\vec{J},\vec{\varphi}\right)\right]\right\} \\ &= \int \mathcal{D}\alpha \exp\left\{ iNS_1[\alpha] + \frac{i}{2}\int d^2x\, d^2y \left(\vec{J}(x), (-\partial^2+\alpha)^{-1}\vec{J}(y)\right)\right\} \end{aligned} \tag{3}$$

$$S_1[\alpha] \equiv \frac{i}{2}\,\mathrm{T}r\ln(-\partial^2+\alpha) + \frac{\mu^{D-2}}{2T}\int d^2x\,\alpha \quad , \quad \partial^2 = \partial^\nu\partial_\nu \tag{4}$$

by expanding the effective α-field action (4) around its constant saddle point $\widehat{\alpha} \equiv m^2$, *i.e.*, $\alpha(x) = m^2 + \frac{1}{\sqrt{N}}\tilde{\alpha}(x)$. From the stationary equation $\delta S_1[\alpha]/\delta\alpha\,\big|_{\alpha=m^2} = 0$ one obtains in the $D = 2$ case:

$$m^2 = \hat{\mu}^2 e^{-4\pi/T}\ , \tag{5}$$

where $\hat{\mu}$ is a renormalization mass scale appearing due to renormalization of the ultraviolet divergence coming from the first term in (4) (see Eq.(23) below). Thus, the "Goldstone" fields $\vec{\varphi}$ acquire dynamically generated mass (squared) $\widehat{\alpha} \equiv m^2$, classical conformal invariance of (3) in the $D = 2$ case is broken due to the *dimensional transmutation* (the dimensionless coupling T is replaced by m^2 via Eq.(5)), and the classically nonlinearly realized $O(N)$ "flavor" symmetry becomes linearly realized on quantum level.

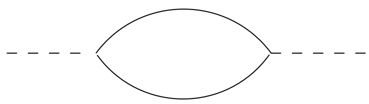
From (4) one arrives at the $1/N$ diagram technique with (free) propagators in momentum space:

$$\left\langle \varphi^a \, \varphi^b \right\rangle_{(0)} = -i \left(m^2 + p^2 \right)^{-1} \delta^{ab} \quad , \quad \langle \tilde{\alpha} \, \tilde{\alpha} \rangle_{(0)} = 2 \Big(\Sigma \left(p^2 \right) \Big)^{-1} \tag{6}$$

with:

$$\Sigma \left(p^2 \right) = - \int \frac{d^D k}{(2\pi)^D} \, \left[\left(m^2 + k^2 \right) \left(m^2 + (p-k)^2 \right) \right]^{-1} \tag{7}$$

and tri-linear $\tilde{\alpha}\varphi\varphi$-vertices, where one-loop φ-tadpoles and subdiagrams of the form in the picture below are forbidden (solid lines depict φ propagators, dashed lines depict $\tilde{\alpha}$ propagators).



A remarkable property of the $1/N$ expansion in nonlinear sigma models has been found in [A1,A2] (see also our paper [17], not included in the thesis). Namely, the nonlinearity of the "Goldstone" field $\vec{\varphi}(x)$ is preserved on the quantum level as an identity on the correlation functions, in spite of the manifest linear $O(N)$ symmetry of the $1/N$ diagrams:

$$\Big\langle \mathcal{N} \Big[\vec{\varphi}^2 P(\vec{\varphi}, \partial\vec{\varphi}) \Big] (x) \ldots \Big\rangle = \text{const} \, \Big\langle \mathcal{N} \Big[P(\vec{\varphi}, \partial\vec{\varphi}) \Big] (x) \ldots \Big\rangle \tag{8}$$

where $P(\vec{\varphi}, \partial\vec{\varphi})$ is arbitrary local polynomial of the fundamental fields and their derivatives, and $\mathcal{N}[\ldots]$ indicates renormalized normal product of the corresponding composite fields (for systematic renormalization of the $1/N$ expansion in arbitrary vector QFT models, see [A1,A2,A6,A8] and subsection 2.5 below)

Another important result contained in our paper [17] (not included in the thesis) is the explicit construction within the properly renormalized $1/N$ expansion of higher quantum local conserved currents in the $D{=}2$ $O(N)$ nonlinear sigma-model which provides the proof of its complete integrability on quantum level.

2.3 $1/N$ Expansion of Generalized Non-Linear Sigma-Models

Now we go over to the main subject of our considerations - $D = 3$ gauge theories with fermions, specifically, $D{=}3$ *gauged nonlinear sigma-models with fermions* $(GNLSM + F)_3$, with internal symmetry $U(N)$("flavor") $\times$ $U(n)$("color" gauge) ($n < N$). Special physically relevant cases of the latter are the *supersymmetric nonlinear sigma-models* on complex projective and Grassmannian manifolds, *e.g.*, by taking $e^2 \to \infty$, $\varepsilon = 1$ in (9) below. On the other hand, $(GNLSM + F)_3$ themselves can be viewed as special fixed points of general

$D = 3$ (non-Abelian) Higgs models with fermions, containing "non-renormalizable" four-fermion couplings (see our papers [18] not included in the thesis). $(GNLSM + F)_3$ are of particular physical interest since their $1/N$ expansion explicitly displays all the fundamental non-perturbative properties listed above under (ii). Below we will discuss for simplicity $1/N$ expansion for the Abelian $(GNLSM + F)_3$ ($n = 1$, the non-Abelian case being a straightforward generalization).

The pertinent Lagrangian reads:

$$\mathcal{L}_{\text{GNLSM+F}} = -\left(\nabla^\nu(A)\varphi_a\right)^* \left(\nabla_\nu(A)\varphi^a\right) + i\bar{\psi}_a\gamma^\nu\nabla^{(\varepsilon)}_\nu(A)\psi^a + \frac{g}{4N\mu}\left(\bar{\psi}_a\psi^a\right)^2 - \frac{N}{4e^2\mu}F_{\nu\lambda}(A)F^{\nu\lambda}(A) \tag{9}$$

with constraints:

$$\varphi^*_a\varphi^a - N\mu/T = 0 \quad , \quad \bar{\psi}_a\varphi^a = \varphi^*_a\psi^a = 0 \; . \tag{10}$$

Here the following notations are used: $\nabla_\nu(A)\varphi^a = (\partial_\nu + iA_\nu)\,\varphi^a$, $\nabla^{(\varepsilon)}_\nu(A)\psi^a = (\partial_\nu + i\varepsilon A_\nu)\,\psi^a$, $F_{\nu\lambda}(A) = \partial_\nu A_\lambda - \partial_\lambda A_\nu$, where the "flavor" indices $a = 1, \ldots, N$ and the space-time indices $\nu, \lambda = 0, 1, 2$; ε is the ratio of fermionic to bosonic electric charges; γ_ν are the standard $D = 3$ Dirac gamma-matrices. Note the presence of the "non-renormalizable" four-fermion (*Gross-Neveu*) term in (9).

Apart from the continuous $U(N)$("flavor")$\times U(1)$(gauge) symmetry, $(GNLSM+F)_3$ (9) is invariant also under the discrete space-time transformations – space (P-) and time (T-) reflections:

$$\varphi^{(P,T)}(x) = \eta_{P,T}\varphi(x_{P,T}) \quad , \quad \psi^{(P,T)}(x) = \eta_{P,T}\gamma_{1,2}\varphi(x_{P,T}) \; , \tag{11}$$

$$A^{(P)}(x) = (A_0, -A_1, A_2)(x_{P,T}) \quad , \quad A^{(T)}(x) = (A_0, -A_1, -A_2)(x_{P,T}) \; , \tag{12}$$

where:

$$x_P = (x^0, -x^1, x^2) \quad , \quad x_T = (-x^0, x^1, x^2) \quad , \quad |\eta_{P,T}| = 1 \; . \tag{13}$$

Note that the fermionic mass term (which does not appear in (9)) reverses sign under P,T-reflection:

$$\bar{\psi}^{(P,T)}\psi^{(P,T)}(x) = -\bar{\psi}\psi(x_{P,T}) \; , \tag{14}$$

and due to its absence in (9) the classical $(GNLSM+F)_3$ is P,T-invariant. Therefore, P,T-reflection symmetries can be viewed as $D = 3$ analogues of the *chiral symmetry* in $D = 4$ gauge theories with massless chiral fermions.

Introducing a set of auxiliary $U(N)$-singlet fields (real scalar α,σ and complex fermionic ρ) one can rewrite the action (9) in the following (classically) equivalent form:

$$\begin{aligned}\mathcal{L}_{\text{GNLSM+F}} = &-\left(\nabla^\nu(A)\varphi_a\right)^* \left(\nabla_\nu(A)\varphi^a\right) - \alpha\left(\varphi^*_a\varphi^a - N\mu/T\right) + i\bar{\psi}_a\gamma^\nu\nabla^{(\varepsilon)}_\nu(A)\psi^a - \sigma\bar{\psi}_a\psi^a \\ &-\frac{N\mu}{g}\,\sigma^2 + \varphi^a\left(\bar{\psi}_a\rho\right) + \left(\bar{\rho}\psi^a\right)\varphi^*_a - \frac{N}{4e^2\mu}F_{\nu\lambda}(A)F^{\nu\lambda}(A) \; .\end{aligned} \tag{15}$$

Derivation of the $1/N$ expansion for the quantum generating functional $Z[J_\Phi]$ of the time-ordered correlation functions of (15) proceeds along similar lines as for the $O(N)$ nonlinear sigma-model (2)–(4). Unlike the $D = 2$ case, in $D \geq 3$ the fundamental N-component scalar field may acquire non-zero vacuum expectation value for certain range of the parameters, therefore, it is appropriate to split it in two parts – parallel and orthogonal w.r.t. direction of the (possible) vacuum expectation value: $\vec{\varphi} = \vec{\varphi}_{||} + \vec{\varphi}_{\perp}$. Without loss of generality one may choose:

$$\vec{\varphi}_{||} = (0, \ldots, 0, N^{\frac{1}{2}}\varphi_{||}) \quad , \quad \vec{\varphi}_{\perp} = (\varphi_1, \ldots, \varphi_{N-1}, 0) \; . \tag{16}$$

Then performing the Gaussian functional integration w.r.t. $\vec{\varphi}_{\perp}$, ψ one gets:

$$Z[J_\Phi] = \int \mathcal{D}\vec{\varphi}_{\perp}\mathcal{D}\psi\mathcal{D}\varphi_{||}\mathcal{D}\alpha\mathcal{D}\sigma\mathcal{D}\rho\mathcal{D}A_\mu \,\exp\Big\{ i \int d^3x \Big[L_{\rm GNLSM+F} + \sum_{\Phi=\varphi,\psi,\ldots} J_\Phi(x)\Phi(x) \Big] \Big\} \tag{17}$$

$$= \int \mathcal{D}\varphi_{||}\mathcal{D}\alpha\mathcal{D}\sigma\mathcal{D}\rho\mathcal{D}A_\mu \,\exp\Big\{ iNS_1\left[\varphi_{||}, \alpha, \sigma, \rho, A\right] + iS_2[J_\Phi] \Big\} \tag{18}$$

Here the effective action reads:

$$S_1\left[\varphi_{||}, \alpha, \sigma, \rho, A\right] = i(1 - 1/N)\,{\rm Tr}\ln\Delta_B - i\,{\rm Tr}\ln\Delta_F$$
$$+ \int d^3x \Big[-\frac{1}{2}\varphi_{||}^*\Delta_B\varphi_{||} + \alpha\,\mu/T - \sigma^2\mu/g - \frac{1}{4e^2\mu}F_{\nu\lambda}(A)F^{\nu\lambda}(A) \Big] \; , \tag{19}$$

where:

$$\Delta_F \equiv i\gamma^\nu\nabla_\nu^{(\varepsilon)}(A) - \sigma \quad , \quad \Delta_B \equiv -\nabla^\nu(A)\nabla_\nu(A) + \alpha + \bar{\rho}\Delta_F^{-1}\rho \; , \tag{20}$$

and $S_2[J_\Phi]$ contains the terms with the sources.

Because of Lorentz invariance of the vacuum only $\varphi_{||}$, α and σ may have non-zero constant stationary values $\widehat{\varphi}_{||} \equiv v$, $\widehat{\alpha} \equiv m_\varphi^2$, $\widehat{\sigma} \equiv m_\psi$, where:

$$\langle\varphi^a\rangle = N^{\frac{1}{2}}\left[v\delta_N^a + O(N^{-1})\right] \quad , \quad \langle\alpha\rangle = m_\varphi^2 + O(N^{-1}) \; ,$$
$$\langle\bar{\psi}\psi\rangle = \frac{2N\mu}{g}\langle\sigma\rangle = \frac{2N\mu}{g}\left[m_\psi + O(N^{-1})\right] \tag{21}$$

Thus, the saddle-point equations acquire the form:

$$\frac{\delta S_1}{\delta\varphi_{||}^*} = -m_\varphi^2\, v = 0 \quad , \quad \frac{\delta S_1}{\delta\alpha} = \frac{m_\varphi}{4\pi} - \left[|v|^2 + \mu\left(\frac{1}{T_c} - \frac{1}{T}\right)\right] = 0 \; ,$$
$$\frac{\delta S_1}{\delta\sigma} = -2m_\psi\left[\frac{m_\psi}{4\pi} - \mu\left(\frac{1}{T_c} - \frac{1}{g}\right)\right] = 0 \tag{22}$$

The dimensionless constant $T_c = 4\pi\mu/\hat{\mu}$ arises in the evaluation of the ultraviolet-divergent integrals appearing in the variational derivatives of S_1 which are renormalized according to the "soft-mass" *BPHZL* subtraction scheme (for details, see subsection 2.5) with arbitrary scale $\hat{\mu}$ (in particular, one may take $\hat{\mu} = \mu$) :

$$i\frac{\delta\,{\rm Tr}\ln\Delta_B}{\delta\alpha}\bigg|_{\widehat{\alpha}=m_\varphi^2,\ldots,\widehat{\rho}=0} = \left\{ i\int d^Dp/(2\pi)^D\left[m_\varphi^2 + p^2\right]^{-1} \right\}^{\rm ren}$$

$$= i\int d^Dp/(2\pi)^D\,\left[\left(m_\varphi^2 + p^2\right)^{-1} - \left(\hat{\mu}^2 + p^2\right)^{-1}\right] = \begin{cases} \frac{1}{4\pi}\ln\left(m_\varphi^2/\hat{\mu}^2\right) & \text{for } D = 2 \\ \frac{1}{4\pi}\left(m_\varphi - \hat{\mu}\right) & \text{for } D = 3 \end{cases} \tag{23}$$

and similarly for $-i\left\{\delta\,{\rm Tr}\ln\Delta_F/\delta\sigma\right\}\big|_{\widehat{\sigma}=m_\psi, A=0}$.

2.4 Structure of Ground States and Phase Transitions. Non-Perturbative Particle Spectra in Different Phases

The solutions of the saddle-point equations (22) yield the following phase structure of $(GNLSM + F)_3$ (9) characterized by the values of *two order parameters* $\langle\vec{\varphi}\rangle$, $\langle\bar{\psi}\psi\rangle$ (21):

(I) $U(N)$("flavor") $\times U(1)$(gauge) and P, T-symmetric "high-temperature" phase for $T > T_c$ and $0 < g < T_c$, where:

$$v = 0 \quad , \quad m_\varphi = 4\pi\mu\,(1/T_c - 1/T) \quad , \quad m_\psi = 0\,. \tag{24}$$

(II) $U(N)$("flavor") $\times U(1)$(gauge) symmetric "high-temperature" phase with spontaneous breakdown of discrete P, T-reflection symmetries due to dynamical generation of fermionic mass m_ψ for $T > T_c$ and either $g < 0$ or $T_c < g < 2T_c$, where:

$$v = 0 \quad , \quad m_\varphi = 4\pi\mu\,(1/T_c - 1/T) \quad , \quad m_\psi = 4\pi\mu\,(1/T_c - 1/g)\ . \tag{25}$$

(III) P, T-symmetric "low-temperature" phase with spontaneous breakdown of internal $U(N)$("flavor") $\times U(1)$(gauge) due to non-zero $\langle\vec{\varphi}\rangle$ (21) for $T < T_c$ and $0 < g < T_c$, where:

$$|v|^2 = \mu\,(1/T - 1/T_c) \quad , \quad m_\varphi = 0 \quad , \quad m_\psi = 0\,. \tag{26}$$

(IV) "Low-temperature" phase with spontaneous breakdown of both the discrete P, T-symmetries (as in phase (II)) and internal symmetry (as in phase (III)) for $T < T_c$ and either $g < 0$ or $T_c < g < 2T_c$, where:

$$|v|^2 = \mu\,(1/T - 1/T_c) \quad , \quad m_\varphi = 0 \quad , \quad m_\psi = 4\pi\mu\,(1/T_c - 1/g)\ . \tag{27}$$

Let us recall that P, T-reflection symmetries are $D = 3$ analogues of the fermionic chiral symmetry in $D = 4$.

The restriction $g < 2T_c$ above originates from the stability requirement for the *quantum effective potential* of $(GNLSM + F)_3$ (9). According to the general definition it is given as a Legendre transform of the logarithm of the quantum generating functional (17):

$$\mathcal{U}\left(\langle\vec{\varphi}\rangle, \langle\bar{\psi}\psi\rangle\right) = -i\ln Z\left[J_\varphi, J_{\bar{\psi}\psi}\right] - \left(J^{*}_{\varphi\,a}\langle\varphi^a\rangle + \langle\varphi^{*}_a\rangle J^a_\varphi + J_{\bar{\psi}\psi}\langle\bar{\psi}\psi\rangle\right)\ . \tag{28}$$

In the large-N limit one obtains (cf. relations (21)):

$$N^{-1}\mathcal{U}\left(\langle\vec{\varphi}\rangle, \langle\bar{\psi}\psi\rangle\right) = \mathcal{U}_1\left(\langle\vec{\varphi}\rangle, \langle\sigma\rangle\right) - g/4\mu\,\left(\delta\mathcal{U}_1/\delta\langle\sigma\rangle\right)^2\ , \tag{29}$$

where:

$$\mathcal{U}_1\left(\langle\vec{\varphi}\rangle, \langle\sigma\rangle\right) = 1/6\pi\,\left(|\langle\sigma\rangle|^3 - \langle\alpha\rangle^{3/2}\right) - \mu|\langle\sigma\rangle|^2\,(1/T_c - 1/g) + \langle\sigma\rangle\left[|\langle\vec{\varphi}\rangle|^2 + \mu\,(1/T_c - 1/T)\right]\ . \tag{30}$$

From (29)–(30) we get:

$$\frac{\delta^2\mathcal{U}}{(\delta\langle\bar{\psi}\psi\rangle)^2} > 0\ \ \text{(stability)}\ \ \text{only}\ \ \text{for}\ \ g < 2T_c \tag{31}$$

All phase transitions between any pair of the above phases are *second-order* on the lines $T = T_c$ and $g = T_c$ on the (T, g) parameter plane. On the other hand, the line $g = 0$ corresponds to *first-order* phase transitions between phases (I) and (II) for $T > T_c$, and between phases (III) and (IV) for $T < T_c$.

All four phases exhibit qualitatively different non-perturbative particle spectra. The spectra are directly derived from the momentum-space pole structure of the propagators in the pertinent $1/N$ diagrams, where the propagators themselves are determined from the quadratic part of the expansion of the large-N effective action (19) around its saddle points. Here we will write down the explicit form of the "free" $1/N$ propagators in momentum space in an unified form valid simultaneously in all different phases (I)–(IV) of $(GNLSM + F)_3$ (9)–(10). For the fundamental matter and gauge fields we have:

$$\langle\varphi_a\varphi_b^*\rangle^{(0)}(p) = (-i)\delta_{ab}\left[m_\varphi^2 + p^2\right]^{-1} + \left(p^2\right)^{-2} v_a v_b^* N\,\langle\alpha\,\alpha\rangle^{(0)}(p)\;, \tag{32}$$

$$\left\langle\psi_a\bar{\psi}_b\right\rangle^{(0)}(p) = (-i)\delta_{ab}\left(m_\psi - \gamma^\nu p_\nu\right)\left[m_\psi^2 + p^2\right]^{-1} + \left(p^2\right)^{-1} v_a v_b^* N\,\langle\rho\,\bar{\rho}\rangle^{(0)}(p)\;, \tag{33}$$

$$\begin{aligned}\langle A_\nu\,A_\lambda\rangle^{(0)}(p) = N^{-1}i\left[\mathcal{F}^2(p;e,\varepsilon) + p^2\varepsilon^4\mathcal{G}^2(p)\right]^{-1}\\ \times\left\{(\eta_{\nu\lambda} - p_\nu p_\lambda/p^2)\mathcal{F}(p;e,\varepsilon) + i\varepsilon^2\varepsilon_{\nu\lambda\kappa}p^\kappa\mathcal{G}(p)\right\}\;,\end{aligned} \tag{34}$$

where the following notations are used:

$$\begin{aligned}\mathcal{F}(p;e,\varepsilon) \equiv \frac{p^2}{e^2\mu} + |v|^2 + \left(4m_\varphi^2 + p^2\right)\frac{1}{2}F(p^2;m_\varphi) - \frac{1}{4\pi}m_\varphi\\ +\varepsilon^2\left[\frac{1}{4\pi}|m_\psi| - \left(4m_\psi^2 - p^2\right)\frac{1}{2}F(p^2;m_\psi)\right]\end{aligned} \tag{35}$$

$$\mathcal{G}(p) \equiv 2m_\psi\,F(p^2;m_\psi) = \frac{1}{4\pi}\mathrm{sign}(m_\psi)f(-p^2/4m_\psi^2) \tag{36}$$

with:

$$f(z) = \begin{cases}(-z)^{-\frac{1}{2}}\arctan[(-z)^{\frac{1}{2}}] & \text{for } z < 0\\ \frac{1}{2}z^{-\frac{1}{2}}\ln[(1 + z^{\frac{1}{2}})(1 - z^{\frac{1}{2}})^{-1}] & \text{for } z > 0\end{cases}\;. \tag{37}$$

The propagators for the auxiliary "flavor"-singlet fields read:

$$\langle\alpha\,\alpha\rangle^{(0)}(p) = iN^{-1}\left[F(p^2;m_\varphi) + 2|v|^2/p^2\right]^{-1}\;, \tag{38}$$

$$\langle\sigma\,\sigma\rangle^{(0)}(p) = -iN^{-1}\Big[\left(4m_\psi^2 + p^2\right)\frac{1}{2}F(p^2;m_\psi) + 2\mu(1/g - 1/T_c)\theta(g)\theta(T_c - g)\Big]\;, \tag{39}$$

$$\langle\rho\,\bar{\rho}\rangle^{(0)}(p) = 2iN^{-1}\Big[(2m_\psi - \gamma^\nu p_\nu)\,F\big(p^2;(m_\varphi + m_\psi)/2\big) + 2i|v|^2\,(m_\psi + \gamma^\nu p_\nu)^{-1}\Big]^{-1}\;. \tag{40}$$

The richest particle spectrum occurs in phase (II). It contains:

(1) N pairs of bosons φ_a and fermions ψ_a with *dynamically generated* masses.

(2) *Massive composite fermion* corresponding to the auxiliary fermionic singlet field ρ in the regime $m_\varphi > |m_\psi|$ with mass $m_\rho = 2|m_\psi|$. This fermion can be viewed as a bound state of the fundamental φ- and ψ-quanta.

(3) Dynamically generated *topologically massive gauge bosons* A_ν – this is due to the presence in phase (II) of P, T-parity breaking term containing $\varepsilon_{\nu\lambda\kappa}$ in their propagator (34). In the low-energy limit ($|p|^2 \ll m^2_{\varphi,\psi}$) the latter term is equivalent to the appearance of an effective gauge-invariant topological Chern-Simmons term:

$$\frac{N}{16\pi} \int d^3x \, \varepsilon^{\nu\lambda\kappa} A_\nu F_{\lambda\kappa}(A) \tag{41}$$

in the $(GNLSM + F)_3$ Lagrangian (9) (in the non-Abelian case Chern-Simmons terms are invariant under gauge transformations up to a shift by the topological charge of the corresponding gauge group element, see next section).

The gauge bosons A_ν disappear completely (they are "confined") in all the remaining phases (I), (III) and (IV) due to the appearance of $\sqrt{p^2}$-singularity in the A_ν-propagators (34). Thus, in spite of the unbroken gauge symmetry in phases (I) and (II), massless gauge bosons are absent there. Also, the standard Higgs mechanism for generating masses of gauge bosons does not operate in phases (III) and (IV) in spite of the spontaneous breakdown of the gauge symmetry there.

In phase (I) there are N massive bosons φ_a, N massless fermions ψ_a and a *massless composite fermion ρ*.

In phase (III) the particle spectrum consists of only $N-1$ massless Goldstone bosons $\varphi_\perp$ and $N-1$ massless fermions $\psi_\perp$ (the splitting $\psi = \psi_{||} + \psi_\perp$ is defined in complete analogy with (16)). All remaining fields are *"confined"* here, in particular, the gauge bosons A_ν and part of the fundamental bosonic $\varphi_{||}$ and fermionic $\psi_{||}$ matter fields.

In phase (IV) there are the same $N-1$ massless Goldstone bosons $\varphi_\perp$ as in phase (III), $N-1$ massive fundamental fermions $\psi_\perp$ and a massless fundamental fermion $\psi_{||}$, all the remaining fields being "confined".

It is also worth mentioning that at the critical point $T = T_c$, $g = T_c$ and upon taking the scaling limit $(GNLSM+F)_3$ (9) becomes the $D = 3$ *supersymmetric non-linear sigma-model* on the complex projective space CP^{N-1}:

$$\mathcal{L}_{\text{susy}-CP^{N-1}} = -\left(\nabla^\nu(A)\varphi_a\right)^* \left(\nabla_\nu(A)\varphi^a\right) + i\bar{\psi}_a\gamma^\nu\nabla_\nu(A)\psi^a + \frac{T_c}{4N\mu}\left(\bar{\psi}_a\psi^a\right)^2 \tag{42}$$

with constraints:

$$\varphi^*_a\varphi^a - N\mu/T_c = 0 \quad , \quad \bar{\psi}_a\varphi^a = \varphi^*_a\psi^a = 0 \; . \tag{43}$$

This is a *non-trivial $D = 3$ superconformal field theory* with a well-defined renormalizable $1/N$ expansion where all relevant *anomalous conformal dimensions* (some of them describing the *critical behaviour* of $(GNLSM + F)_3$ (9) in the vicinity of the second-order phase transitions) can be explicitly computed order by order in $1/N$ from our $1/N$ diagram techniques ($1/N$ propagators (32)–(40) with $m_{\varphi,\psi} = 0$, $v = 0$, $g = T_c$, $e^2 \to \infty$, $\varepsilon = 1$).

For further details, see [A7,A8,A9] and also our papers [18] not included in the thesis.

2.5 Renormalization of $1/N$ Expansion. Renormalization of Non-Renormalizable Theories

Although the diagrams of the $1/N$ expansion of vector QFT models have vastly improved ultraviolet behaviour in comparison with the Feynman diagrams of ordinary "naive" perturbation theory, they still contain ultraviolet divergences which can be systematically renormalized both in $D = 2$ and $D \geq 3$ by a version (developed in our papers [A1,A2,A6,A8]) of the mass-independent ("soft-mass") momentum-space subtraction procedure of Zimmermann-Lowenstein [19], which in turn is based on the general Bogoliubov-Parasiuk-Hepp-Zimmermann [20] renormalization scheme (*BPHZL* scheme for short). The BPHZL renormalization scheme has the advantage over other renormalization schemes in that it can be applied simultaneously and in an uniform way in all phases of the pertinent QFT models, especially in those of them with phases containing massless particles where particular care is needed to avoid possible *infrared singularities*. The general idea is to perform all subtractions in the integrands of ultraviolet-divergent Feynman diagrams at zero external momenta and at *zero values of the mass parameters* except for those which by naive power counting would give rise to infrared divergences, so that the latter subtractions are performed at zero external momenta but at *non-zero values of the mass parameters.*

Technically, this is accomplished in the following way:

(a) One rescales temporarily all dimensionfull (mass) parameters M entering the propagators and vertices of any $1/N$ diagram $M \rightarrow s^{d_M} M$, where d_M is the canonical mass dimension of M. In particular, in the "free" $1/N$ propagators for the gauge and auxiliary "flavor" singlet fields (34)–(40) $m_{\varphi,\psi} \rightarrow s\, m_{\varphi,\psi}$ and the low-temperature vacuum expectation value $|v| \rightarrow s^{1/2}|v|$. At the end of the subtraction procedure the auxiliary parameter s is set to $s = 1$.

(b) For the masses in the propagators of the fundamental N-component matter fields (32)–(33) one assigns temporarily a slightly more complex dependence on the auxiliary parameter s:

$$\Big[m_\varphi^2 + p^2\Big]^{-1} \quad \longrightarrow \quad \Big[(s m_\varphi + (1-s)\hat{\mu})^2 + p^2\Big]^{-1} , \tag{44}$$

$$(m_\psi - \gamma^\nu p_\nu)\Big[m_\psi^2 + p^2\Big]^{-1} \quad \longrightarrow \quad (s m_\psi - \gamma^\nu p_\nu)\Big[(s m_\psi + (1-s)\hat{\mu})^2 + p^2\Big]^{-1} , \tag{45}$$

where $\hat{\mu}$ is arbitrary renormalization mass scale and again at the end of the subtraction procedure one sets $s = 1$.

(c) The momentum space Taylor subtraction operators $\tau^{\delta(\Gamma),\rho(\Gamma)}$ in the fundamental *"forest formula"* [20, 19] of the recursive *BPHZL* subtraction scheme, acting on the integrand of a UV-divergent (sub)diagram Γ, are now defined as:

$$1 - \tau^{\delta(\Gamma),\rho(\Gamma)} = \Big(1 - t^{\rho(\Gamma)-1}_{\{p\},s-1}\Big)\Big(1 - t^{\delta(\Gamma)}_{\{p\},s}\Big) \ . \tag{46}$$

Here $\{p\}$ is the set of all external momenta, $\delta(\Gamma)$ and $\rho(\Gamma)$ are the ultraviolet and infrared indices of the (sub)diagram Γ. $\delta(\Gamma)$ is determined from the asymptotic behaviour of the integrand of Γ for large internal momenta, whereas $\rho(\Gamma)$ is determined from the asymptotic behaviour of the Γ-integrand for small internal momenta at vanishing external momenta and all masses set to zero. $t^n_{x,y}$ denotes the usual Taylor subtraction operator:

$$t^n_{x,y} F(x,y) \equiv \sum_{k,l=0\,,\,k+l\leq n}^{n} \frac{x^k}{k!} \frac{y^l}{l!} \frac{\partial^{k+l} F}{\partial x^k \partial y^l} \Big|_{x=0,y=0} \ . \qquad (47)$$

Further details are contained in the included papers [A1,A2,A6,A8].

3 Anomalies in Odd-Dimensional Quantum Field Theory and in Stochastic Quantization Scheme

Quantum mechanical breaking of classical symmetries, *a.k.a.* quantum anomalies, are rather a rule than an exception in quantum field theory. First of all, breaking of the "naive" scaling of dimensionfull quantities (particle momenta, masses *etc*) is the ubiquitous anomaly present in *any* quantized field-theoretic model. It is due to the inevitable introduction of an additional (not present on classical level) mass scale in the process of renomalizing the inherent ultraviolet divergencies. Thus, scaling anomalies (anomalous dimensions of quantized fields) control high-energy behaviour of particle scattering processes (*e.g.*, "asymptotic freedom" in quantum chromodynamics (for a review, see Ref.[22]) as well as critical behaviour of statistical mechanical systems (*a.k.a.* Euclidean quantum field theories) in the vicinity of second order phase transition critical points, where the basic instrument are the renormalization group equations [15]. See our papers [A1–A8] from the previous section for non-perturbative description of critical behaviour of $D = 3$ gauge theories (with fermions).

The next most important class of quantum anomalies are the chiral anomalies in D=even dimensional gauge theories with massless fermions. They also play fundamental role in determining low-energy pseudo-scalar meson processes (Adler-Bell-Jackiw anomaly) as well as providing of guiding principles for determining the correct fermionic multiplets in (grand) unified and string theory inspired gauge theories of fundamental particle interactions ("anomaly cancellation" mechanisms). For review, see [23, 24, 25]. In subsection 3.4 below we will study systematically the mechanism through which chiral anomalies arise in the relatively new alternative to standard quantum field theory – the so called *stochastic quantization scheme*.

We will first start with discussion of D=odd dimensional gauge theories with massless fermions where the concept of chiral gauge symmetry is absent. However, there occurs a different type of anomaly – quantum anomalous breaking of the discrete space-time reflection (P- and T-reflection) symmetries. The behaviour of bilinear fermionic composite fields, *e.g.*, fermionic mass terms under P- and T-reflections in odd D is analogous to their transformation properties under chiral gauge symmetries in even D. Therefore, odd-dimensional parity-violating anomalies provide a nontrivial "toy" model for deeper understanding of anomalous chiral symmetry breaking in quantum chromodynamics.

3.1 Parity-Violating Anomalies

The effective action of quantized fermions in background gauge (and scalar) fields is of the form (in this and next subsections we will consider *Euclidean* quantum field theories):

$$\exp\{-S_{eff}(A,\varphi)\} = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\,\exp\{-\int d^D x\,\bar{\psi}\,i\,\nabla\!\!\!/(A,\varphi)\psi\}$$
$$-S_{eff}(A,\varphi) = \ln\det[-i\,\nabla\!\!\!/(A,\varphi)] \qquad (48)$$

yielding the induced fermionic current:

$$J^\mu_a(x) = \langle\bar{\psi}(x)T_a(-i\gamma^\mu)\psi(x)\rangle = -i[\delta/\delta A^a_\mu(x)]\ln\det[-i\,\nabla\!\!\!/(A,\varphi)] \qquad (49)$$

where the Dirac operator:

$$\not{\nabla}(A,\varphi) \equiv \gamma^\mu[\partial_\mu + iA_\mu(x)] + \varphi(x) \quad ; \quad A_\mu = T_a A^a_\mu \ , \ \varphi(x) = \varphi_1(x) + \gamma^5\varphi_2(x) \ , \tag{50}$$

($\varphi_2 \equiv 0$ for odd D), $\{T^a\}_{a=0,1,\ldots,n^2-1}$ are hermitian generators of $G = U(n)$.

A systematic non-perturbative gauge-invariant regularization and renormalization of the one-loop fermionic effective action (48) [B2,B3,B4,B1] (see also [26] and our paper [27] not included in the thesis) is achieved via the *heat-kernel method* for elliptic operators (see Ref.[28] for mathematical background, and Ref.[29] for a physicist's perspective). Let B be an elliptic operators with spectral asymmetry, *e.g.*, $B = \not{\nabla}(A,\varphi)$:

$$\mathcal{B} = \int d\lambda\, \lambda\, \mathcal{P}_B(\lambda) \quad , \quad \mathcal{P}_B(-\lambda) \neq \mathcal{P}_B(\lambda) \ , \tag{51}$$

where $\mathcal{P}_B(\lambda)$ is the spectral density operator. Using the well-known *heat-kernel* representation of fractional powers of non-negative elliptic operators $\mathcal{B}$, *e.g.*, $\mathcal{B} = B^2 = \not{\nabla}^2(A,\varphi)$:

$$\mathcal{B}^{-s} = \int_0^\infty d\tau\, \frac{\tau^{s-1}}{\Gamma(s)} \exp\{-\tau\mathcal{B}\} \tag{52}$$

we have the following heat-kernel representations for zeta-functions and eta-functions:

$$\zeta_{B^2}(s) = \sum_{\lambda\neq 0} (\lambda^2)^{-s} = \mathrm{T}r \int d\lambda\, \lambda^{-2s}\mathcal{P}_B(\lambda) = \mathrm{T}r(B^2)^{-s} = \mathrm{T}r \int_0^\infty d\tau\, \frac{\tau^{s-1}}{\Gamma(s)} \exp\{-\tau\, B^2\} \ , \tag{53}$$

$$\begin{aligned} \eta_B(s) &= \sum_{\lambda\neq 0} \mathrm{sign}(\lambda)|\lambda|^{-s} = \int d\lambda\, \mathrm{sign}(\lambda)|\lambda|^{-s}\, \mathrm{T}r[\mathcal{P}_B(\lambda)] \\ &= \mathrm{T}r\Big[B(B^2)^{\frac{-s+1}{2}}\Big] = \int_0^\infty d\tau \frac{\tau^{\frac{s-1}{2}}}{\Gamma(\frac{s+1}{2})} \mathrm{T}r\Big[Be^{-\tau B^2}\Big] \ . \end{aligned} \tag{54}$$

In particular:

$$\eta_B \equiv \eta_B(0) = \int_0^\infty d\tau\, (\pi\tau)^{-1/2}\, \mathrm{T}r\left[B \exp\{-\tau\, B^2\}\right] \tag{55}$$

is the spectral asymmetry measuring eta-invariant of Atiyah-Patodi-Singer [30].

Using the heat-kernel representation (55) we derive the following explicit formula for the variation of the eta-invariant with respect to variation of the coefficients of the underlying elliptic operator denoted by δB below:

$$\delta\eta_B(0) = -\frac{2}{\sqrt{\pi}}\, \mathrm{T}r\Big[(\delta B)\Phi_{-\frac{1}{2}}(B^2;\cdot)\Big] + 2\,\mathrm{T}r\Big[(\delta B)\mathcal{P}_B(0)\Big] \tag{56}$$

The first term on the right hand side of (56) is due to the short proper-time (the ultraviolet) behaviour of the heat kernel. The object $\Phi_{-\frac{1}{2}}(B^2;\cdot)$ is the coefficient in front of $\tau^{-1/2}$ in the *Seeley-De Witt expansion* of the heat-kernel [28]:

$$\exp\{-\tau B^2\}(x,x) = \sum_{j=0}^\infty \tau^{(j-D)/2}\Phi_{(j-D)/2}(B^2;x) \tag{57}$$

Let us emphasize that all Seeley coefficients $\Phi_{(j-D)/2}(B^2;x)$ are local functionals of dimension j with respect to the coefficients of B. In particular, Seeley coefficients for $B = \nabla\!\!\!/(A,\varphi)$ (the Dirac operator (50)) are local gauge-invariant functionals of the corresponding fields.

The second term on the right hand side of (56) is due to the large proper-time (the infrared) behaviour of the heat kernel. The object $\mathcal{P}_B(0)$ is the operator spectral density for the corresponding zero-modes and, accordingly, it vanishes in the absence of zero-modes of B.

Formula (56) plays fundamental role in deriving all new results related to parity-violating anomalies in [B2,B3,B4,B1].

Now, with the help of (51)–(55) and using the identity $\ln\lambda = \frac{1}{2}\ln\lambda^2 - \frac{i\pi}{2}\big(\mathrm{sign}(\lambda)-1\big)$ inside operator spectral integrals, we obtain the following gauge invariant renormalized expression for the determinant of the Dirac operator (50):

$$\ln\det[-i\nabla\!\!\!/(A,\varphi)] = \frac{1}{2}\ln\det[\nabla\!\!\!/^2(A,\varphi)] - i\frac{\pi}{2}\eta_{\nabla\!\!\!/(A,\varphi)} - S_{\mathrm{c.t.}}[A,\varphi] \tag{58}$$

The last term in (58) is local gauge-invariant *counterterm* accounting for the renormalization ambiguity.

The first "normal" term on the right hand side of (58) is given by the *zeta-function* regularization formula (cf. (53)):

$$\ln\det[\nabla\!\!\!/^2(A,\varphi)] = -\int_0^\infty \frac{d\tau}{\tau}\,\mathrm{T}r\{\exp[-\tau\nabla\!\!\!/^2(A,\varphi)]\} = \frac{\partial}{\partial s}\zeta_{\nabla\!\!\!/^2(A,\varphi)}(s)\,\big|_{s=0} \tag{59}$$

Quantum parity-violating anomalies are due to the second spectral asymmetry measuring term on the right hand side of (58), which is up to a coefficient the Atiyah-Patodi-Singer eta-invariant (55) of the Dirac operator (50):

$$\eta_{\nabla\!\!\!/(A,\varphi)} = \int_0^\infty d\tau\,(\pi\tau)^{-\frac{1}{2}}\,\mathrm{T}r\{\nabla\!\!\!/(A,\varphi)\exp[-\tau\nabla\!\!\!/^2(A,\varphi)]\} \tag{60}$$

Taking functional derivative of (60) with respect to A_μ (henceforth we shall discard for simplicity the dependence on the scalar field vp), applying the general variation formula (56) and integrating back we obtain:

$$\eta_{\nabla\!\!\!/(A)} = (-1)^{(D+1)/2}2\Big\{W^{(D)}_{\mathrm{Ch-S}}[A] + \mathcal{N}[A]\Big\}\,. \tag{61}$$

Here $W^{(D)}_{\mathrm{Ch-S}}$ denotes the well-known Chern-Simmons term, whose explicit form in $D=3$ reads:

$$W^{(3)}_{\mathrm{Ch-S}}[A] = \frac{1}{16\pi^2}\int d^3x\,\varepsilon^{\mu\nu\lambda}\;\mathrm{tr}\{A_\mu F_{\nu\lambda} - i\frac{2}{3}A_\mu A_\nu A_\lambda\} \tag{62}$$

(cf. (41) for the Abelian case; here an additional normalization factor $1/pi$ has been introduced for notational convenience). For general odd dimensions see Eq.(125) below.

Next, $\mathcal{N}[A]$ is a non-trivial functional of A_m determined from:

$$[\delta/\delta A^a_\mu(x)]\mathcal{N}[A] = (-1)^{(D+1)/2}i\,\mathrm{tr}[T_a\gamma^\mu\mathcal{P}_{\nabla\!\!\!/(A)}(0;x,x)] \tag{63}$$

When A_μ obey standard boundary conditions $A_\mu(x) = -ig^{-1}(\hat{x})\partial_\mu g(\hat{x}) + \mathrm{O}(|x|^{-1-\varepsilon}$ for $|x| \to \infty$ (*e.g.*, A_μ – quantum fluctuating) we have [B1] (see also our paper [27] not included in the thesis):

$$\mathcal{P}_{\nabla(A)}(0;x,x') = \begin{cases} \delta(0)\Pi_0^{\nabla(A)}(x,x') = \infty & \text{if } \lambda = 0 \text{ is a discrete eigenvalue} \\ 0 & \text{otherwise} \end{cases} \tag{64}$$

with $\Pi_0^{\nabla(A)}(x,x')$ being the kernel of the zero-mode projector. In this case $\mathcal{N}[A]$ is piecewise constant functional taking integer values (it can be associated with an index of an appropriate $(D+1)$-dimensional Dirac operator [30]).

Let us stress that both $\eta_{\nabla(A)}$, as well as $W^{(D)}_{\text{Ch-S}}[A]$ and $\mathcal{N}[A]$ are *odd* under P,T-reflections. On the other hand $\eta_{\nabla(A)}$ is manifestly gauge invariant under arbitrary gauge transformations as evident from the heat-kernel representation (60), whereas its constituents both change under topologically non-trivial ("large") gauge transformations as:

$$W^{(D)}_{\text{Ch-S}}[A^g] = W^{(D)}_{\text{Ch-S}}[A] + n_D[g] \quad , \quad \mathcal{N}[A^g] = \mathcal{N}[A] - n_D[g] \tag{65}$$

with $A^g_\mu = g^{-1}A_\mu g - g^{-1}i\partial_\mu g$, where:

$$n_D[g] = -(i/2\pi)^{(D+1)/2}\big(\frac{1}{2}(D-1)!\big)(D!)^{-1}\varepsilon^{\mu_1\ldots\mu_D}\int d^Dx\, tr\left[\big(g^{-1}\partial_{\mu_1}g\big)\ldots\big(g^{-1}\partial_{\mu_D}g\big)\right] \tag{66}$$

is the topological charge of $g(x) \in U(n)$. The corresponding homotopy group determining the topological charge of $g(x)$:

$$\pi_D(U(n)) = \begin{cases} \mathbb{Z} & \text{for odd } D < 2n \\ 0 & \text{for odd } D > 2n\ . \end{cases} \tag{67}$$

In the second topologically trivial case the Chern-Simmons term is gauge invariant and the functional $\mathcal{N}[A^g]$ identically vanishes.

We conclude this subsection with the following observation concerning the possible choices for the local gauge invariant counterterm $S_{\text{c.t.}}[A]$ in (58):

(A) For either a trivial homotopy group (odd $D > 2n$) or if odd $D < 2n$ (cf. (67)) and the number of fermion "flavors" N_f =even simultaneously, an admissible choice is:

$$N_f S_{\text{c.t.}}[A] = i\pi(-1)^{(D-1)/2}N_f W^{(D)}_{\text{Ch-S}}[A] \tag{68}$$

which cancels the parity-violating part from the eta-term (61) and, therefore, in this case parity-violating anomalies are absent.

(B) In the opposite case, *i.e.* for nontrivial homotopy of the gauge group (67) and odd N_f simultaneously, the only admissible choice for the counterterm is:

$$S_{\text{c.t.}}[A] = i\pi(-1)^{(D-1)/2}N_f W^{(D)}_{\text{Ch-S}}[A^{a=0}] \tag{69}$$

where $A^{a=0}$ is the Abelian $U(1)$ part of the gauge field. Therefore, in this case the Chern-Simmons term for the non-Abelian $SU(n)$ part cannot be cancelled in (61) and the parity-violating anomaly is unavoidable.

3.2 Boundary Effects and Interplay Between Spontaneous and Anomalous Breaking of Parity in Odd Dimensions

In the case when A_μ is a non-trivial background field, *e.g.*, when A_μ is static (with zero electric field) or when the field strength $F_{\mu\nu}(A)$ is uniformly non-vanishing at infinity (in either direction), then $\mathcal{N}[A]$ becomes a non-trivial smooth functional contributing a *spontaneous* (not anomalous) parity breakdown *in addition* to the already present parity violating anomaly due to the first Chern-Simmons term in (61). This issue has been studied in detail in [B3].

First, let us consider the static case:

$$A_0 = 0 \quad , \quad A_k = A_k(\underline{x}) \quad , \quad k = 1, \ldots, D-1$$
$$A_k(\underline{x}) = -ih^{-1}(\hat{x})\partial_k h(\hat{x}) + O\big(|\underline{x}|^{-1-\varepsilon}\big) \quad , \quad |\underline{x}| \to \infty \quad , \quad h : S^{D-2} \to U(n) \tag{70}$$

where we get:

$$\mathcal{P}_{\nabla\!\!\!\!/(A)}(0; x, x) = \frac{1}{2\pi}\,\Pi_0^{\nabla\!\!\!\!/_{D-1}}(x, x) \tag{71}$$

$$\nabla\!\!\!\!/(A) = \gamma_0\partial_0 + \nabla\!\!\!\!/_{D-1} \quad , \quad \nabla\!\!\!\!/_{D-1} = \gamma_k\left[\partial_k + iA_k(\underline{x})\right] \; , \tag{72}$$

Here $\Pi_0^{\nabla\!\!\!\!/_{D-1}}(x, x')$ denotes the kernel of the zero-mode projector of the $(D-1)$=even dimensional Dirac operator $\nabla\!\!\!\!/_{D-1}$ (72).

With the help of the above machinery we derive an entirely parity-odd expression for the induced fermionic current (49) in a static gauge field background (we confine ourselves to the singlet current, *i.e.*, for $a = 0$):

$$N_f^{-1} J_0^{a=0}(\underline{x}) = \begin{cases} (-1)^{(D+1)/2}\frac{1}{2}\mathrm{index}(\nabla\!\!\!\!/_{D-1}; \underline{x}) \\ \qquad\qquad \text{if conditions (A) above hold ,} \\ (-1)^{(D+1)/2}\frac{1}{2}\left\{\mathcal{C}_{(D-1)/2}(F^{a=0}; \underline{x}) - \left[\mathcal{C}_{(D-1)/2}(F; \underline{x}) - \mathrm{index}(\nabla\!\!\!\!/_{D-1}; \underline{x})\right]\right\} \\ \qquad\qquad \text{if conditions (B) above hold .} \end{cases} \tag{73}$$

Here:

$$\mathrm{index}(\nabla\!\!\!\!/_{D-1}; \underline{x}) = \mathrm{tr}\left[\gamma^5 \Pi_0^{\nabla\!\!\!\!/_{D-1}}(x, x)\right. \tag{74}$$

indicates the index density of $\nabla\!\!\!\!/_{D-1}$, $F_{\mu\nu}^{a=0}$ is the Abelian part of the field strength $F_{\mu\nu}(A)$, and $\mathcal{C}_{(D-1)/2}(F; \underline{x})$ denotes the denisty of the well-known Chern characteristic class – "instanton" number of $A_k(\underline{x})$ in $(D-1)$ = even dimensions (cf. *e.g.* Ref.[31]).

Thus, accounting for the standard index theorem:

$$\int d^{D-1}x\, \mathrm{index}(\nabla\!\!\!\!/_{D-1}; \underline{x}) = \int d^{D-1}x\, \mathcal{C}_{(D-1)/2}(F; \underline{x}) = n_{D-2}[h] \tag{75}$$

where $n_{D-2}[h]$ is the topological charge (66) of $h(\hat{x})$ appearing in (70), relations (73) yield the following result for the induced static charge:

$$Q_{ind} = \int d^{D-1}x\, J_0^{a=0}(\underline{x}) = N_f(-1)^{(D+1)/2}\frac{1}{2}\, n_{D-2}[h] \tag{76}$$

when conditions (A) above hold, *i.e.* we obtain a *fractional* (half-integer) induced charge.

Next, let us consider the case of non-trivial boundary conditions:

$$F_{\mu\nu}(A) \to F_{\mu\nu}(A^{\rm as}) = {\rm const} \quad {\rm for} \quad |x| \to \infty \ . \tag{77}$$

For the spectral density at $\lambda = 0$ we obtain:

$$\mathcal{P}_{\nabla\!\!\!/(A)}(0;x,x) = \left[U_\pm \mathcal{P}_{\nabla\!\!\!/(A^{\rm as})}(0) U_\pm^*\right](x,x) \ , \ \ \mathcal{P}_{\nabla\!\!\!/(A^{\rm as})}(0) = \left(16\pi^2\right)^{-1} \varepsilon^{\mu\nu\lambda}(-i\gamma_\mu) F_{\nu\lambda}(A^{\rm as}) \ , \tag{78}$$

where $U_\pm$ denote scattering wave operators for $H = \nabla\!\!\!/^2(A)$ and $H_0 = \nabla\!\!\!/^2(A^{\rm as})$ ("total" and "free" quantum mechanical "hamiltonians").

Using (78), for either $G = U(1)$ or $G = SU(n)$ and N_f =even we get:

$$N_f^{-1} J_\mu^a(x) = \frac{n}{8\pi} \varepsilon^{\kappa\lambda\nu} F_{\lambda\nu}^b(A^{\rm as}) w_{\mu\kappa}^{ab}(x) + \text{parity-normal terms} \ , \tag{79}$$

$$w_{\mu\nu}^{ab}(x) \equiv -\frac{1}{2n} \,{\rm tr}\{T^a \gamma_\mu \Big(\int d^3y U_\pm(x,y) \Big) T^b \gamma_\nu \Big(\int d^3y U_\pm^*(y,x) \Big)\}$$
$$= \delta_{\mu\nu}\delta^{ab} + \text{nonlocal functional of } (A_\mu - A_\mu^{\rm as}) \ .$$

Accordingly, for $G = SU(n)$ and N_f =odd we obtain:

$$N_f^{-1} J_\mu^a(x) = N_f^{-1} J_\mu^a(x) \left(\text{Eq.(79)}\right) - n(8\pi)^{-1} \varepsilon^{\mu\nu\lambda} F_{\nu\lambda}^a(A(x)) \ . \tag{80}$$

Our result (79) differs from the result in the special case $G = U(1)$ presented in Refs.[4] where the local form $n(8\pi)^{-1}\varepsilon^{\mu\nu\lambda} F_{\nu\lambda}^a(A(x))$ for the parity breaking term, *i.e.*, a *would-be* parity violating anomaly was claimed. Our analysis shows that, upon careful treatment of the non-local functional $\mathcal{N}[A]$ in the systematic eta-function renormalization of the fermionic determinant (61)–(63), the correct parity-odd term in the induced current (79) is *non-local* and it is entirely due to the *spontaneous* breakdown of parity through the non-trivial boundary conditions (77) for the background field, therefore, it does *not* represent a parity violating anomaly.

3.3 Conserved Noether Currents in Stochastic Quantization

By its introduction [7] (for a comprehensive review, see Ref.[8]) the stochastic quantization scheme (SQS) was in intended to give an alternative way of quantization of field theory models, in particular, to provide new invariant regularizations [32] which presumably respects simultaneously all symmetries of the underlying models.

In usual field theory the symmetries of the classical action correspond via the Noether theorem to conserved currents which, upon quantization, yield Ward identities for the correlation functions of the quantum fields. Unlike this, the original formulation of SQS [7] in terms of stochastic Langevin equations is *not based* on an action principle and, therefore, the underlying symmetries of SQS averages (the stochastic analogues of the quantum field theory correlation functions) are not expressed in terms of Ward-like identities.

In our paper [B8] we have proposed a systematic general procedure for deriving *stochastic Noether conserved currents* by employing the superspace formulation of SQS (first proposed in [33] and further developed in our paper [B8]).

Let us briefly recall the stochastic Langevin equation for a general D-dimensional (Euclidean) field theory with a classical action $S[\varphi] = \int d^D x\, \mathcal{L}(\varphi)$:

$$\partial_\tau \varphi_\eta = -\mathcal{K} \frac{\delta S}{\delta \varphi}\bigg|_{\varphi=\varphi_\eta} + \eta \quad , \quad \langle \eta(\tau, x)\, \eta(\tau', x') \rangle = 2\mathcal{K} \delta(\tau - \tau')\, \delta^{(D)}(x - x') \,. \tag{81}$$

Here τ denotes the stochastic evolution time, $\mathcal{K} \equiv \mathcal{K}[\partial_x]$ is an appropriate differential operator which ensures the positiveness of the corresponding Fokker-Planck Hamiltonian [32].

The generating functional of SQS averages $\langle \varphi_\eta(\tau, x_1) \ldots \varphi_\eta(\tau, x_n) \rangle$ (choosing initial conditions $\varphi_\eta(\tau = -\infty, x) = 0$ in (81)) reads:

$$Z[h] = \exp\Big(\mp \frac{1}{2} \, \mathrm{Tr} \ln \mathcal{K} \Big) \int \mathcal{D}\eta \, \exp\Big\{ \int d^D x d\tau \, \big[-\frac{1}{4} \eta \mathcal{K}^{-1} \eta + h \, \varphi_\eta \big] \Big\} \,. \tag{82}$$

The signs $\mp$ refer to φ , η being bosonic/fermionic. $Z[h]$ turns out to be (formally) equal to the generating functional of correlation functions of the following *supersymmetric* field theory in $(D+1|2)$-dimensional superspace [33],[B8]:

$$Z[h] = \mathcal{Z}[H] = \int \mathcal{D}\Phi \, \exp\Big\{ -\Sigma[\Phi] + \int d^{D+1|2} z \; H \Phi \Big\} \,, \tag{83}$$

$$\Sigma[\Phi] = \int d^{D+1|2} z \, \Big[\frac{1}{2} \overline{D} \Phi \mathcal{K}^{-1} D \Phi - \frac{1}{2} D \Phi \mathcal{K}^{-1} \overline{D} \Phi - i \mathcal{L}(\Phi) \Big] \,, \tag{84}$$

where:

$$z \equiv \big(\tau, x; \theta, \bar\theta \big) \quad , \quad D = \partial / \partial \theta \quad , \quad \overline{D} = \partial / \partial \bar\theta - i \partial_\tau \quad , \quad \big\{ D \, , \, \overline{D} \big\} = i \partial_\tau \tag{85}$$

and the pertinent superfields are of the form:

$$\Phi(z) = \varphi(\tau, x) + \bar\theta \, \chi(\tau, x) + \bar\chi(\tau, x) \, \theta + \bar\theta \theta \, F(\tau, x) \quad , \quad H(z) = \bar\theta \theta \, h(\tau, x) \,. \tag{86}$$

It is easy to show [B8] that, due to the τ-translation invariance and the manifest supersymmetry, all stochastic correlation functions for equal τ's and equal $\theta, \bar\theta$'s given by (82) are in fact τ- and θ-independent and, therefore, they are equal to their equilibrium limits:

$$\big\langle \Phi(\tau, x_1; \theta, \bar\theta) \ldots \Phi(\tau, x_n; \theta, \bar\theta) \big\rangle = \lim_{\tau \to \infty} \langle \varphi_\eta(\tau, x_1) \ldots \varphi_\eta(\tau, x_n) \rangle = \langle \varphi(x_1) \ldots \varphi(x_n) \rangle$$
$$\equiv \big(\delta^n / \delta j^n \big) \int \mathcal{D}\varphi \, \exp\Big\{ \int d^D x \, [-\mathcal{L}(\varphi) + j \varphi] \Big\} \, \Big|_{j=0} \tag{87}$$

The stochastic superspace action (84) respects all symmetries G of the original D-dimensional field-theory action (provided $\mathcal{K}$ has been appropriately chosen to be G-covariant). Therefore, we can apply to (84) the standard Noether procedure in order to derive conserved currents $J_M(z) \equiv (J_\mu, J_\theta, J_{\bar\theta})\,(z)$ in the $(D+1|2$-dimensional supersymmetric theory which precisely represent the SQS analogues of the usual Noether conserved currents $j_\mu(x)$ in D dimensions. We find (using the same notations as in (85)):

$$\overline{D} J_\theta[\Phi] + D J_{\bar\theta}[\Phi] + \partial^\mu J_\mu[\Phi] = 0 \,, \tag{88}$$

$$J_\theta[\Phi] = i \delta \Phi \, \big(\mathcal{K}^{-1} D \Phi \big) \quad , \quad J_{\bar\theta}[\Phi] = -i \delta \Phi \, \big(\mathcal{K}^{-1} \overline{D} \Phi \big) \,, \tag{89}$$

$$J_\mu[\Phi] = j_\mu[\Phi] - i/2 \, (D\Phi) \Big(\frac{\delta}{\delta B_\mu} \mathcal{K}^{-1} [\partial_x + B] \, \big|_{B=0} \Big) (\overline{D} \Phi)$$
$$+ i/2 \, (\overline{D} \Phi) \Big(\frac{\delta}{\delta B_\mu} \mathcal{K}^{-1} [\partial_x + B] \, \big|_{B=0} \Big) (D \Phi) \,. \tag{90}$$

Here $\mathcal{K}[\partial_x + B]$ is the gauge-covariant counterpart of $\mathcal{K}[\partial_x]$ with B_μ being an auxiliary G-gauge potential. The first term on the right hand side of (90) formally resembles the ordinary D-dimensional field theory conserved current:

$$j_\mu[\Phi] = \delta\Phi \left(\frac{\partial}{\partial(\partial_m \Phi)} \mathcal{L} - \mathcal{R}_\mu \right) \tag{91}$$

where $\delta\Phi$ and $\delta\mathcal{L} = \partial^\mu \mathcal{R}_\mu$ are the standard Noether variations under the corresponding G-symmetry transformations.

When calculating stochastic averages (87) and, in particular, when studying the Ward identities for the stochastic conserved currents (88)–(91) via the stochastic superspace functional integral (83), we encounter as expected ultraviolet divergencies. In our paper [B8] we have proposed a new ultraviolet regularization in SQS, which manifestly preserves all symmetries of the original D-dimensional field theory as well as the stochastic supersymmetry of (84). The latter is introduced by modifying the free Φ-propagator entering the pertinent stochastic superspace diagrams:

$$\langle \Phi(z)\, \Phi(z') \rangle^{(0)}_{\rm reg} = i\mathcal{K} \int_0^\infty d\alpha\, \rho_\Lambda(\alpha) \exp\left(-\alpha \mathcal{K} S''\right)(x,x') \, \exp\left(-i\alpha\, [D, \overline{D}]\right) \, \delta(\tau - \tau') \delta^{(2)}(\theta - \theta') \, , \tag{92}$$

where:

$$S'' \equiv \delta^2 S / \delta\varphi^2 \,\Big|_{\varphi=0} \quad , \quad \delta^{(2)}(\theta - \theta') \equiv (\bar\theta - \bar\theta')(\theta - \theta') \, , \tag{93}$$

and the regularizing function $\rho_\Lambda(\alpha)$ obeys the properties:

$$\lim_{\Lambda\to\infty} \rho_\Lambda(\alpha) = 1 \quad , \quad (d^k/d\alpha^k) \rho_\Lambda(\alpha) \,\Big|_{\alpha=0} = 0 \;\; , \;\; k = 0, 1, \ldots, L \, , \tag{94}$$

L being an appropriate integer depending on the space-time dimension D. A particular choice for $\rho_\Lambda(\alpha)$ is:

$$\rho_\Lambda(\alpha) = 1 - \exp(-\Lambda\alpha) \left(\sum_{k=0}^{L} \frac{1}{k!} (\Lambda\alpha)^k \right) \quad \text{for } \alpha \geq 0 \;\; ; \;\; \rho_\Lambda(-\alpha) = -\rho_\Lambda(\alpha) \, . \tag{95}$$

With the regularized propagator (92), all ultraviolet divergencies in the superspace diagram expansion of (83) manifesting themselves as singularities $O(\alpha^{-k})$, $k \geq 1$, in the proper-time integrals entering the diagram expressions are regulated by $\rho_\Lambda(\alpha)$ (94).

In fact, one can show that our manifestly supersymmetric stochastic regularization (92) yields the same final results as the standard Breit-Gupta-Zaks stochastic regularization [32], which smears out the "white-noise" correlator in (81):

$$\langle \eta(\tau, x)\, \eta(\tau', x') \rangle_\Lambda = 2\mathcal{K} \delta_L(\tau - \tau')\, \delta^{(D)}(x - x') \, , \tag{96}$$

(cf. Eq.(103) below), provided we choose [B8]:

$$2\delta_\Lambda(\alpha) = \frac{d}{d\alpha} \rho_\Lambda(\alpha) \tag{97}$$

3.4 Chiral Anomalies in Stochastic Quantization

Let us now apply the machinery developed in the previous section to the problem of reproducing in the *equilibrium limit* of the well-known chiral anomalies in stochastically quantized gauge theories with massless fermions bearing in mind that SQS manifestly preserves all symmetries of the underlying classical field theory at *finite* stochastic times.

We start with stochastic quantization of massive Dirac fermions in a background gauge field, whose (Euclidean) action reads:

$$S[\psi, \bar{\psi}] = \int d^D x \, \bar{\psi}(x) \, (m + i \not{\nabla}(A)) \, \psi(x) \tag{98}$$

and take the massless limit at the end. Here and below the Dirac operator $\not{\nabla}(A)$ is of the same form as (50). In the present context the general objects (83)–(86) specialize as follows:

$$\mathcal{Z}[H, \bar{H}] = \int \mathcal{D}\Psi\mathcal{D}\overline{\Psi} \; \exp\Big\{-\Sigma[\Psi, \overline{\Psi}] + \int d^{D+1|2}z \; \bar{H}\Psi + \overline{\Psi}H\Big\} \,, \tag{99}$$

$$\Sigma[\Psi, \overline{\Psi}] = \int d^{D+1|2}z \Big[D\overline{\Psi} \; (m - i \not{\nabla}(A))^{-1} \, \overline{D}\Psi$$
$$-\overline{D}\,\overline{\Psi} \; (m - i \not{\nabla}(A))^{-1} \, D\Psi + i\overline{\Psi} \; (m + i \not{\nabla}(A)) \, \Psi\Big] \,. \tag{100}$$

Here the stochastic fermionic superfield has the component decomposition:

$$\Psi(z) = \psi(\tau, x) + \bar{\theta} \, \omega(\tau, x) + \bar{\omega}(\tau, x) \, \theta + \bar{\theta}\theta \, \Omega(\tau, x) \tag{101}$$

and similarly for $\overline{\Psi}(z)$, where $\omega(\tau, x)\,, \bar{\omega}(\tau, x)$ and $\Omega(\tau, x)$ are bosonic/fermionic auxiliary stochastic fields which are irrelevant in the equilibrium limit.

Equivalently, the stochastic quantization of (massive) fermions in a background gauge field in terms of stochastic Langevin equations reads:

$$\partial_\tau \psi(\tau, x) = -[\not{\nabla}^2(A) + m^2]\psi(\tau, x) + \eta(\tau, x) \tag{102}$$
$$\partial_\tau \bar{\psi}(\tau, x) = -[\not{\nabla}^2(A) + m^2]\bar{\psi}(\tau, x) + \bar{\eta}(\tau, x)$$
$$\langle \eta(\tau, x)\bar{\eta}(\tau', x')\rangle = 2\delta_\Lambda(\tau - \tau')[m - i \not{\nabla}(A)]\delta^{(D)}(x - x') \tag{103}$$

Here the standard stochastic regularization of Breit-Gupta-Zaks [32] for the correlator of the fermionic random "white noise" $\eta(\tau, x)$ is employed via $\delta_\Lambda(\tau) \to \delta(\tau)$ for $\Lambda \to \infty$ and $\partial_\tau^k \delta_\Lambda(\tau) \mid_{\tau=0} = 0$ for $k = 0, \ldots L - 1$ (cf. Eqs.(96)–(97)).

Stochastic Noether theorem (88)–(91) yields the following axial (chiral) stochastic superspace current conserved when $m = 0$ (classical axial symmetry):

$$J_\theta^{(D+1)\,a}(z) = i\overline{\Psi}T^a\gamma^{(D+1)}\big[(m - i \not{\nabla}(A))^{-1} \, D\Psi\big] - i \big[\overline{D}\,\overline{\Psi} \; (m - i \not{\nabla}(A))^{-1}\big]T^a\gamma^{(D+1)}\Psi \tag{104}$$

$$J_{\bar{\theta}}^{(D+1)\,a}(z) = -i\overline{\Psi}T^a\gamma^{(D+1)}\big[(m - i \not{\nabla}(A))^{-1} \, \overline{D}\Psi\big] + i \big[D\,\overline{\Psi} \; (m - i \not{\nabla}(A))^{-1}\big]T^a\gamma^{(D+1)}\Psi \tag{105}$$

$$J_\mu^{(D+1)\,a}(z) = \overline{\Psi}T^a(-i\gamma_\mu)\gamma^{(D+1)}\Psi$$
$$+ \big[\overline{D}\,\overline{\Psi} \; (m - i \not{\nabla}(A))^{-1}\big]T^a\gamma_\mu\gamma^{(D+1)}\big[(m - i \not{\nabla}(A))^{-1} \, D\Psi\big]$$
$$- \big[D\overline{\Psi} \; (m - i \not{\nabla}(A))^{-1}\big]T^a\gamma_\mu\gamma^{(D+1)}\big[(m - i \not{\nabla}(A))^{-1} \, \overline{D}\Psi\big] \,. \tag{106}$$

Here $\gamma^{(D+1)} = \gamma_0 \dots \gamma_{D-1}$ is "γ^5" in D=even dimensions. The classical (*i.e.*, before taking stochastic averages (87)) conservation law for (104)–(106) reads:

$$\begin{aligned} &\overline{D} J_\theta^{(D+1)\,a} + D J_{\bar\theta}^{(D+1)\,a} + \nabla_\mu^{ab}(A) J_\mu^{(D+1)\,a} = -2m \overline{\Psi} T^a \gamma^{(D+1)} \Psi \\ &+2im \left[\overline{D}\,\overline{\Psi}\,(m - i\,\nabla\!\!\!/(A))^{-1}\right] T^a \gamma^{(D+1)} \left[(m - i\,\nabla\!\!\!/(A))^{-1} D\Psi\right] \\ &-2im \left[D\overline{\Psi}\,(m - i\,\nabla\!\!\!/(A))^{-1}\right] T^a \gamma^{(D+1)} \left[(m - i\,\nabla\!\!\!/(A))^{-1}\,\overline{D}\Psi\right] \end{aligned} \tag{107}$$

In order to calculate the stochastic average, *i.e.*, the quantum version of (107) we need the regularized form of the free propagator for the stochastic fermionic superfield. The general formula (92) specializes in this case as follows:

$$\begin{aligned} \left\langle \Psi(z)\,\overline{\Psi}(z')\right\rangle^{(0)}_{\rm reg} &= -\frac{i}{2}\,(m - i\,\nabla\!\!\!/(A))\,\rho_\Lambda(\tau - \tau')\,e^{-\left(m^2 + \nabla\!\!\!/^2(A)\right)|\tau-\tau'|}(x,x')\delta^{(2)}(\theta-\theta') \\ &+\frac{1}{2}\left[D\,,\,\overline{D}\right]\delta^{(2)}(\theta-\theta')\int_{|\tau-\tau'|}^{\infty} d\alpha\,\rho_\Lambda(\alpha)\exp\left[-\alpha\left(m^2 + \nabla\!\!\!/^2(A)\right)\right](x,x') \end{aligned} \tag{108}$$

Due to the general property of stochastic averages (87) we find that only the lowest component in the θ-expansion of the stochastic superspace axial current divergence survives in the equilibrium limit. Also due to the manifest preservation of stochastic supersymmetry in the stochastic averages we have:

$$D\left\langle J_{\bar\theta}^{(D+1)\,a}(z)\right\rangle = 0 \quad , \quad \overline{D}\left\langle J_\theta^{(D+1)\,a}(z)\right\rangle = 0 \tag{109}$$

Therefore, in the stochastic average of (107) only the last term on the left hand side survives:

$$\nabla_\mu^{ab}(A)\left\langle J_\mu^{(D+1)\,a}(z)\right\rangle = \nabla_\mu^{ab}(A)\left\langle \bar\psi(x,\tau) T^a(-i\gamma_\mu)\gamma^{(D+1)}\psi(x,\tau)\right\rangle \tag{110}$$

Upon using (108) we get:

$$\left\langle J_\mu^{(D+1)\,a}(z)\right\rangle = \int_0^\infty d\alpha\,\rho_\Lambda(\alpha)\,{\rm Tr}\left[T^a\gamma_\mu\gamma^{(D+1)}\,(im + \nabla\!\!\!/(A))\,e^{-\alpha\left\{m^2 + \nabla\!\!\!/^2(A)\right\}}(x,x)\right. \tag{111}$$

and for the covariant divergence:

$$\nabla_\mu^{ab}(A)\left\langle J_\mu^{(D+1)\,a}(z)\right\rangle = -4\int_0^\infty d\tau\,\delta_\Lambda(\tau)e^{-\tau m^2}\,{\rm tr}[T^a\gamma^{(D+1)}e^{-\tau\nabla\!\!\!/^2(A)}(x,x)] \tag{112}$$

$$+2m^2\int_0^\infty d\tau\,e^{-\tau m^2}\,{\rm tr}[T^a\gamma^{(D+1)}e^{-\tau\nabla\!\!\!/^2(A)}(x,x)]\int_{-\tau}^{\tau} d\tau'\delta_\Lambda(\tau')\,, \tag{113}$$

where $\delta_\Lambda(\tau)$ is related to $\delta_\Lambda(\tau)$ as in (97).

Removing stochastic regularization $\Lambda \to \infty$ and taking zero-mass limit $m \to 0$ the above equation reduces to:

$$\nabla_\mu^{ab}(A)\left\langle J_\mu^{(D+1)\,a}(z)\right\rangle = -2\,{\rm Tr}\left[T^a\gamma^{(D+1)}\Phi_0^{(D)}(\nabla\!\!\!/^2(A);x)\right] \tag{114}$$

$$+\lim_{m\to 0} 2\int_0^\infty d\alpha\,e^{-\alpha}\,{\rm Tr}\left[T^a\gamma^{(D+1)}e^{-(\alpha/m^2)\nabla\!\!\!/^2(A)}(x,x)\right]. \tag{115}$$

Here $\Phi_0^{(D)}(\nabla\!\!\!/^2(A);x)$ is the zero-order Seeley coefficient in the Seeley-De Witt expansion (57) for the heat kernel in (112), whose explicit form can be easily derived using the formalism described in Appendix A of [B1] (see also [34]):

$$\mathrm{Tr}\Big[T^a\gamma^{(D+1)}\Phi_0^{(D)}(\nabla\!\!\!/^2(A);x)\Big] = \frac{\varepsilon^{\mu_1\ldots\mu_D}}{(D/2)!\,(4\pi)^{D/2}}\,\mathrm{Tr}\left[T^a F_{\mu_1\mu_2}\ldots F_{\mu_{D-1}\mu_D}\right]\ . \tag{116}$$

The Abelian part ($a=0$) of the last term in (116) equals $\mathcal{C}_{D/2}(F;x)$ – the density of the $D/2$-th Chern topological characteristic class (see *e.g.* [31]).

For the zero-mass limit of the last term in (115) using the formulas from Appendix B of [B1] we find:

$$2\,\mathrm{Tr}\left[T^a\gamma^{(D+1)}\Pi_0^{\nabla\!\!\!/(A)}(x,x)\right] = \mathrm{index}(\nabla\!\!\!/(A);x)\ , \tag{117}$$

where $\Pi_0^{\nabla\!\!\!/(A)}$ is the projector of zero-modes of the Dirac operator $\nabla\!\!\!/(A)$ and index$(\nabla\!\!\!/(A);x)$ is the corresponding index density (see *e.g.* [34]).

Collecting (116) and (117) we recover from (114)–(115) the correct covariant form of the Abelian and non-Abelian chiral anomalies within the SQS framework:

$$\partial^\mu J_\mu^{(D+1)\,a=0} = 2\left[\mathrm{index}(\nabla\!\!\!/(A);x) - \mathcal{C}_{D/2}(F;x)\right]\ , \tag{118}$$

$$\nabla_\mu^{ab} J_\mu^{(D+1)\,b} = 2\,\mathrm{tr}[T^a\gamma^{(D+1)}\Pi_0^{\nabla\!\!\!/(A)}(x,x)]$$
$$-2[(D/2)!(4\pi)^{D/2}]^{-1}\varepsilon^{\mu_1\ldots\mu_D}\,\mathrm{tr}[T^a F_{\mu_1\mu_2}\ldots F_{\mu_{D-1}\mu_D}]\ , \tag{119}$$

where in the last equation $a,b=1,\ldots,n^2-1$.

Further details are contained in [B1,B5,B6,B8,B9].

3.5 Topological Quantization of Physical Parameters and Global Anomalies in Stochastic Quantization

Another crucial test for the self-consistency of SQS is to show that SQS correctly reproduces in the equilibrium limit the topological quantization of physical parameters, which is an inherent non-perturbative feature of various physically relevant field theory models such as:

(i) Chiral field model with a Wess-Zumino term in D=even space-time dimensions [35]:

$$S_\xi^{(1)} = S_{\mathrm{chiral}}[U] + i\xi\Gamma_{\mathrm{WZ}}[U] \tag{120}$$

where:

$$S_{\mathrm{chiral}}[U] = -\frac{1}{2}f^2\int d^Dx\ \mathrm{Tr}\Big[L_\mu^2(U)\Big]\quad,\quad L_\mu(U)\equiv U^{-1}\partial_m U\ , \tag{121}$$

$$\Gamma_{\mathrm{WZ}}[U] = c_{D+1}\int_0^\infty dx^{D+1}\int d^Dx\,\varepsilon^{\mu_1\ldots\mu_{D+1}}\ \mathrm{Tr}\left[L_{\mu_1}(\widetilde{U})\ldots L_{\mu_{D+1}}(\widetilde{U})\right]\ , \tag{122}$$

$$c_{D+1}\equiv -(i/2\pi)^{(D+2)/2}\,(D/2)!\,[(D+1)!]^{-1}\ . \tag{123}$$

Here $\widetilde{U} = \widetilde{U}(x, x^{D=1})$ is a continuation of $U(x)$ to $(D+1)$ dimensions, such that $\widetilde{U}(x, 0) = U(x)$ and $\widetilde{U}(x, x^{D+1}) \to \mathbb{1}$ for $x^{D+1} \to \infty$.

(ii) Non-Abelian Yang-Mills gauge theories with Chern-Simmons terms in D=odd space-time dimensions [36]:

$$S_\xi^{(2)} = \frac{1}{4ng^2} \int d^D x \ \mathrm{Tr} \left[F_{\mu\nu}^2(A) \right] + i\xi W_{\mathrm{Ch-S}}^{(D)}[A] \tag{124}$$

where:

$$W_{\mathrm{Ch-S}}^{(D)}[A] = \int d^D x \, \varepsilon^{\mu_1 \ldots \mu_D} \, \mathrm{Tr} \left[b_D A_{\mu_1} F_{\mu_2\mu_3} \ldots F_{\mu_{D-1}\mu_D} - \ldots + (-i)^D c_D A_{\mu_1} \ldots A_{\mu_D} \right] \tag{125}$$

Here:

$$b_D \equiv 2 \Big[\Big(\frac{D+1}{2} \Big)! \, (4\pi)^{(D+1)/2} \Big]^{-1} \tag{126}$$

and c_D is the same as in (123)

Due to the non-trivial topology of the configuration spaces of field theories (120) and (124), namely, due to the existence of closed *non-contractable loops*, the corresponding actions are *multi-valued* along these loops (shifted by integers) and, therefore, the parameters ξ must be quantized as follows:

$$\xi = 2\pi k \quad , \quad k \in \mathbb{Z} \tag{127}$$

Let us now quantize the above models with multi-valued actions within the stochastic scheme. In generic notations the corresponding stochastic Langevin equations read (cf. (81); here we can choose $\mathcal{K} = \mathbb{1}$):

$$\partial_\tau \varphi_\eta = - \frac{\delta S_\xi}{\delta \varphi} \Big|_{\varphi = \varphi_\eta} + \eta \quad , \quad \langle \eta(\tau, x) \, \eta(\tau', x') \rangle = 2\delta(\tau - \tau') \, \delta^{(D)}(x - x') \, . \tag{128}$$

where:

$$S_\xi[\varphi(\cdot; 1)] - S_\xi[\varphi(\cdot; 0)] = i\xi N \quad , \quad N \in \mathbb{Z} \tag{129}$$

for every non-contractable closed contour C in the configuration space:

$$C = \big\{ \varphi(x; s) \, , \ 0 \le s \le 1 \mid \varphi(x; 0) = \varphi(x; 1) = \varphi(x) \big\} \, . \tag{130}$$

Specifically, the Langevin equation (128) acquires the following form for model (120):

$$\begin{gathered} U^{-1} \partial_\tau U = -f^2 \partial^\mu L_\mu(U) - i\xi(D+1) c_D \varepsilon^{\mu_1 \ldots \mu_D} L_{\mu_1}(U) \ldots L_{\mu_D}(U) + \eta \\ \big\langle \eta^a(\tau, x) \, \eta^b(\tau', x') \big\rangle = 2\delta^{ab} \delta(\tau - \tau') \, \delta^{(D)}(x - x') \quad , \quad \eta = i T^a \eta^a \end{gathered} \tag{131}$$

and for model (124):

$$\begin{gathered} \partial_\tau A_\mu^a = -\frac{1}{g^2} (\nabla^\nu F_{\mu\nu})^a - \frac{1}{2} i\xi (D+1) b_D \varepsilon^{\mu\nu_1 \ldots \nu_{D-1}} \, \mathrm{Tr} \left[T^a F_{\nu_1\nu_2} \ldots F_{\nu_{D-2}\nu_{D-1}} \right] + \eta_\mu^a \\ \big\langle \eta_\mu^a(\tau, x) \, \eta_\nu^b(\tau', x') \big\rangle = 2\delta^{ab} \delta_{\mu\nu} \delta(\tau - \tau') \, \delta^{(D)}(x - x') \end{gathered} \tag{132}$$

It is obvious from (131)–(132), that unlike the multi-valued actions $S_\xi[\varphi]$, their functional derivatives $\frac{\delta S_\xi}{\delta\varphi}$ defining the "drift" force in the stochastic Langevin equations (128) are smooth *single-valued* functionals for any value of ξ. Therefore, (128) yield well-defined (after appropriate stochastic regularization, cf. above) stochastic averages for *any* ξ, *i.e.*, no topological quantization (127) of ξ is enforced at finite stochastic time τ:

$$\left\langle \mathcal{F}\left[\varphi^{(\xi)}(\tau,\cdot)\right]\right\rangle_\eta = \int \mathcal{D}\varphi\, \mathcal{F}[\varphi]\, \mathcal{P}_\xi[\varphi;\tau] \; . \tag{133}$$

Here $\mathcal{F}$ is arbitrary (gauge-invariant) functional and $\mathcal{P}_\xi[\varphi;\tau]$ is the Fokker-Planck distribution corresponding to the solutions $\varphi_\eta^{(\xi)}(\tau,x)$ of the stochastic Langevin Eqs.(128):

$$\mathcal{P}_\xi[\varphi;\tau] = \int \mathcal{D}\eta\, \exp\Big\{-\int d^Dx\, d\tau\, \eta^2\Big\} \prod_x \delta\left(\varphi(x) - \varphi_\eta^{(\xi)}(\tau,x)\right) \tag{134}$$

Thus, the important question arises as to how SQS enforces the topological quantization (127) of ξ in the equilibrium limit. The answer is given by observing that the equilibrium Fokker-Planck distribution $\mathcal{P}_\xi[\varphi]$ must satisfy the following equation:

$$\Big(\frac{\delta}{\delta\varphi(x)} + \frac{\delta S_\xi[\varphi]}{\delta\varphi(x)}\Big)\, \mathcal{P}_\xi[\varphi] = 0 \quad , \quad \mathcal{P}_\xi[\varphi] = \lim_{\tau\to\infty} \mathcal{P}_\xi[\varphi;\tau] \; , \tag{135}$$

with the obvious solution:

$$\mathcal{P}_\xi[\varphi] = \text{const}\, \exp\Big\{-\int_{C_{(\varphi,\varphi_0)}} \delta S_\xi/\delta\varphi\Big\} \tag{136}$$

Here $C_{(\varphi,\varphi_0)}$ denotes an open path in the field configuration space:

$$C_{(\varphi,\varphi_0)} = \Big\{\, \varphi(x;s) \mid \varphi(x;0) = \varphi_0(x) - -\ \text{some reference point}\ ,\ \varphi(x;1) = \varphi(x) \,\Big\} \tag{137}$$

and the functional line integral in (136) is defined as:

$$\int_{C_{(\varphi,\varphi_0)}} (\ldots) = \int_0^1 ds \int d^Dx\, \partial_s\varphi(x;s)\, (\ldots) \; . \tag{138}$$

Due to the multi-valuedness (129)–(130) of $S_\xi[\varphi]$, the equilibrium solution (136) for the Fokker-Planck distribution is in general *path-dependent*, *i.e.*, $\mathcal{P}_\xi[\varphi]$ is not well-defined for generic values of ξ, unless ξ obeys the quantization condition (127) in which case:

$$\mathcal{P}_\xi[\varphi] = \text{const}\, \exp\big\{-S_\xi[\varphi]\big\} \quad (\, \xi = 2\pi N \,) \tag{139}$$

becomes well-defined single-valued (Euclidean) functional integral weight. Further details are contained in [B7,B9].

4 Super-Poincare Covariant Quantization of Supersymmetric Strings

4.1 Canonical Hamiltonian Formulation of Green-Schwarz Superstrings: Covariant Separation of First and Second Class Constraints

Realization of the importance of manifest space-time supersymmetry for superstring theory (anomaly cancellations, ultraviolet finiteness, vanishing cosmological constant, *etc.* [24]) attributed a fundamental role to the Green-Schwarz formulation of superstrings [37, 24]. There exist several types of Green-Schwarz superstrings, the most important being type *IIB* (with $\mathcal{N}=2$ chiral space-time supersymmetry), type *IIA* (with $\mathcal{N}=2$ non-chiral space-time supersymmetry) and heterotic (with $\mathcal{N}=1$ space-time supersymmetry) Green-Schwarz superstrings. For simplicity here we shall concentrate on the last mentioned type.

The standard Lagrangian form of the heterotic Green-Schwarz superstring (in flat $D=10$ embedding space-time) reads:

$$S_{\rm GS}^{\rm heterotic} = \int d\tau\, d\xi \sqrt{-\gamma}\Big[-\frac{1}{2}\gamma^{mn}\partial_m X^\mu \partial_n X_\mu - 2i\left(P_-^{nm}\partial_m X^\mu\right)\left(\theta\sigma_\mu\partial_n\theta\right) + \frac{1}{2}\gamma^{mn}\left(\theta\sigma_\mu\partial_n\theta\right)\left(\theta\sigma^\mu\partial_m\theta\right) + S_{\rm internal}\ , \tag{140}$$

where $S_{\rm internal}$ is the action for the left-moving modes describing the internal string degrees of freedom. The precise form of $S_{\rm internal}$ does not affect the present analysis and, therefore, it will be suppressed. The objects appearing in (140) have the following meaning. (τ,ξ) are the standard string world-sheet parameters; $\partial_0 \equiv \partial/\partial\tau$, $\partial_1 \equiv \partial/\partial\xi$; γ_{mn} $(m,n=0,1)$ denotes the intrinsic Riemannian world-sheet metric. The string bosonic coordinates X^μ $(\mu=0,1,\ldots,9)$ transform as $D=10$ vectors and $D=2$ world-sheet scalars. The anti-commuting string coordinates θ_α $(\alpha=1,\ldots,16)$ transform as a left-handed Majorana-Weyl $SO(1,9)$ spinor and as $D=2$ world-sheet scalars. P_-^{mn} denotes $D=2$ chiral projector:

$$P_\pm^{mn} \equiv \frac{1}{2}\Big(\gamma^{mn} \pm \frac{\varepsilon^{mn}}{\sqrt{-\gamma}}\Big) \tag{141}$$

Passing to the Hamiltonian formalism via the standard way of introducing the canonically conjugated momenta $(P_\mu\, ,\, p_\theta^\alpha)$ the action (140) acquires the form:

$$S_{\rm GS}^{\rm heterotic} = \int d\tau\, d\xi \left[P_\mu \partial_\tau X^\mu + p_\theta^\alpha \partial_\tau\theta - \Lambda_L T_L - \Lambda_R T_R - \Lambda_\alpha \mathcal{D}^\alpha\right] \tag{142}$$

Here $\Lambda_L, \Lambda_R, \Lambda_\alpha$ are Lagrange multipliers for the corresponding Hamiltonian constraints. $T_{L,R}$ are the left (right) reparametrization (Virasoro) constraints (primes indicate differentiation with respect to the space-like worldsheet parameter ξ):

$$T_L \equiv \left(P_\mu - X'_\mu\right)^2 \quad , \quad T_R \equiv \Pi^2 - 4i\theta'_\alpha \mathcal{D}^\alpha\ , \tag{143}$$

where:

$$\Pi_\mu \equiv P_\mu + X'_\mu + 2i\theta\sigma_\mu\theta' \; . \tag{144}$$

The spinorial fermionic constraints $\mathcal{D}^\alpha$ appearing in (142) are:

$$\mathcal{D}^\alpha \equiv -ip^\alpha_\theta - \left(P^\mu + X'_\mu + i\theta\sigma^\mu\theta'\right)(\sigma_\mu\theta)^\alpha \tag{145}$$

The Poisson-bracket algebra of the constraints (143)–(145) takes the form:

$$\{T_{L,R}(\xi),\, T_{L,R}(\eta)\}_{\rm PB} = \pm 8\Big[T_{L,R}(\xi)\,\delta'(\xi-\eta) + \frac{1}{2}T'_{L,R}(\xi)\delta(\xi-\eta)\Big] \; , \tag{146}$$

$$\{T_L(\xi),\, T_R(\eta)\}_{\rm PB} = 0 \quad , \quad \{\mathcal{D}^\alpha(\xi),\, T_R(\eta)\}_{\rm PB} = -4\mathcal{D}^\alpha(\eta)\,\delta'(\xi-\eta) \; , \tag{147}$$

$$\left\{\mathcal{D}^\alpha(\xi),\, \mathcal{D}^\beta(\eta)\right\}_{\rm PB} = 2i\delta(\xi-\eta)\,(\sigma_\mu)^{\alpha\beta}\Pi^\mu(\xi) \tag{148}$$

From (146)–(148) one finds that $T_{L,R}$ (143) are first class bosonic constraints, whereas $\mathcal{D}^\alpha$ is a mixture of 8 first class and 8 second class fermionic constraints (the 16×16 matrix on the right hand side of (148) has rank 8 on the constraint surface). This is the famous problem of Lorentz *non-covariant* mixing of first and second class constraints in the Green-Schwarz superstring formulation: both 8 first and 8 second class parts of $\mathcal{D}^\alpha$ cannot be represented as covariant Lorentz spinor objects since the lowest Lorentz-spinor dimensionality in $D=10$ space-time is 16.

In our series of papers [C1–C13] we have provided a detailed and systematic solution to the above notorious problem. The covariant separation of the first and second class parts of $\mathcal{D}^\alpha$ (145) is achieved through introduction of a set of two auxiliary "pure gauge" *bosonic Majorana-Weyl spinor* variables $v^{\pm\frac{1}{2}}_\alpha$ together with a further set of 7 auxiliary "pure gauge" vector variables u^p_μ ($p=1,\ldots,7$) which are simultaneously reparametrization scalars on the string world-sheet.

Before proceeding let us note the following remarkable properties involving the auxiliary bosonic spinor variables $v^{\pm\frac{1}{2}}_\alpha$ which are due to the famous $D=10$ Fierz identity:

$$(\sigma_\mu)^{\alpha\beta}(\sigma^\mu)^{\gamma\delta} + (\sigma_\mu)^{\beta\gamma}(\sigma^\mu)^{\alpha\delta} + (\sigma_\mu)^{\gamma\alpha}(\sigma^\mu)^{\beta\delta} = 0 \; . \tag{149}$$

As a result of (149) the following 3 vectors, bilinear composites of the auxiliary bosonic spinors:

$$u^+_\mu \equiv v^{+\frac{1}{2}}\sigma_\mu v^{+\frac{1}{2}} \quad , \quad u^-_\mu \equiv v^{-\frac{1}{2}}\sigma_\mu v^{-\frac{1}{2}} \quad , \quad u^8_\mu \equiv \sqrt{2} v^{+\frac{1}{2}}\sigma_\mu v^{-\frac{1}{2}} \tag{150}$$

obey the orthogonality identities:

$$\left(u^\pm\right)^2 = 0 \;\; (\text{lightlike}) \quad , \quad u^8_\mu u^{\pm\,\mu} = 0 \quad , \quad \left(u^8\right)^2 + u^+_\mu u^{-\,\mu} = 0 \tag{151}$$

The "internal" index p of u^p_μ transforms as a $SO(7)$-vector, whereas the internal indices $\pm\frac{1}{2}$ of $v^{\pm\frac{1}{2}}_\alpha$ transform as $SO(1,1)$ charge $\pm\frac{1}{2}$. Furthermore, the internal index a of the combination $u^a_\mu \equiv \left(u^p_\mu, u^8_\mu \equiv \sqrt{2}v^{+\frac{1}{2}}\sigma_\mu v^{-\frac{1}{2}}\right)$ ($a=(p,8)=1,\ldots,8$) transforms as $SO(8)$-vector.

With the help of the auxiliary variables $\left(v_\alpha^{\pm\frac{1}{2}}\,,\,u_\mu^p\right)$ the fermionic constraints $\mathcal{D}^\alpha$ (145) are covariantly separated into independent first-class part (generators of the fermionic "kappa"-symmetry):

$$\mathcal{D}^{+\frac{1}{2}a} \equiv v^{+\frac{1}{2}}\sigma^\mu u_\mu^a \sigma^\nu \Pi_\nu \mathcal{D} \quad , \quad a = 1,\ldots,8\,, \tag{152}$$

and independent second class part:

$$G^{+\frac{1}{2}a} \equiv \frac{1}{2} v^{-\frac{1}{2}} u_\mu^a \sigma^{[\mu}\sigma^{\nu]} u_\nu^+ \mathcal{D} \quad , \quad a = 1,\ldots,8\,, \tag{153}$$

where u_μ^+ , u_μ^8 are the bilinear composites (150).

4.2 Covariant Formulation of Green-Schwarz Superstrings in Extended Phase Space

As mentioned above, the auxiliary bosonic spinor and vector world-sheet scalar variables $\left(v_\alpha^{\pm\frac{1}{2}}\,,\,u_\mu^p\right)$ must be "pure-gauge" degrees of freedom not altering the dynamics of the original Green-Schwarz superstring (140). Therefore, their dynamics must be governed by a Hamiltonian, which contains only a linear combination of independent first-class constraints equal in number with the number (=102) of the auxiliary variables.

The resulting action for the Green-Schwarz (heterotic) superstring in the extended phase space with canonical coordinates and their canonically conjugated momenta:

$$\left(X^\mu\,,\,\theta_\alpha\,,\,v_\alpha^{\pm\frac{1}{2}}\,,\,u_\mu^p\right) \quad , \quad \left(P_\mu\,,\,p_\theta^\alpha\,,\,\pi_v^{\mp\frac{1}{2}\,\alpha}\,,\,\pi_u^{p,\mu}\right) \tag{154}$$

reads [C12,C13]:

$$\widetilde{S} = \widetilde{S}_{\rm GS}^{\rm heterotic} + S_{\rm auxiliary}\,. \tag{155}$$

Here:

$$\widetilde{S}_{\rm GS}^{\rm heterotic} = \int d\tau\, d\xi \left[P_\mu \partial_\tau X^\mu + p_\theta^\alpha \partial_\tau \theta - \Lambda_L \widetilde{T}_L - \Lambda_R T_R - \Lambda_a^{-\frac{1}{2}} \mathcal{D}^{+\frac{1}{2}a} - M_a^{-\frac{1}{2}} G^{+\frac{1}{2}a}\right]\,, \tag{156}$$

where now the left Virasoro constraint T_L (143) acquires contribution from the auxiliary variables:

$$\widetilde{T}_L \equiv \left(P_\mu - X'_\mu\right)^2 - 4\left(\pi_u^p u'_p + \pi_v^{\mp\frac{1}{2}} (v^{\pm\frac{1}{2}})'\right)\,, \tag{157}$$

T_R is the same as in (143), and $\Lambda_a^{-\frac{1}{2}}$, $M_a^{-\frac{1}{2}}$ denote the Lagrange multipliers for the covariantly disentangled fermionic constraints (152)–(153).

The part of the action governing the "pure-gauge" dynamics of the auxiliary variables is:

$$S_{\rm auxiliary} = \int d\tau\, d\xi \left[\pi_v^{\mp\frac{1}{2}} \partial_\tau v^{\pm\frac{1}{2}} + \pi_u^p \partial_\tau u_p - \Lambda_{MN} D^{MN} - \mathcal{M}_{AB} \Psi^{AB}\right]\,, \tag{158}$$

where Λ_{MN}, $\mathcal{M}_{AB}$ are Lagrange multipliers corresponding to the first-class constraints D^{MN}, Ψ^{AB} on the auxiliary variables. The latter constraints have a transparent geometrical meaning. First, Ψ^{AB} represent 50 orthonormality constraints $(A, B = p, 8, +, -)$:

$$\begin{aligned} \Psi^{pq} &\equiv u^p_\mu u^{q\,\mu} - \delta^{pq} = 0 \quad , \quad \Psi^{p8} \equiv u^p_\mu u^{8\,\mu} = 0 \\ \Psi^{p\pm} &\equiv u^p_\mu u^{\pm\,\mu} = 0 \quad , \quad \Psi^{+-} \equiv u^+_\mu u^{-\,\mu} + 1 = 0 \end{aligned} \tag{159}$$

(recall that u^8, $u^\pm$ are bilinear composites (150)). The constraints (159) imply that on-shell the auxiliary variables form an orthonormal frame of ten $SO(1,9)$ vectors (the "missing" orthonormality conditions are automatically fulfilled *off-shell* by construction (151) due to the $D = 10$ Fierz identities (149)).

The remaining 52 constraints D^{MN} $(M, N = p, 8, +, -)$ imply that the dynamics is invariant (a) under local $SO(1,9)$ rotations of the orthonormal frame u^p, u^8, $u^\pm$, and (b) under transformations of the bosonic spinors $|^{\pm\frac{1}{2}}$, which leave this frame invariant (the latter gauge invariance being expressed by the constraints $\widetilde{D}^{8p}$ below):

$$D^{pq} \equiv -u^p \pi^q_u + u^q \pi^p_u - \frac{1}{2} \sum_{+,-} v^{\pm\frac{1}{2}} u^p_\mu u^q_\nu \sigma^{[m} \sigma^{n]} \pi_v^{\mp\frac{1}{2}} \ , \tag{160}$$

$$D^{8p} \equiv -u^8 \pi^p_u - \frac{1}{2} \sum_{+,-} v^{\pm\frac{1}{2}} u^8_\mu u^p_\nu \sigma^m \sigma^n \pi_v^{\mp\frac{1}{2}} \ , \tag{161}$$

$$D^{+p} \equiv -u^+ \pi^p_u - \frac{1}{2} \sum_{+,-} v^{\pm\frac{1}{2}} u^+_\mu u^p_\nu \sigma^{[m} \sigma^{n]} \pi_v^{\mp\frac{1}{2}} \ , \tag{162}$$

$$D^{-p} \equiv -u^- \pi^p_u - \frac{1}{2} \sum_{+,-} \big(v^{\pm\frac{1}{2}} u^-_\mu u^p_\nu \sigma^{[m} \sigma^{n]} \pi_v^{\mp\frac{1}{2}} \ , \tag{163}$$

$$D^{+8} \equiv -\frac{1}{2} \sum_{+,-} v^{\pm\frac{1}{2}} u^+_\mu u^8_\nu \sigma^{[m} \sigma^{n]} \pi_v^{\mp\frac{1}{2}} \quad , \quad D^{-8} \equiv -\frac{1}{2} \sum_{+,-} v^{\pm\frac{1}{2}} u^-_\mu u^8_\nu \sigma^{[m} \sigma^{n]} \pi_v^{\mp\frac{1}{2}} \ , \tag{164}$$

$$D^{-+} \equiv -\frac{1}{2} \sum_{+,-} v^{\pm\frac{1}{2}} u^-_\mu u^+_\nu \sigma^{[m} \sigma^{n]} \pi_v^{\mp\frac{1}{2}} \quad , \quad \widetilde{D}^{8p} \equiv -\frac{1}{2} \sum_{+,-} (\pm) v^{\pm\frac{1}{2}} u^8_\mu u^p_\nu \sigma^{[m} \sigma^{n]} \pi_v^{\mp\frac{1}{2}} \ . \tag{165}$$

Let us stress once again that all constraints in (156),(158) except $G^{+\frac{1}{2}a}$ (153) are first class. Further details are contained in [C12,13].

Because of the geometric meaning of the constraints (159)–(165) on the auxiliary variables $\left(v_\alpha^{\pm\frac{1}{2}} \ , \ u^p_\mu \right)$ we call our covariant formulation of Green-Schwarz superstrings in extended phase space (154) –"covariant spinor-harmonic superspace formalism".

4.3 Covariant Quantum Green-Schwarz Superstrings: Path Integral

Taking into account the results in the previous subsection we find that the path integral for the covariantly quantized (heterotic) Green-Schwarz superstring will contain the following elements [C12,13]:

(a) functional integration over the canonical variables $\left(X^\mu\,,\,\theta_\alpha\,,\,v_\alpha^{\pm\frac{1}{2}}\,,\,u_\mu^p\right)$ and their canonically conjugated momenta $\left(P_\mu\,,\,p_\theta^\alpha\,,\,\pi_v^{\mp\frac{1}{2}\,\alpha}\,,\,\pi_u^{p,\mu}\right)$;

(b) functional integration over the Lagrange multipliers Λ_L, Λ_R, $\Lambda^{-\frac{1}{2}a}$, $M^{-\frac{1}{2}a}$, Λ_{MN}, $\mathcal{M}_{AB}$;

(c) delta-functions imposing the gauge-fixing conditions $\chi^{(\ldots)}$ for the first class constraints: $\big(\widetilde{T}_L,\,T_R\big) \leftrightarrow \chi^{(\mathrm{rep})}_{L,R}$ (reparametrization (Virasoro)), $\mathcal{D}^{+\frac{1}{2}a} \leftrightarrow \chi^{(\kappa)}_a$ (fermionic "kappa"-symmetry), $\Psi^{AB} \leftrightarrow \chi^{(\mathrm{norm})}_{AB}$ (orthonormality for auxiliary variables), $D^{MN} \leftrightarrow \chi^{(\mathrm{rot})}_{MN}$ (local frame rotations for auxiliary variables);

(d) Faddeev-Popov "ghost" determinants of the matrices formed by the Poisson brackets among the first class constraints and their respective gauge fixing conditions: $\Delta^{(\mathrm{rep})}_{\mathrm{FP}}$, $\Delta^{(\kappa)}_{\mathrm{FP}}$, $\Delta^{(\mathrm{norm})}_{\mathrm{FP}}$ and $\Delta^{(\mathrm{rot})}_{\mathrm{FP}}$;

(e) inverse square root of the determinant of the Poisson brackets among the second-class constraints $G^{+\frac{1}{2}a}$ (153), which equals $\det^{-4}\left[u^+_\mu \Pi^\mu\right]$ (Π^μ is the same as in (144)).

In our covariant formalism we are able to impose a Lorentz-invariant gauge fixing condition for the fermionic "kappa"-gauge symmetry:

$$\delta\big(\chi^{(\kappa)}_a\big) \equiv \delta\big(v^{+\frac{1}{2}} u^a_\mu \sigma^\mu \theta\big)\ , \tag{166}$$

which implies a local Faddeev-Popov determinant:

$$\Delta^{(\kappa)}_{\mathrm{FP}} = \det{}^{-8}\big[u^+_\mu \Pi^\mu\big]\ . \tag{167}$$

For $\chi^{(\mathrm{rep})}$ we take the standard conformal gauge.

For $\chi^{(\mathrm{norm})}_{AB} \equiv \Omega^{AB}$ we choose:

$$\Omega^{pq} = \frac{1}{2}\pi_u^{(p} u^{q)}\ \ ,\ \ \Omega^{p8} = \pi^p_u u^8\ \ ,\ \ \Omega^{p\pm} = \pi^p_u u^\pm\ \ ,\ \ \Omega^{+-} = \frac{1}{4}\big(v^{+\frac{1}{2}}\pi_v^{-\frac{1}{2}} + v^{-\frac{1}{2}}\pi_v^{+\frac{1}{2}}\big)\ , \tag{168}$$

(recall that u^8, $u^\pm$ are bilinear composites (150)), and for $\chi^{(\mathrm{rot})}_{MN}$ we take:

$$\chi^{(\mathrm{rot})}_{8p} = \Lambda_{8p}\ \ ,\ \ \chi^{(\mathrm{rot})}_{\widetilde{8p}} = \widetilde{\Lambda}_{8p}\ \ ,\ \ \chi^{(\mathrm{rot})}_{+-} = \Lambda_{+-}\ \ ,\ \ \chi^{(\mathrm{rot})}_{pq} = \chi^{SO(7)}\ , \tag{169}$$

the latter meaning that we take some fixed choice for the seven-frame spanned by u^p_μ.

Thus, taking into account (a)-(e) above and (166)–(169), the hamiltonian path integral acquires the form [C12]:

$$Z = \int \mathcal{D}X^\mu\,\mathcal{D}\theta_\alpha\,\mathcal{D}u\,\mathcal{D}v\,\mathcal{D}P_\mu\,\mathcal{D}p^\alpha_\theta\,\mathcal{D}\pi_u\,\mathcal{D}\pi_v\,\mathcal{D}\Lambda_L\,\mathcal{D}\Lambda_R\,\mathcal{D}\Lambda^{-\frac{1}{2}a}\,\mathcal{D}M^{-\frac{1}{2}a}\,\mathcal{D}\Lambda_{MN}\,\mathcal{D}\mathcal{M}_{AB}$$
$$\times \exp\Big\{i\widetilde{S}\Big\}\ \delta\big(\chi^{(\mathrm{rep})}\big)\,\delta\big(v^{+\frac{1}{2}}u^a_\mu\sigma^\mu\theta\big)\,\delta\big(\Omega^{AB}\big)\,\delta\big(\chi^{(\mathrm{rot})}\big)\,\Delta^{(\mathrm{rep})}_{\mathrm{FP}}\Delta^{(\mathrm{norm})}_{\mathrm{FP}}\Delta^{(\mathrm{rot})}_{\mathrm{FP}}\det{}^{-12}\big[u^+_\mu\Pi^\mu\big]\ . \tag{170}$$

In [C12,C13] we have systematically performed the passage from the Hamiltonian to the Lagrangian form of the functional integral (170). In the course of derivation we make change of fermionic space-time variables:

$$\theta_\alpha \longrightarrow \theta^{\pm\frac{1}{2}a} = v^{\pm\frac{1}{2}} u^a_\mu \sigma^\mu \theta \tag{171}$$

with a subsequent nonlinear rescaling:

$$\theta^{-\frac{1}{2}a} \longrightarrow \psi^a = -2\left(\partial_{\bar z} X^\mu u^+_\mu\right)^{\frac{1}{2}} \theta^{-\frac{1}{2}a} \tag{172}$$

(recall again that that u^8, $u^\pm$ are bilinear composites (150)). As a final result we obtain *quadratic* covariant gauge-fixed path integral in terms of a finite number of conformal fields and ghosts:

$$Z = \int \mathcal{D}X^\mu \mathcal{D}\psi^a \mathcal{D}u\, \mathcal{D}v\, \mathcal{D}(\pi_u)_z \mathcal{D}(\pi_v)_z \mathcal{D}b^z\, \mathcal{D}c_{zz} \mathcal{D}\bar b^{\bar z}\, \mathcal{D}\bar c_{\bar z\bar z} \mathcal{D}\bar\eta_z \mathcal{D}\eta\, \mathcal{D}\bar\zeta_z D\zeta \\ \times \exp\{iS^{\text{bilinear}}_{GS} + S_{\text{internal}}\}\ , \tag{173}$$

where:

$$S^{\text{bilinear}}_{GS} = -2\int d\tau d\xi\, [\partial_z X^\mu \partial_{\bar z} X_\mu + i\psi^a \partial_z \psi_a + b^z \partial_{\bar z} c_{zz} + \bar b^{\bar z} \partial_z \bar c_{\bar z\bar z} \\ + (\pi_v^{\mp\frac{1}{2}})_z \partial_{\bar z} v^{\pm\frac{1}{2}} + (\pi^p_u)_z \partial_{\bar z} u_p + \bar\eta^{MN}_z \partial_{\bar z}\eta_{MN} + \bar\zeta^{AB}_z \partial_{\bar z}\zeta_{AB}]\ . \tag{174}$$

Here (b, c) are the standard reparametrization ghosts, whereas $\left(\bar\zeta, \zeta\right)$ and $(\bar\eta, \eta)$ are the ghosts for the orthonormality (159) and frame-rotation (160)–(165) gauge constraints, respectively.

Moreover we have explicitly shown that apart from the standard conformal anomalies, which cancel in the present $D = 10$ case, there are *no further anomalies* in (173). Further details are contained in [C11,C12,C13].

Eqs.(173)–(174) are the basic starting point for calculations of Green-Schwarz superstring amplitudes in a manifestly super-Poincare covariant formalism upon inserting in (173) the appropriate covariant vertex operators which in general will depend also on the auxiliary variables $\left(v^{\pm\frac{1}{2}}_\alpha,\, u^p_\mu\right)$. For further details, see [C8,C10].

4.4 Action Principle for Over-Determined Systems of Non-Linear Field Equations

The zero-mode (point-particle) limit of the Green-Schwarz superstring (140)–(142) is the Brink-Schwarz superparticle (with $N = 1$ space-time supersymmetry) whose action reads (in Hamiltonian form):

$$S_{\text{BS}} = \int d\tau\, \left[p_\mu \partial_\tau x^\mu + p^\alpha_\theta \partial_\tau \theta_a - \lambda p^2 - \lambda_\alpha d^\alpha\right]\ , \tag{175}$$

$$d^\alpha \equiv -ip^\alpha_\theta - \not p^{\alpha\beta}\theta_\beta \quad , \quad \left\{d^\alpha,\, d^\beta\right\}_{\text{PB}} = 2i\not p^{\alpha\beta}\ , \tag{176}$$

the fermionic spinor constraint d^α again being a Lorentz non-covariant mixture of first and second class constraints. Our formalism developed in the previous two subsections can naturally be applied for super-Poincare covariant quantization of the Brink-Schwarz superparticle as a simple limiting case of the Green-Schwarz superstring (see next subsection).

Indeed, we have shown in [C5,C10] that second quantization of Brink-Schwarz $N = 1$ superparticle in $D = 10$ space-time dimensions yields a covariant off-shell superspace formulation for the linearized version of the $D = 10$ $N = 1$ super-Yang-Mills gauge theory. Similar results have been obtained in [C3] for the second quantized Brink-Schwarz $N = 2$ superparticle in $D = 10$ as a covariant off-shell superspace formulation for the linearized $D = 10$ type IIB supergravity. See also [C2] for second quantization of $D = 4$ $N = 1, 2$ superparticles yielding (linearized) superfield theories of $N = 1$ chiral and vector supermultiplets and $N = 2$ matter and super-Maxwell multiplets, respectively.

From a mathematical point of view, the quantum first-class Dirac constraint equations constitute a *consistent overdetermined set* of $\mathcal{N}$ (= number of Dirac first class constraint) linear (matrix) equations for one (vector-valued) wave function, *i.e.*, $\mathcal{N}$ classical (matrix) free-field equations for one classical (vector-valued) field. Similar structure will arise in the full non-linear field theory. In particular, the well-known geometrical (Nilsson's) constraints [38] on the superfield gauge potential in the full non-linear $D = 10$ $N = 1$ super-Yang-Mills theory, which are in fact equivalent to the equations of motion, can be cast [C10] in the above mentioned form of a *consistent overdetermined system* of non-linear field equations by exclusive use of our spinor-harmonic superspace formalism.

In our works [C9,C10] we have proposed a general scheme for constructing an action principle for arbitrary consistent overdetermined systems of non-linear field equations. As a main application the latter scheme combined with our spinor-harmonic superspace formalism yields a manifestly covariant action for the $D = 10$ $N = 1$ super-Yang-Mills theory in terms of unconstrained off-shell superfields.

Here below we will describe the main steps of the above mentioned general scheme. Let us consider the following general overdetermined system of $\mathcal{N}$ non-linear equations:

$$\mathcal{L}_A(\phi|z) \equiv L_A\phi(z) + V_A(\phi|z) = 0 \quad , \quad A = 1, \ldots, \mathcal{N} \; , \tag{177}$$

$$V_A(\phi|z) \equiv \sum_{n\geq 2} \int dz_1 \ldots dz_n V_A^{(n)}(z; z_1, \ldots, z_n)\, \phi(z_1) \ldots, \phi(z_n) \; . \tag{178}$$

In (177) the function ("field") $\phi(z)$ is defined on a (graded) linear space $\mathcal{R}$ and it takes values in another (graded) vector space $\mathcal{U}$, *i.e.*, $\phi(z)$ has a vector index ($\phi^a(z)$) and may contain both bosonic and fermionic components. L_A are (graded) linear differential operators of at most second degree and are, correspondingly, matrices $\left(L_A \equiv \left(L_A^{ab}\right)\right)$ in the vector space $\mathcal{U}$. Similarly $V_A(\phi|z) \equiv ([V_A(\phi|z)]^a)$ are also vectors in $\mathcal{U}$. The vector indices (a, b) will be suppressed for brevity.

The necessary conditions for consistency of the overdetermined system (177) are obtained by repeated application of (graded) antisymmetrized products of the linear operators L_A on $\mathcal{L}_B(\phi|z)$ (177) and by requiring the result to vanish when Eqs.(177) are fulfilled. The first consistency condition yields for the linear and non-linear parts of (177), respectively:

$$\Big[L_A,\, L_B\Big\} \equiv L_A L_B + (-1)^{\varepsilon_A\varepsilon_B+1} L_B L_A = f_{AB}^C L_C \; , \tag{179}$$

$$\begin{aligned} &L_A V_B(\phi|z) + (-1)^{\varepsilon_A\varepsilon_B+1} L_B V_A(\phi|z) - f_{AB}^C V_C(\phi|z) \\ &= \int dz' \Big[\frac{\delta V_B(\phi|z)}{\delta\phi(z')}\mathcal{L}_A(\phi|z')(-1)^{\varepsilon_A\varepsilon_B+1}\frac{\delta V_A(\phi|z)}{\delta\phi(z')}\mathcal{L}_B(\phi|z')\Big] \\ &= 0 \;\; (\text{on the surface of Eq.}(177)) \; . \end{aligned} \tag{180}$$

In (179)–(180) f_{AB}^{C} are in general linear differential operators and $\varepsilon_{A,B}$ are the Grassmann parities of $L_{A,B}$. In Eq.(180) the operators L_A act on $V_B(\phi|z)$ as on functions of z.

The next level consistency condition gives using (179)–(180):

$$\left[f_{ABC}^{(2)\,DE}(-1)^{\varepsilon_D} f_{ED}^{G}\right]_{\text{antisymm}(A,B,C)} V_G(\phi|z) = 0 \;, \tag{181}$$

where $f_{ABC}^{(2)\,DE}$ are in general operators defined through:

$$f_{ABC}^{(2)\,DE} L_E = \left((-1)^{\varepsilon_B+\varepsilon_D+1}\left\{(-1)^{\varepsilon_D\varepsilon_C}\left[f_{AB}^{D}, L_C\right] + f_{AB}^{G} f_{GC}^{D}\right\}\right)_{\text{antisymm}(A,B,C)} \;, \tag{182}$$

and "antisymm $(\ldots)$" means graded antisymmetrization. For most interesting systems, in particular for the case at hand ($D = 10$, $N = 1$ super-Yang-Mills theory), it turns out that:

$$f_{ABC}^{(2)\,DE} = 0 \;, \tag{183}$$

i.e., there are no further independent consistency conditions for the overdetermined system (177) apart from (179)–(180).

In what follows we will employ the "ghost" formalism of Batalin-Fradkin-Vilkovisky (BFV) [39], which allows us to rewrite the whole consistent overdetermined system (177) of $\mathcal{N}$ (matrix) field equations as a single (matrix) equation in terms of a (vector-valued) field $\Phi(z, \eta)$ depending on additional "ghost" variables collectively denoted by $\eta \equiv (\eta^A)$ $(A = 1, \ldots, \mathcal{N})$:

$$\Phi(z, \eta) = \phi(z) + \sum_{n\geq 1} \frac{1}{n!} \eta^{A_1} \ldots \eta^{A_n} \widetilde{\phi}_{A_1 \ldots A_n}(z) \;, \tag{184}$$

where η^A has opposite Grassmann parity with respect to L_A: $\varepsilon(\eta^A) = \varepsilon_A + 1$.

The new single (matrix) equation for $\Phi(z, \eta)$ replacing the system (177) reads:

$$Q(\Phi|z, \eta) \equiv Q_0 \Phi(z, \eta) + \mathcal{V}(\Phi|z, \eta) = 0 \;, \tag{185}$$

where:

$$Q_0 = \eta^A L_A + \frac{1}{2}(-1)^{\varepsilon_B} \eta^B \eta^C f_{CB}^{A} \frac{\partial}{\partial \eta^A} \tag{186}$$

$$\mathcal{V}(\Phi|z, \eta) = \sum_{n\geq 2} \int dz_1 \ldots dz_n \eta^A V_A^{(n)}(z; z_1, \ldots, z_n)\, \Phi(z_1, \eta) \ldots, \Phi(z_n, \eta) \;, \tag{187}$$

the kernels $V_A^{(n)}(\ldots)$ being the same as in (178). The original overdetermined system (177) is contained in (185) as the lowest order term in the "ghost" η- expansion of $Q(\Phi|z, \eta)$.

Note that Q_0 (186) entering (185) is the BRST charge corresponding to the algebra (179), where the operators L_A may be viewed as quantized Dirac first-class Hamiltonian constraints. Moreover, in the BFV language $f_{ABC}^{(2)\,DE}$ is precisely the so called second order BFV structure function [39] and the condition (183) means that the corresponding constrained Hamiltonian system is BFV first-rank (no higher-order ghost terms in the BRST charge Q_0 (186)).

In [C9,C10] we have derived the following covariant action producing (185) as equation of motion:

$$S = \int dz\, d\eta\, \hat{H}\Phi(z,\eta)\, \overline{Q}(\Phi|z,\eta) \tag{188}$$

with notations as follows. The linear operator $\hat{H}$ is defined to fulfill ("T" denotes operator transposition):

$$\hat{H}^T = \hat{H} \quad , \quad Q_0^T\, \hat{H} = \hat{H}\, Q_0 \tag{189}$$

A typical form of $\hat{H}$ is:

$$\hat{H}\Phi(z,\eta) = R\Phi(\rho_1 z, \rho_2 \eta) \quad , \quad \rho_{1,2} = \pm 1, \pm i \; , \tag{190}$$

where R is a matrix acting on vector indices of $\Phi(z,\eta)$ (see next subsection). The functional $\overline{Q}(\Phi|z,\eta)$ has the following explicit form:

$$\overline{Q}(\Phi|z,\eta) \equiv \frac{1}{2} Q_0 \Phi(z,\eta) + \overline{\mathcal{V}}(\Phi|z,\eta) \; , \tag{191}$$

$$\overline{\mathcal{V}}(\Phi|z,\eta) = \sum_{n\geq 2} \int dz_1 \ldots dz_n \frac{1}{n+1}\, \eta^A V_A^{(n)}(z; z_1, \ldots, z_n)\, \Phi(z_1,\eta) \ldots, \Phi(z_n,\eta) \; , \tag{192}$$

with $V_A^{(n)}(\ldots)$ – the same as in (178) and (187).

In [C9,C10] we have also shown that the action (188) possesses Witten-type [40] gauge symmetry:

$$\delta_\Lambda \Phi(z,\eta) = \int dz'\, d\eta'\, \Lambda(z',\eta') \frac{\delta Q(\Phi|z,\eta)}{\delta\Phi(z',\eta')} \; . \tag{193}$$

The original field $\phi(z)$ does not change under the gauge transformation (193).

Concluding this subsection let us emphasize that the action principle for arbitrary consistent overdetermined systems of non-linear field equations proposed in our papers [C9,C10] has an independent value by itself beyond the present context of covariant superstring quantization.

4.5 Off-Shell Superspace $D = 10$ Super-Yang-Mills from a Covariantly Quantized Green-Schwarz Superstring

Here we shall illustrate the general scheme developed in the previous subsection by deriving the off-shell superspace action for the linearized $N = 1$ super-Yang-Mills gauge theory in $D = 10$ as a second quantization of the $D = 10$, $N = 1$ Brink-Schwarz superparticle (175). About the detailed derivation of the off-shell superspace action for the full non-linear $D = 10$, $N = 1$ super-Yang-Mills theory, see [C9,C10].

For a covariant quantization of the Brink-Schwarz superparticle we will use here a version of our spinor-harmonic superspace formalism first proposed in [C6,C7] (see also [C10]). The main idea is to further extend the phase space of the system by introducing additional fermionic variables Ψ^a ($a = 1, \ldots, 8$ – "internal" $SO(8)$ index) which convert the original fermionic constraint d^α (176) (non-covariant mixture of first and second class constraints) into fully covariant first class ones $\widehat{d}^{\alpha}$ (see (197) below). This idea was first proposed in a different context in Refs.[41]. The main advantage of having an extended constrained Hamiltonian system with first class constraints only is that, while preserving the physical content of the original system, the passage from Poisson to Dirac brackets as a prerequisite for quantization is avoided.

Before proceeding let us introduce the following short-hand notations which will turn very useful in the sequel:

$$B^a \equiv u^a_\mu B^\mu \quad , \quad B^\pm \equiv u^a_\mu B^\mu \equiv v^{\pm\frac{1}{2}}\sigma_\mu B^m v^{\pm\frac{1}{2}} \quad , \quad \sigma^{ab} \equiv u^a_\mu \sigma^{[\mu}\sigma^{\nu]}u^b_\nu \quad , \quad \sigma^{\pm a} \equiv u^\pm_\mu \sigma^{[\mu}\sigma^{\nu]}u^a_\nu \; , \tag{194}$$

where B^μ is an arbitrary $D = 10$ Lorentz vector.

The action of $D = 10$, $N = 1$ Brink-Schwarz superparticle (175) in the extended spinor-harmonic superspace formalism reads [C6,C7,C10]:

$$\widetilde{S}_{\rm BS} = \int d\tau \left[p_\mu \partial_\tau x^\mu + p^\alpha_\theta \partial_\tau \theta_a + i\Psi^a \partial_\tau \Psi_a - \lambda p^2 - \lambda_\alpha \widehat{d}^\alpha \right] + \widehat{S}_{\rm auxiliary} \; , \tag{195}$$

$$\widehat{S}_{\rm auxiliary} = \int d\tau \left[p^\mu_{u,a}\, \partial_\tau u^a_\mu + p_v^{-\frac{1}{2}\alpha} \partial_\tau v_\alpha^{+\frac{1}{2}} + p_v^{+\frac{1}{2}\alpha} \partial_\tau v_\alpha^{-\frac{1}{2}} \right.$$
$$\left. - \lambda_{ab} \widehat{D}^{ab} - \lambda^{+-} D^{-+} - \lambda^-_a D^{+a} - \lambda^+_a \widehat{D}^{-a} \right] . \tag{196}$$

The explicit expressions for the first class constraints entering (195)–(196), apart from the standard reparametrization constraint p^2, are as follows:

$$\widehat{d}^\alpha = d^\alpha \, (\text{Eq.}(176)) + (p^+)^{-\frac{1}{2}} \left(\not{p} \sigma^+ \sigma^a v^{-\frac{1}{2}} \right)^\alpha \Psi_a \; , \tag{197}$$

$$D^{-+} = \frac{1}{2}\Big(v^{+\frac{1}{2}} \frac{\partial}{\partial v^{+\frac{1}{2}}} - v^{-\frac{1}{2}} \frac{\partial}{\partial v^{-\frac{1}{2}}} \Big) \; , \tag{198}$$

$$D^{+a} = u^+_\mu \frac{\partial}{\partial u_{\mu\, a}} + \frac{1}{2} v^{-\frac{1}{2}} \sigma^+ \sigma^a \frac{\partial}{\partial v^{-\frac{1}{2}}} \; , \tag{199}$$

$$\widehat{D}^{-a} = D^{-a} - \frac{p_b}{p^+} \widehat{R}^{ab} \quad , \quad D^{-a} \equiv u^-_\mu \frac{\partial}{\partial u_{\mu\, a}} + \frac{1}{2} v^{+\frac{1}{2}} \sigma^- \sigma^a \frac{\partial}{\partial v^{+\frac{1}{2}}} \; , \tag{200}$$

where:

$$\widehat{R}^{ab} \equiv \frac{1}{4} \big(v^{-\frac{1}{2}} \sigma_c \sigma^{[a} \sigma^{b]} \sigma^+ \sigma_d v^{-\frac{1}{2}} \big) \Psi^c \Psi^d \; , \tag{201}$$

and:

$$\widehat{D}^{ab} = D^{ab} + \widehat{R}^{ab} \tag{202}$$

with:

$$D^{ab} \equiv u^a_\mu \frac{\partial}{\partial u_{\mu\, b}} - u^b_\mu \frac{\partial}{\partial u_{\mu\, a}} + \frac{1}{2} \big(v^{+\frac{1}{2}} \sigma^{[a}\sigma^{b]} \frac{\partial}{\partial v^{+\frac{1}{2}}} + v^{-\frac{1}{2}} \sigma^{[a}\sigma^{b]} \frac{\partial}{\partial v^{-\frac{1}{2}}} \big) \; . \tag{203}$$

From (198)–(203) one immediately recognizes $\widehat{D}^{ab}$, D^{-+} as generators of "internal" $SO(8) \times SO(1,1)$, whereas D^{+a}, $\widehat{D}^{-a}$ are recognized as the coset generators corresponding to $SO(1,9)/SO(8) \times SO(1,1)$. Both terms entering (202) may be interpreted as harmonic "orbital" and harmonic "spin" $SO(8)$ rotations.

The quantum counterpart of the classical graded Poisson bracket for the auxiliary fermionic variables $\left\{\Psi^a, \Psi^b\right\}_{\rm PB} = -i\delta^{ab}$ is the anticommutator:

$$\left\{\Psi^a, \Psi^b\right\} = \delta^{ab} \ , \tag{204}$$

meaning that the quantized Ψ^a are represented as 16×16 Dirac $SO(8)$ Γ-matrices:

$$\Psi^a = \frac{1}{\sqrt{2}}\Gamma^a_8 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & (\gamma^a)_{bc} \\ (\tilde{\gamma}^a)_{bc} & 0 \end{pmatrix} \tag{205}$$

where:

$$(\gamma^a)_{bc} \equiv \sqrt{2}v^{+\frac{1}{2}}\sigma_b\sigma^a\sigma_c v^{-\frac{1}{2}} \quad , \quad (\tilde{\gamma}^a)_{bc} \equiv \sqrt{2}v^{-\frac{1}{2}}\sigma_b\sigma^a\sigma_c v^{+\frac{1}{2}} \ . \tag{206}$$

Therefore, the wave function:

$$\phi(z) \equiv \phi\big(x^\mu, \theta_\alpha, u^a_\mu, v^{\pm\frac{1}{2}}_\alpha\big) \tag{207}$$

will be 16-component vector with upper fermionic and lower bosonic halfs.

As shown in [C10], upon appropriate similarity transformation we can bring $\phi(z)$ (207) in the following form:

$$\phi(z) = \begin{pmatrix} Y^{+\frac{1}{2}a}(z) \\ B^a(z) \end{pmatrix} = \begin{pmatrix} \big(v^{+\frac{1}{2}}\sigma^a\big)^\alpha Y_\alpha(z) \\ u^a_\mu B^\mu(z) \end{pmatrix} \ . \tag{208}$$

Correspondingly, the (first-quantized) Dirac constrained equations corresponding to the Hamiltonian constraints (197)–(203) in (195)–(196) acquire the form:

$$(-\partial^2)\phi \equiv \begin{pmatrix} (-\partial^2)Y^{+\frac{1}{2}a} \\ (-\partial^2)B^a \end{pmatrix} = 0 \ , \tag{209}$$

$$\widehat{D}^\alpha\phi \equiv \begin{pmatrix} D^\alpha Y^{+\frac{1}{2}a} - i(\partial\!\!\!/\sigma^b\sigma^a v^{+\frac{1}{2}})^\alpha B_b \\ D^\alpha B^a - (\partial^+)^{-1}(\partial\!\!\!/\sigma^b\sigma^a v^{+\frac{1}{2}})^\alpha Y^{+\frac{1}{2}}_b \end{pmatrix} = 0 \ , \tag{210}$$

where $D^\alpha = \frac{\partial}{\partial\theta_a} + i(\partial\!\!\!/\theta)^\alpha$;

$$D^{+a}\phi \equiv \begin{pmatrix} D^{+a}Y^{+\frac{1}{2}b} \\ D^{+a}B^b \end{pmatrix} = 0 \ , \tag{211}$$

$$\widehat{D}^{-a}\phi \equiv \begin{pmatrix} \big(D^{-a} - \frac{1}{2}\partial^a(\partial^+)^{-1}\big)Y^{+\frac{1}{2}b} - (S^{ac})^b_d\partial_c(\partial^+)^{-1}Y^{+\frac{1}{2}d} \\ D^{-a}B^b - (V^{ac})^b_d\partial_c(\partial^+)^{-1}B^d \end{pmatrix} = 0 \ , \tag{212}$$

where:

$$(S^{ab})_{cd} \equiv \frac{1}{2}v^{+\frac{1}{2}}\sigma_c\sigma^{[a}\sigma^{b]}\sigma^-\sigma_d v^{+\frac{1}{2}} \quad , \quad (V^{ab})_{cd} \equiv \delta^a_c\delta^b_d - \delta^a_d\delta^b_c \ , \tag{213}$$

and the constraint equations $\widehat{D}^{ab}\phi = 0$, $\widehat{D}^{-+}\phi = 0$ are identically satisfied for (208).

Now, notice that Eqs.(209)–(212) represents specific case of a consistent overdetermined system of (linear) equations $L_A\phi(z) = 0$ as in (177) with:

$$L_A \equiv \left(-\partial^2 \ , \ \widehat{D}^\alpha \ , \ D^{+a} \ , \ \widehat{D}^{-a}\right) \quad , \quad \eta^A \equiv \left(c \ , \ \chi_\alpha \ , \ \eta^{-a} \ , \ \eta^{+a}\right) \tag{214}$$

Therefore, we can use the formalism of the previous subsection to write down a covariant action yielding (209)–(212) as variational equations of motion. This action has the form (cf. (188)):

$$S_0 = \frac{1}{2}\int dz\, d\eta\, \hat{H}\Phi(z,\eta)\, Q_0\Phi(z,\eta) \tag{215}$$

where the objects are as follows:

$$\Phi(z,\eta) \equiv \begin{pmatrix} \mathcal{Y}^{+\frac{1}{2}a}(z,\eta) \\ \mathcal{B}^a(z,\eta) \end{pmatrix} \quad , \quad (z,\eta) \equiv \left(x^\mu, \theta_\alpha, u^a_\mu, v^{\pm\frac{1}{2}}_\alpha \ ; \ c, \chi_\alpha, \eta^{-a}, \eta^{+a}\right) , \tag{216}$$

Q_0 is given by (186) with L_A, η^A as in (214), and the operator $\hat{H}$ satisfying conditions (189) acts on functions of (z,η) (216) as:

$$\hat{H} = \begin{pmatrix} -\frac{1}{2}\left(K_1 + K_1^T\right)(\partial^+)^{-1} & 0 \\ 0 & \frac{1}{2}\left(K_2 + K_2^T\right) \end{pmatrix} \tag{217}$$

$$K_1 \ : \ v^{\pm\frac{1}{2}}_\alpha \to i \pm v^{\pm\frac{1}{2}}_\alpha \quad , \quad c \to -c \quad , \quad \eta^{\pm a} \to -\eta^{\pm a} \tag{218}$$

$$K_2 \ : \ v^{\pm\frac{1}{2}}_\alpha \to i \pm v^{\pm\frac{1}{2}}_\alpha \quad , \quad \chi_\alpha \to -\phi_\alpha \tag{219}$$

Formula (215) is the superspace action for the linearized $D = 10$ super-Yang-Mills field theory in terms of unconstrained (off-shell) superfields. The connection with the fundamental super-Yang-Mills gauge potentials $A^\mu(x,\theta)$, $A^\alpha(x,\theta)$ is given by:

$$\Phi(z,\eta) \equiv \begin{pmatrix} \mathcal{Y}^{+\frac{1}{2}a}(z,\eta) \\ \mathcal{B}^a(z,\eta) \end{pmatrix} = \begin{pmatrix} \frac{i}{2}\partial^+(v^{+\frac{1}{2}}\sigma^a\sigma^-)_\beta\left[\mathcal{A}^\beta(z,\eta) + iD^\beta\lambda(z,\eta)\right] \\ u^a_\mu\left[\mathcal{A}^\mu(z,\eta) + \partial^\mu\lambda(z,\eta)\right] \end{pmatrix} \tag{220}$$

where:

$$\lambda(z,\eta) \equiv -\int^{x^-} dy^- \, u^+_\mu \mathcal{A}^\mu\left(x(y^-;u,v),\theta,u,v;\eta\right) \tag{221}$$

$$x^- \equiv u^-_\mu x^\mu \quad , \quad x^\mu(y^-;u,v) \equiv \left(\eta^{\mu\nu} + u^{+\,\mu}u^{-\,\nu}\right)x_\nu - u^{+\,\mu}y^-$$

As in (184) we have ((z,η) as in (216)):

$$\mathcal{A}^\alpha(z,\eta) = A^\alpha(z) + \sum_{n\geq 1}\frac{1}{n!}\eta^{B_1}\ldots\eta^{B_n}A^\alpha_{B_1\ldots B_n}(z) \tag{222}$$

$$\mathcal{A}^\mu(z,\eta) = A^\mu(z) + \sum_{n\geq 1}\frac{1}{n!}\eta^{B_1}\ldots\eta^{B_n}A^\mu_{B_1\ldots B_n}(z) \tag{223}$$

and:

$$A^{\alpha}(z) = A^{\alpha}(x,\theta) + \ldots \quad , \quad A^{\mu}(z) = A^{\mu}(x,\theta) + \ldots \ , \tag{224}$$

where the dots indicate higher-order terms in the harmonic expansion with respect to $\left(u^a_\mu,\, v^{\pm\frac{1}{2}}\right)$. As shown in [C10], the lowest components $A^{\alpha}(x,\theta)$, $A^{\mu}(x,\theta)$ in (224) are precisely the super-Yang-Mills gauge potentials. The proof goes by showing that, upon substituting of (220) into the equations of motion for (215) which are equivalent to the Dirac quantized constrained equations (209)–(212), the latter reduce to the (linearized) Nilsson constraints for $A^{\alpha}(x,\theta)$, $A^{\mu}(x,\theta)$, *i.e.*, the equations of motion for the (linearized) $D = 10$ super-Yang-Mills theory.

Further details, including complete derivation of covariant off-shell superspace action for the full non-linear $D = 10$ super-Yang-Mills theory, are contained in [C10].

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6 LIST of selected scientific papers whose full text is included as part of the present thesis

- Total number of included selected papers: **31** (out of 92 papers in the complete list of author's papers)

- Total number of selected papers published in international scientific journals: **25**
  Phys.Lett. B – 9 ; *Nucl.Phys. B* - 4 ; *Comm.Math.Phys.* - 1 ; *Mod.Phys.Lett. A* - 1 ; *Int.Journ.Mod.Phys. A* - 1 ; *Lett.Math.Phys.* – 6 ; *Theor.Math.Phys.* – 3
  Total number of selected papers published in Bulgarian scientific journals: **3**
  Bulg.J.Phys. – 2 ; *Comp.Rend.Acad.Sci.Bulg.* – 1
  Total number of selected papers published as full-text articles in proceedings of international conferences: **3**

- Total number of independent citations of the included selected papers: **599** (out of 854 independent citations of all author's papers)

- Total impact-factor of the included selected papers: **75.400** (out of the total impact-factor 165.730 of all author's papers)

A1. CHIRAL FIELD MODEL AND UNIVERSALITY IN THREE-DIMENSIONAL SPACE. I. *Theor. Math. Phys.* **41** (1979) 882-891. (*Teor. Mat. Fiz.* **41** (1979) 55-68).
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7 Principal Contributions

In the context of three-dimensional quantum gauge theories with fermions and within the framework of the non-perturbative $1/N$ expansion we have found for the first time in the literature explicit realizations of the following non-perturbative mechanisms ((A1)–(A6) modelling the principal physical properties of the realistic four-dimensional gauge theories:

(A1) Dynamical mass generation, including dynamical generation of gauge-invariant masses for the gluons – this is a radically new mechanism significantly different from the standard Higgs mechanism.

(A2) Multiple phases defined via *more than one* order parameters, which are related to dynamical spontaneous breakdown (and restoration) not only of continuous internal symmetries, but also discrete space-time reflection symmetries. The latter represents an explicit solvable three-dimensional model of the dynamical chiral symmetry breaking in quantum chromodynamics.

(A3) Non-perturbative particle spectra, qualitatively different in the various phases, including particle "confinement" in some of the phases and their "deconfinement" in other phases.

(A4) A systematic quantum field theoretic approach is developed for description of the pertinent phase transitions and critical behaviour of the three-dimensional gauge theories with fermions within the non-perturbative $1/N$ expansion. All result in (A1)–(A4) are generalized to the case of three-dimensional supersymmetric gauge theories.

(A5) For the first time in the literature we have proved explicitly the renormalizability of naively non-renormalizable (within the standard perturbation theory) quantum field theory models, including those containing four-fermion interactions.

(A6) Explicit construction of the critical theories at the second order phase transition points, which turn out to be three-dimensional supersymmetric nonlinear sigma-models. The latter are nontirival examples of *three-dimensional conformal gauge theories with fermions* whose anomalous operator dimensions are explicitly calculable with our $1/N$ expansion techniques.

Furthermore:

(B1) We have found and thoroughly studied the explicit mechanisms for dynamical *anomalous* (not spontaneous) breaking of discrete space-time symmetries in three-dimensional gauge theories, which significantly contributes to the deeper understanding of the dynamical *anomalous* chiral symmetry breaking in four-dimensional quantum chromodynamics. Here for the first time in the literature we have proposed an adequate systematic approach for studying the anomalous breaking of discrete $P-$ and $T-$reflection parities by using the so called Atiyah-Patodi-Singer topological invariant. These results are generalized to the case of quantum anomalies of discrete symmetries in supersymmetric three-dimensional gauge theories.

(B2) For the first time in the literature we have found and studied in detail the explicit mechanisms for dynamical spontaneous and dynamical anomalous symmetry breaking within the framework of the non-perturbative stochastic approach to the quantum gauge field theories which is of fundamental importance for the self-consistency of stochastic quantization.

(B3) For the first time in the literature we have proposed a general scheme for construction of conserved Noether currents within the framework of stochastic quantization which are counterparts of the symmetries in the corresponding equilibrium quantum field theory.

(B4) Within the framework of stochastic quantization we have identified the explicit mechanisms of topological quantization of physical parameters.

(C1) We have provide a systematic solution to the problem of manifestly super-Poincare covariant quantization of strings with space-time supersymmetry (Green-Schwarz superstrings). This is achieved with the help of bosonic spinorial auxiliary (pure-gauge) degrees of freedom introduced in our works for the first time in the literature.

(C2) For the first time in the literature we have succeeded to explicitly construct generating functional integral for the Green-Schwarz superstring correlation functions with an explicitly Lorentz-covariant gauge-fixing of the fermionic kappa-symmetry. It is shown that this generating functional acquires the form of a functional integral for a free two-dimensional conformal field theory with a finite number of matter and ghost fields on the superstring world-sheet, which is free of conformal anomalies in ten-dimensional embedding space-time.

(C3) As a nontrivial application of the results in (A1)–(A2) we have found a new explicitly covariant formulation of supersymmetric gauge theories with extended supersymmetry in terms of new types of off-shell unconstrained superfields.